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Robustly chain transitive diffeomorphisms

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Abstract

In this paper, we discuss the robustly chain transitive set, and show that the robustly chain transitive set is hyperbolic if and only if every periodic points in the set is hyperbolic and has the same index.

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1 Introduction

In the theory of dynamical systems one has been to describe and characterize systems exhibiting dynamical properties that are preserved under small perturbations. It is related to the stability theory. In fact, structurally stable systems and Ω-stable systems have been the main objects of interests in the global qualitative theory of dynamical systems and they are characterized as the hyperbolic ones (see [1–4]). Thus, in differentiable dynamical systems, the robustness property is a very interesting topic. Let us consider more details. Let M be a closed C^∞ Riemannian manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$ and Λ be a closed f-invariant set.

The set Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. Here $\omega(x)$ is the forward limit set of x. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

We say that the set Λ is *robustly transitive* if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is transitive. Here $f|_\Lambda$ is transitive means that f is transitive in Λ. If $\Lambda = M$ then we say that f is robustly transitive. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. If $\Lambda = M$ then we say that f is Anosov. Although f is robustly transitive, we can find that f is not Anosov. In fact, Mañé [5] showed that there exists a diffeomorphism f on the three-dimensional torus $\mathbb{T}^3$ that satisfies: there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ is transitive, but not Anosov.

In [6], Mañé proved that if a diffeomorphism on two-dimensional C^∞ manifolds is robustly transitive, then it is hyperbolic, and Díaz *et al.* [7] proved that if a diffeomorphism on three-dimensional C^∞ manifolds is robustly transitive then it is partially hyperbolic. Also, in [8], the authors proved that for C^∞ manifolds of any dimension, if a diffeomorphism is robustly transitive, then it admits a dominated splitting.

From the facts, we study the relation between the robustly chain transitive and hyperbolicity. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^n$ ($n \geq 1$) of f such that $x_0 = x$ and $x_n = y$. For any $x, y \in \Lambda$, we write $x \rightsquigarrow_\Lambda y$ if $x \rightsquigarrow y$ and $\{x_i\}_{i=0}^n \subset \Lambda$ ($n \geq 1$). We say that the set Λ is *chain transitive* (or, $f|_\Lambda$ is *chain transitive*) if for any $x, y \in \Lambda$, $x \rightsquigarrow_\Lambda y$. Note that by the definition, a transitive set is a chain transitive set, but the converse is not true (see Example 1.5 in [9]). In this paper, we study robustly chain transitive sets for a diffeomorphism. It is weaker notion of the robustly transitivity. Let $p \in P(f)$ be a hyperbolic point. Denote by $\mathrm{index}(p) = \dim W^s(p)$. We say that the set Λ is *robustly chain transitive* if there are a C^1-neighborhood $\mathcal{U}(f)$ and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is chain transitive. Then we have the following.

Theorem 1.1 *Let $f|_\Lambda$ be robustly chain transitive in U. Then following conditions are equivalent*:

(a) *there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, any periodic point of $\Lambda_g(U)$ is hyperbolic and has the same index*;

(b) *there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U)$ is hyperbolic.*

2 Proof of Theorem 1.1

It is clear that (a) follows from (b) by the local stability of hyperbolic basic set (see Theorem 7.4 in [10]). To prove Theorem 1.1, we show from (a) to (b). We say that $p \in P(f)$ with period $\pi(p)$ is a *sink* if all the eigenvalues of $D_pf^{\pi(p)}$ are less than 1, and $p \in P(f)$ with period $\pi(p)$ is a *source* if all eigenvalues of $D_pf^{\pi(p)}$ is greater than 1. The following is the version for diffeomorphisms of the result by Lemma 6 in [11].

Lemma 2.1 *If $f|_\Lambda$ is chain transitive, then $f|_\Lambda$ has neither sinks nor sources.*

Proof Let p be a sink. Then there exist $\epsilon > 0$ and $\lambda < 1$ such that if $d(x,p) < \epsilon$ then $d(f^i(x),p) < \lambda d(x,p)$ for all $i \geq 1$. Take $y \in \Lambda$ such that $d(y,p) \geq 2\epsilon$. For any $\delta > 0$, let $\xi = \{p = x_0, x_1, \ldots, x_m = y\}$ ($m \geq 1$) be a δ-pseudo orbit of f such that $x_i \in \Lambda$. For simplicity, we may assume that $f(p) = p$. Then we have $d(p,x_1) < \delta$, and $d(p,x_2) \leq d(p,f(x_1)) + d(f(x_1),x_2) < \lambda d(p,x_1) + \delta < \delta(\lambda + 1)$. Thus we obtain

$$\begin{aligned} d(p,x_i) &\leq d\bigl(p,f(x_{i-1})\bigr) + d\bigl(f(x_{i-1}),x_i\bigr) < d\bigl(p,f(x_{i-1})\bigr) + \delta \\ &< \lambda d(p,x_{i-1}) + \delta < \cdots < \delta\bigl(1 + \lambda + \cdots + \lambda^i\bigr) \\ &\leq \frac{\delta}{1-\lambda}. \end{aligned}$$

Put $\eta = \delta/(1-\lambda)$. Then if δ is sufficiently small then we can make $\eta < \epsilon$. This is a contradiction since $d(y,p) \geq 2\epsilon$. □

Theorem 2.2 (Corollary 2.19 in [12]) *For any $\epsilon > 0$, there are two integers l and n such that for any periodic point x of period $p(x) \geq n$:*

(1) *either f admits an l-dominated splitting along the orbit of x;*
(2) *or, for any neighborhood U of the orbit of x, there exists an ϵ-perturbation g of f in the C^1-topology, coinciding with f outside U and on the orbit of x, and such that x is a source or a sink of g for which the differential $D_x g^{p(x)}$ has all eigenvalues real with the same modulus.*

We say that the *Hausdorff distance* between two closed subsets A and B of M is given by

$$d_H(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\}.$$

Lemma 2.3 (Theorem 4 in [9]) *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$, a compact invariant set Λ is the Hausdorff limit of a sequence of periodic points if and only if Λ is chain transitive.*

Lemma 2.4 *If $f|_\Lambda$ is robustly chain transitive, then Λ admits a dominated splitting.*

Proof Let $f|_\Lambda$ be robustly chain transitive. Then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ is chain transitive. By Lemma 2.3, take $\mathcal{U}_0(f) = \mathcal{U}(f) \cap \mathcal{G}$. Then there exist $f_n \in \mathcal{U}_0(f)$ with $f_n \to f$ and $\mathrm{Orb}_{f_n}(p_n)$ of f_n such that

$$\Lambda = \limsup_{n\to\infty} \mathrm{Orb}_{f_n}(p_n).$$

Since $f|_\Lambda$ is robustly chain transitive, by Lemma 2.1, $f|_\Lambda$ does not contains neither sinks nor sources. By Theorem 2.2, f_n admits an l-dominated splitting over $\mathrm{Orb}_{f_n}(p_n)$ with l independent n. Thus Λ admits an l-dominated splitting. □

By Mañé (see [6]), the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg (g close to f) along the hyperbolic periodic point $q \in \Lambda_g(U) \cap P(g)$ is uniformly hyperbolic. This means that there is $\epsilon > 0$ such that for any g C^1-nearby f, $q \in \Lambda_g(U) \cap P(g)$ and any sequence of linear maps $A_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$ with $\|A_i - D_{g^i(q)}g\| < \epsilon$ ($i = 1, 2, \ldots, \pi(q)$), $\prod_{i=0}^{\pi(q)-1} A_i$ is hyperbolic. By Proposition II.1 in [6], we have the following.

Lemma 2.5 *Suppose that there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, any periodic point of $\Lambda_g(U)$ is hyperbolic. Then there are constants $C > 0$, $0 < \lambda < 1$ and $m > 0$ such that if for any $p \in \Lambda_g(U) \cap P(g)$ has minimum period $\pi(p) \geq m$ then*

$$\prod_{i=0}^{k-1} \big\|D_{g^{im}(p)} g^m|_{E^s(g^{im}(p))}\big\| < C\lambda^k \quad \textit{and}$$

$$\prod_{i=0}^{k-1} \big\|D_{g^{-im}(p)} g^m|_{E^u(g^{-im}(p))}\big\| < C\lambda^k,$$

where $k = [\pi(p)/m]$.

Let us recall Mañé's ergodic closing lemma in [6]. For any $\epsilon > 0$, let $B_\epsilon(f,x)$ an ϵ-tubular neighborhood of f-orbit of x, *i.e.*, $B_\epsilon(f,x) = \{y \in M : d(f^n(x), y) < \epsilon$ for some $n \in \mathbb{Z}\}$. Let Σ_f be the set of points $x \in M$ such that for any C^1-neighborhood $\mathcal{U}(f)$ of f and $\epsilon > 0$, there are $g \in \mathcal{U}(f)$ and $y \in P(g)$ satisfying $g = f$ on $M \setminus B_\epsilon(f,x)$ and $d(f^i(x), g^i(y)) \leq \epsilon$ for $0 \leq i \leq \pi(y)$.

Remark 2.6 (Theorem A in [6]) For any f-invariant probability measure μ, we have $\mu(\Sigma_f) = 1$.

Proof of Theorem 1.1 By assumption, $f|_\Lambda$ is robustly chain transitive. Then by Lemma 2.4, Λ admits a dominated splitting $E \oplus F$. We will finish the proof of Theorem 1.1, we show that

$$\liminf_{n\to\infty} \|D_x f^n|_{E_x}\| = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|D_x f^{-n}|_{F_x}\| = 0$$

for all $x \in \Lambda$. Then the splitting is hyperbolic. To prove, we consider $\liminf_{n\to\infty} \|D_x f^n_{|E}\| = 0$ (other case is similar). It is enough to show that for any $x \in \Lambda$, there exists $n = n(x) > 0$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E_{f^{mj}(x)}}\| < 1.$$

We will derive a contraction. If it is not true, then there is $x \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E_{f^{mj}(x)}}\| \geq 1$$

for all $n \geq 0$. Thus

$$\frac{1}{n}\sum_{j=0}^{n-1} \log\|Df^m|_{E_{f^{mj}(x)}}\| \geq 0$$

for all $n \geq 0$. Define a probability measure

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}.$$

Then there exists μ_{n_k} ($k \geq 0$) such that $\mu_{n_k} \to \mu_0 \in \mathcal{M}_f(M)$, as $k \to \infty$, where M is compact metric space. Thus

$$\begin{aligned}\int \log\|Df|_{E_x}\|\, d\mu_0 &= \lim_{k\to\infty} \int \log\|Df|_{E_x}\|\, d\mu_{n_k} \\ &= \lim_{k\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df|_{E_{f^j(x)}}\| \geq 0.\end{aligned}$$

By Mañé ([6], p.521),

$$\int_\Lambda \log\|Df|_{E_x}\|\, d\mu_0 = \int_\Lambda \frac{1}{n}\sum_{j=0}^{n-1} \log\|D_{f^j(x)} f|_{E_{f^j(x)}}\|\, d\mu_0 \geq 0,$$

where μ_0 is a f-invariant measure. Let

$$B_\epsilon(f,x) = \{y \in M : d(f^n(x), y) < \epsilon \text{ for some } n \in \mathbb{Z}\},$$

and $\Sigma_f = \{x \in M : d(f^n(x), y) < \epsilon$, there exist $g \in \mathcal{U}(f)$ and $y \in P(g)$ such that $g = f$ on $M \setminus B_\epsilon(f,x)$ and $d(f^i(x), f^i(y)) \leq \epsilon$ for $0 \leq i \leq \pi(y)\}$.

For any $\mu \in \mathcal{M}_f(M)$, $\mu(\Sigma_f) = 1$. Then, for any $\mu \in \mathcal{M}_f(\Lambda)$,

$$\mu(\Lambda \cap \Sigma_f) = 1,$$

since $\mu(\Lambda) = 1$ and $\mu(\Sigma_f) = 1$. Thus, $\Lambda = \Lambda \cap \Sigma(f)$ almost everywhere. Therefore,

$$\int_{\Lambda\cap\Sigma(f)} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df|_{E_{f^j(x)}}\| \, d\mu \geq 0.$$

By Birkhoff's theorem, and the ergodic closing lemma, we can take $z_0 \in \Lambda \cap \Sigma(f)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df|_{E_{f^j(z_0)}}\| \geq 0.$$

By Lemma 2.5, this is a contradiction. Thus by Lemma 2.5, $z_0 \notin P(f)$.

Let $C > 0$, $m > 0$, and $\lambda \in (0,1)$ be given by Lemma 2.5 and take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^m|_{E_{f^{mj}(z_0)}}\| \geq \log \lambda_0, \quad \text{if } n \geq n_0.$$

Then, by Mañé's ergodic closing lemma, we can find $g \in \mathcal{V}_0(f)$, $g = f$ on $M \setminus U_j$ and $r_g \in \Lambda_g \cap P(g)$ near by z_0. By assumption, for any $p \in \Lambda_g(U) \cap P(g)$ we know that

$$\operatorname{index}(r_g) = \operatorname{index}(p)$$

since $g = f$ on $M \setminus U_j$. Since $f|_\Lambda$ is robustly chain transitive, we can construct $h \in \mathcal{V}_0(f)$ ($\subset \mathcal{V}(f)$) C^1-nearby g such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} \|D_{h^{im}(r_h)} h^m|_{E_{h^{im}(r_h)}}\|$$

(see [6], pp.523-524). On the other hand, by Lemma 2.5, we see that

$$\prod_{i=0}^{k-1} \|D_{h^{im}(r_h)} h^m|_{E_{h^{im}(r_h)}}\| < C\lambda^k.$$

We can choose the period $\pi(r_h)$ ($> n_0$) of r_h as large as $\lambda_0^k \geq C\lambda^k$. Here $k = [\pi(r_h)/m]$. This is a contradiction. Thus,

$$\liminf_{n\to\infty} \|D_x f^n|_{E_x}\| = 0.$$

for all $x \in \Lambda$. Therefore, Λ is hyperbolic. This completes the proof of Theorem 1.1. □

Competing interests
The author declares to have no competing interests.

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