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On Levinson's operator inequality and its converses

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Abstract
We give new results on Levinson's operator inequality and its converse for normalized positive linear mappings and some large class of '3-convex functions at a point *c*'.

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Keywords: self-adjoint operator; positive linear mapping; convex function; Levinson's inequality; Mond-Pečarić method

1 Introduction and preliminary results

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. We denote by $\mathcal{B}_h(H)$ the real subspace of all self-adjoint operators on H. Bounds of $X \in \mathcal{B}_h(H)$ are defined by $m := \inf\{\langle X\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}$ and $M := \sup\{\langle X\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}$.

A continuous real valued function f defined on an interval I is said to be operator convex if $f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda)f(Y)$ for all self-adjoint operators X, Y with spectra contained in I and all $\lambda \in [0,1]$. If the function f is operator convex, then the so-called Jensen operator inequality $f(\Phi(X)) \leq \Phi(f(X))$ holds for any unital positive linear mapping Φ on $\mathcal{B}(H)$ and any $X \in \mathcal{B}_h(H)$ with spectrum contained in I. Many other versions of Jensen's operator inequality can be found in [1, 2].

Assume furthermore that $(\Phi_1, \dots, \Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$. If in addition $\sum_{i=1}^n \Phi_i(1_H) = 1_K$, we say that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$ is *unital*.

Now we give the definition of classes of functions for which we observe Levinson's operator inequality.

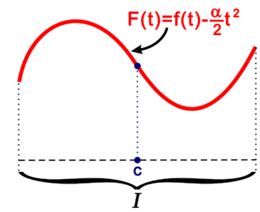
Definition 1 ([3]) Let $f \in \mathcal{C}(I)$ be a real valued function on an arbitrary interval I in $\mathbb{R}$ and $c \in I^\circ$, where I° is the interior of I.

We say that $f \in \mathcal{K}_1^c(I)$ (resp. $f \in \mathcal{K}_2^c(I)$) if there exists a constant α such that the function $F(t) = f(t) - \frac{\alpha}{2}t^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$. (See Figure 1.)

Moreover, we say that $f \in \dot{\mathcal{K}}_1^c(I)$ (resp. $f \in \dot{\mathcal{K}}_2^c(I)$) if there exists a constant α such that the function F is operator concave (resp. operator convex) on $I \cap (-\infty, c]$ and operator convex (resp. operator concave) on $I \cap [c, \infty)$.

The class of functions $\mathcal{K}_1^c(I)$ can be interpreted as functions that are '3-convex at a point c' and extends 3-convex functions in the following sense: a function is 3-convex on I if and only if it is at every $c \in I^\circ$.

Figure 1 Continous 3-convex function at a point *c*.



Next, we will review the history of research of Levison's inequality.

Levinson [4] considered an inequality as follows:

If $f : (0, 2c) \to \mathbb{R}$ *satisfies* $f''' \geq 0$ *and* p_i, x_i, y_i, $i = 1, 2, \ldots, n$, *are such that* $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $0 \leq x_i \leq c$, *and*

$$x_1 + y_1 = x_2 + y_2 = \cdots = x_n + y_n = 2c, \tag{1}$$

then the inequality

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^{n} p_i f(y_i) - f(\bar{y}) \tag{2}$$

holds, where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ *and* $\bar{y} = \sum_{i=1}^{n} p_i y_i$ *denote the weighted arithmetic means.*

Numerous papers have been devoted to generalizations and extensions of Levinson's result. Popoviciu [5] showed that the assumptions on the differentiability of f can be weakened for (2); to hold it is enough to assume that f is 3-convex. Bullen [6] gave another proof of Popoviciu's result rescaled to a general interval $[a, b]$.

Mercer [7] made a significant improvement by replacing (1) with the weaker condition that the variances of the two sequences are equal: $\sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} p_i (y_i - \bar{y})^2$.

Witkowski [8, 9] extended this result in several ways. Firstly, he showed that Levinson's inequality can be stated in a more general setting with random variables. Furthermore, he showed that it is enough to assume that f is 3-convex and that the assumption of equality of the variances can be weakened to inequality in a certain direction.

Baloch *et al.* [10] built on and extended the methods of Witkowski [8]. They introduced a new class of functions $\mathcal{K}_1^c((a, b))$ as in Definition 1.

Mićić *et al.* [3] built on the methods given in [11] on operators. We give Levinson's operator inequality for unital fields of positive linear mappings and classes of functions given by Definition 1. Moreover, we considered order among quasi-arithmetic means under similar conditions.

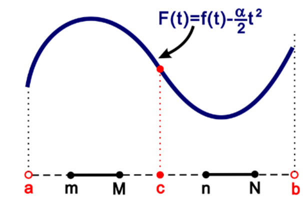
Next, we give the main result in [3] for two operators and $f \in \dot{\mathcal{K}}_i^c(I)$, $i = 1, 2$.

Theorem 1 *Let* $X, Y \in \mathcal{B}_h(H)$ *be self-adjoint operators with spectra contained in* $[m, M]$ *and* $[n, N]$, *respectively, such that* $a < m \leq M \leq c \leq n \leq N < b$. (*See Figure* 2.) *Let* Φ, Ψ *be normalized positive linear mappings* $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(K)$.

If $f \in \dot{\mathcal{K}}_1^c((a, b))$ *and* $C_1 \leq C_2$, *then*

$$\Phi\big(f(X)\big) - f\big(\Phi(X)\big) \leq C_1 \leq C_2 \leq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big), \tag{3}$$

Figure 2 Spectra conditions in Levinson's inequality for two operators and an operator 3-convex function.



where

$$C_1 := \frac{\alpha}{2}\left[\Phi\left(X^2\right) - \Phi(X)^2\right], \qquad C_2 := \frac{\alpha}{2}\left[\Psi\left(Y^2\right) - \Psi(Y)^2\right]. \tag{4}$$

But, if $f \in \overset{\bullet}{\mathcal{K}}{}_2^c((a,b))$ *and* $C_1 \geq C_2$ *holds, then the reverse inequalities are valid in* (3).

Proof This theorem is special case of [3], Theorem 1, for $k = n = 1$. For the sake of completeness, we give the proof.

Let $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$. So there is a constant α such that $F(t) = f(t) - \frac{\alpha}{2}t^2$ is operator concave on $[m,M] \subset (a,c]$. Jensen's inequality for an operator concave function implies

$$0 \leq F\left(\Phi(X)\right) - \Phi\left(F(X)\right) = f\left(\Phi(X)\right) - \frac{\alpha}{2}\Phi(X)^2 - \Phi\left(f(X)\right) + \frac{\alpha}{2}\Phi\left(X^2\right).$$

It follows that

$$\Phi\left(f(X)\right) - f\left(\Phi(X)\right) \leq C_1. \tag{5}$$

Similarly, since F is operator convex on $[n,N] \subset [c,b)$, then Jensen's inequality for an operator convex function implies

$$0 \leq \Psi\left(F(Y)\right) - F\left(\Psi(Y)\right) = \Psi\left(f(Y)\right) - \frac{\alpha}{2}\Psi\left(Y^2\right) - f\left(\Psi(Y)\right) + \frac{\alpha}{2}\Psi(Y)^2.$$

It follows that

$$C_2 \leq \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right). \tag{6}$$

Combining inequalities (5) and (6) and taking into account that $C_1 \leq C_2$ we obtain the desired inequality (3). □

Applying Theorem 1 we obtain a version of Levinson's inequality with more operators as follows.

Corollary 2 ([3], Theorem 1) *Let* $(X_1,\ldots,X_{k_1})$ *be a* k_1*-tuple and* $(Y_1,\ldots,Y_{k_2})$ *be a* k_2*-tuple of self-adjoint operators* $X_i, Y_j \in \mathcal{B}_h(H)$ *with spectra contained in* $[m,M]$ *and* $[n,N]$, *respectively, such that* $a < m \leq M \leq c \leq n \leq N < b$. *Let* $(\Phi_1,\ldots,\Phi_{k_1})$ *be a unital* k_1*-tuple and* $(\Psi_1,\ldots,\Psi_{k_2})$ *be a unital* k_2*-tuple of positive linear mappings* $\Phi_i, \Psi_j : \mathcal{B}(H) \to \mathcal{B}(K)$.

If $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$ *and* $D_1 \leq D_2$, *then*

$$\sum_{i=1}^{k_1}\Phi_i\left(f(X_i)\right) - f\left(\sum_{i=1}^{k_1}\Phi_i(X_i)\right) \leq D_1 \leq D_2 \leq \sum_{i=1}^{k_2}\Psi_i\left(f(Y_i)\right) - f\left(\sum_{i=1}^{k_2}\Psi_i(Y_i)\right) \tag{7}$$

holds, where

$$D_1 := \frac{\alpha}{2}\left[\sum_{i=1}^{k_1}\Phi_i\big(X_i^2\big) - \left(\sum_{i=1}^{k_1}\Phi_i(X_i)\right)^2\right],$$
$$D_2 := \frac{\alpha}{2}\left[\sum_{i=1}^{k_2}\Psi_i\big(Y_i^2\big) - \left(\sum_{i=1}^{k_2}\Psi_i(Y_i)\right)^2\right]. \tag{8}$$

If $f \in \dot{\mathcal{K}}_2^c((a,b))$ and $D_1 \geq D_2$ holds, then the reverse inequalities are valid in (7).

Proof This result is proven directly in [3], Theorem 1, using Jensen's operator inequality on the sum of the operators. We will give the proof by applying Theorem 1. We set $\tilde{X} = \operatorname{diag}(X_1,\dots,X_{k_1})$ and $\tilde{Y} = \operatorname{diag}(Y_1,\dots,Y_{k_2})$. Then $\tilde{X} \in \mathcal{B}_h(\underbrace{H\oplus\cdots\oplus H}_{k_1})$ and $\tilde{Y} \in \mathcal{B}_h(\underbrace{H\oplus\cdots\oplus H}_{k_2})$, with spectra contained in $[m,M]$ and $[n,N]$, respectively. Also, we set $\tilde{\Phi}(\operatorname{diag}(A_1,\dots,A_{k_1})) = \sum_{i=1}^{k_1}\Phi_i(A_i)$ and $\tilde{\Psi}(\operatorname{diag}(B_1,\dots,B_{k_2})) = \sum_{i=1}^{k_2}\Psi_i(B_i)$. Then $\tilde{\Phi} : \mathcal{B}(\underbrace{H\oplus\cdots\oplus H}_{k_1}) \to \mathcal{B}(K)$ and $\tilde{\Psi} : \mathcal{B}(\underbrace{H\oplus\cdots\oplus H}_{k_2}) \to \mathcal{B}(K)$ are normalized positive linear mappings. We have

$$\tilde{C}_1 = \frac{\alpha}{2}\big[\tilde{\Phi}\big(\tilde{X}^2\big) - \tilde{\Phi}(\tilde{X})^2\big] = \frac{\alpha}{2}\left[\sum_{i=1}^{k_1}\Phi_i\big(X_i^2\big) - \left(\sum_{i=1}^{k_1}\Phi_i(X_i)\right)^2\right] = D_1$$

and

$$\tilde{C}_2 = \frac{\alpha}{2}\big[\tilde{\Psi}\big(\tilde{Y}^2\big) - \tilde{\Psi}(\tilde{Y})^2\big] = \frac{\alpha}{2}\left[\sum_{i=1}^{k_2}\Psi_i\big(Y_i^2\big) - \left(\sum_{i=1}^{k_2}\Phi_i(X_i)\right)^2\right] = D_2.$$

Applying Theorem 1 on $\tilde{X}$, $\tilde{Y}$, $\tilde{\Phi}$, $\tilde{\Psi}$ and taking into account that $D_1 \lesseqgtr D_2$ implies $\tilde{C}_1 \lesseqgtr \tilde{C}_2$, we obtain the desired inequalities (7) or their reverse inequalities. □

In this paper, as a continuation of the above consideration, we will observe other results as regards Levinson's operator inequality and its converse. We give a few examples for power functions.

2 Converse of Levinson's operator inequality

In this section we give the converse of inequalities (3) and (7) for $f \in \mathcal{K}_i^c(I)$, $i = 1,2$. First, for convenience we introduce some abbreviations.

Let $f : [m,M] \to \mathbb{R}$, $m < M$, such that $F(t) = f(t) - \frac{\alpha}{2}t^2$, $\alpha \in \mathbb{R}$, be a convex or a concave function. We denote a linear function through the points $(m,F(m))$ and $(M,F(M))$ by $f_{\alpha,[m,M]}^{\text{line}}$, *i.e.*

$$f_{\alpha,[m,M]}^{\text{line}}(t) = \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) - \frac{\alpha}{2}\big((M+m)t - mM\big), \quad t \in \mathbb{R},$$

and the slope of the line through $(m,F(m))$ and $(M,F(M))$ by $k_{\alpha,f[m,M]}$, *i.e.*

$$k_{\alpha,f[m,M]} = \frac{f(M)-f(m)}{M-m} - \frac{\alpha}{2}(M+m).$$

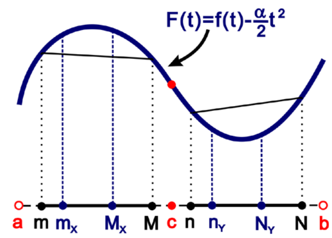


Figure 3 Spectra conditions in the converse of Levinson's inequality for two operators and a 3-convex function.

Next, we give the converse of Levinson's operator inequality for two operators.

Theorem 3 *Let X, Y, m, M, n, N, Φ, Ψ, C_1, C_2 be as in Theorem* 1 *and $m < M$, $n < N$. Let m_x, M_x ($m_x \le M_x$), and n_y, N_y ($n_y \le N_y$) be bounds of the operators $\Phi(X)$ and $\Psi(Y)$, respectively.* (*See Figure* 3.)

If $f \in \mathcal{K}_1^c((a,b))$ and $C_1 \ge C_2$, then

$$\Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K \ge C_1 \ge C_2 \ge \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K, \tag{9}$$

where

$$\beta_1 = \max_{m_x \le t \le M_x} \left\{ f(t) - \frac{\alpha}{2}t^2 - f_{\alpha,[m,M]}^{\text{line}}(t) \right\} \ge 0, \tag{10}$$

$$\beta_2 = \min_{n_y \le t \le N_y} \left\{ f(t) - \frac{\alpha}{2}t^2 - f_{\alpha,[n,N]}^{\text{line}}(t) \right\} \le 0. \tag{11}$$

The constants β_1, β_2 exist for any α, m, M, m_x, M_x and n, N, n_y, N_y.

The value of the constant β_1 is $\beta_1 = f(t_0) - \frac{\alpha}{2}t_0^2 - f_{\alpha,[m,M]}^{\text{line}}(t_0)$, where t_0 may be determined as follows:

- *if $f'_-(t) - \alpha t \le k_{\alpha,f[m,M]}$ for every $t \in (m_x, M_x)$, then $t_0 = m_x$,*
- *if $f'_-(t_1) - \alpha t_1 \ge k_{\alpha,f[m,M]} \ge f'_+(t_1) - \alpha t_1$ for some $t_1 \in (m_x, M_x)$, then $t_0 = t_1$,*
- *if $f'_+(t) - \alpha t \ge k_{\alpha,f[m,M]}$ for every $t \in (m_x, M_x)$, then $t_0 = M_x$.*

The value of β_2 can be determined as β_1 if we replace m, M, m_x, M_x by n, N, n_y, N_y, respectively, and with reverse inequality signs.

In the dual case, if $f \in \mathcal{K}_2^c((a,b))$ and $C_1 \le C_2$ holds, then the reverse inequalities are valid in (9), *where $\beta_1 \le 0$ with* min *instead of* max *in* (10) *and $\beta_2 \ge 0$ with* max *instead of* min *in* (11). *The value of the constants β_1 and β_2 can be determined as above with reverse inequality signs.*

Proof We will give the proof for $f \in \mathcal{K}_1^c((a,b))$. So there is a constant α such that $F(t) = f(t) - \frac{\alpha}{2}t^2$ is concave on $[m,M] \subset (a,c]$. The converse of Jensen's operator inequality gives (see [12], Theorem 3.4)

$$\begin{aligned}
&\Phi\big(F(X)\big) - F\big(\Phi(X)\big) \ge \min_{m_x \le t \le M_x} \left\{ f_{\alpha,[m,M]}^{\text{line}}(t) - f(t) - \frac{\alpha}{2}t^2 \right\} 1_K \\
&\quad \Rightarrow \quad \Phi\big(f(X)\big) - \frac{\alpha}{2}\Phi\big(X^2\big) - f\big(\Phi(X)\big) + \frac{\alpha}{2}\Phi(X)^2 + \beta_1 1_K \ge 0 \\
&\quad \Rightarrow \quad \Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K \ge C_1.
\end{aligned} \tag{12}$$

Similarly, since F is operator convex on $[n,N] \subset [c,b)$, then Jensen's operator inequality gives (see [12], Theorem 3.4)

$$\begin{aligned} &\Psi\big(F(Y)\big) - F\big(\Psi(Y)\big) \le \max_{n_y \le t \le N_y} \left\{ f^{\mathrm{line}}_{\alpha,[n,N]}(t) - f(t) - \frac{\alpha}{2}t^2 \right\} \\ &\quad\Rightarrow\quad \Psi\big(f(Y)\big) - \frac{\alpha}{2}\Psi\big(Y^2\big) - f\big(\Psi(Y)\big) + \frac{\alpha}{2}\Psi(Y)^2 + \beta_2 1_K \le 0 \\ &\quad\Rightarrow\quad C_2 \ge \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K. \end{aligned} \tag{13}$$

Combining inequalities (12) and (13) and taking into account $C_1 \ge C_2$ we obtain the desired inequality (9). We obtain $\beta_1 = f(t_0) - \frac{\alpha}{2}t_0^2 - f^{\mathrm{line}}_{\alpha,[m,M]}(t_0)$, where t_0 is determined as in the statement of Theorem 3, by applying [12], Theorem 3.4, to $\beta_1 = -\min_{m_x \le t \le M_x}\{f^{\mathrm{line}}_{\alpha,[m,M]}(t) - f(t) + \frac{\alpha}{2}t^2\}$. Analogously we get $\beta_2 = f(t_0) - \frac{\alpha}{2}t_0^2 - f^{\mathrm{line}}_{\alpha,[n,N]}(t_0)$. □

Remark 1 Let the assumptions of Theorem 3 be satisfied. If $C_1 \ge C_2$, f is strictly concave differentiable on $[m,c]$ and strictly convex differentiable on $[c,N]$, then (9) holds for

$$\begin{aligned} \beta_1 &= f(x_0) - \frac{\alpha}{2}x_0^2 - f^{\mathrm{line}}_{\alpha,[m,M]}(x_0) \le f(\bar{x}_0) - \frac{\alpha}{2}\bar{x}_0^2 - f^{\mathrm{line}}_{\alpha,[m,M]}(\bar{x}_0), \\ \beta_2 &= f(y_0) - \frac{\alpha}{2}y_0^2 - f^{\mathrm{line}}_{\alpha,[n,N]}(y_0) \ge f(\bar{y}_0) - \frac{\alpha}{2}\bar{y}_0^2 - f^{\mathrm{line}}_{\alpha,[n,N]}(\bar{y}_0), \end{aligned}$$

where x_0 may be determined as follows:

- if $f'(m_x) - \alpha m_x \le k_{\alpha,f[m,M]}$, then $x_0 = m_x$,
- if $f'(m_x) - \alpha m_x \ge k_{\alpha,f[m,M]} \ge f'(M_x) - \alpha M_x$, then x_0 is the unique solution of the equation $f'(t) - \alpha t = k_{\alpha,f[m,M]}$,
- if $f'(M_x) - \alpha M_x \ge k_{\alpha,f[m,M]}$, then $x_0 = M_x$,

and $\bar{x}_0$ is the unique solution in (m,M) of the equation $f'(t) - \alpha t = k_{\alpha,f[m,M]}$.

The values of y_0, $\bar{y}_0$ can be determined as x_0, $\bar{x}_0$, if we replace m, M, m_x, M_x by n, N, n_y, N_y, respectively, and with reverse inequality signs.

Example 1 Let $\Phi, \Psi, X, Y, m, M \ge 0$, $n, N \ge 0$, m_x, M_x, n_y, N_y be as in Theorem 3.

We will apply Theorem 3 putting $f(t) = t^p$ on $(0,c]$ and $f(t) = dt^q$ on $[c,\infty)$, where $c > 0$ and $d = c^{p-q}$.

(i) If $p \in (-\infty,0] \cup [1,\infty)$, $q \in [0,1]$, and $\alpha = 0$, then $f \in \mathcal{K}_2^c([0,\infty))$. So, (5) and the reverse of (9) give

$$\Phi\big(X^p\big) - \Phi(X)^p + \beta_1^\circ 1_K \le 0 \le d\Psi\big(Y^q\big) - d\Psi(Y)^q + \alpha_2 1_K \le \beta_2^\circ 1_K,$$

where

$$\begin{aligned} \beta_1^\circ &= \min_{m_x \le t \le M_x} \left\{ t^p - \frac{M-t}{M-m}m^p - \frac{t-m}{M-m}M^p \right\} \le 0, \\ \beta_2^\circ &= d \cdot \max_{n_y \le t \le N_y} \left\{ t^q - \frac{N-t}{N-n}n^q - \frac{t-n}{N-n}N^q \right\} \ge 0. \end{aligned}$$

(ii) If $p, q \in (-\infty, 0] \cup [1,2]$, $p^2 - p \geq q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then

$$\frac{d^2}{dt^2}\left(t^p - \frac{\alpha}{2}t^2\right) = p(p-1)t^{p-2} - \alpha \geq p(p-1)c^{p-2} - \alpha \geq 0, \quad \text{if } 0 \leq t \leq c,$$
$$\frac{d^2}{dt^2}\left(c^{p-q}t^q - \frac{\alpha}{2}t^2\right) = q(q-1)c^{p-q}t^{q-2} - \alpha \leq q(q-1)c^{p-2} - \alpha \leq 0, \quad \text{if } t \geq c.$$

So, $f \in \mathcal{K}_2^c([0,\infty))$. If

$$(0 <) \quad C_1 := \frac{\alpha}{2}\left[\Phi\left(X^2\right) - \Phi(X)^2\right] \leq C_2 := \frac{\alpha}{2}\left[\Psi\left(Y^2\right) - \Psi(Y)^2\right],$$

then the reverse of (9) gives

$$\Phi\left(X^p\right) - \Phi(X)^p + \beta_1 1_K \leq C_1 \leq C_2 \leq d\Psi\left(Y^q\right) - d\Psi(Y)^q + \beta_2 1_K,$$

where

$$\beta_1 = \min_{m_x \leq t \leq M_x}\left\{t^p - \frac{M-t}{M-m}m^p - \frac{t-m}{M-m}M^p + \frac{\alpha}{2}\left((M+m)t - mM - t^2\right)\right\} \leq 0,$$
$$\beta_2 = \max_{n_y \leq t \leq N_y}\left\{d\left(t^q - \frac{N-t}{N-n}n^q - \frac{t-n}{N-n}N^q\right) + \frac{\alpha}{2}\left((N+n)t - nN - t^2\right)\right\} \geq 0.$$

(iii) If $p, q \in [0,1] \cup [2,\infty)$, $p^2 - p \leq q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then

$$\frac{d^2}{dt^2}\left(t^p - \frac{\alpha}{2}t^2\right) \leq p(p-1)c^{p-2} - \alpha \leq 0, \quad \text{if } 0 \leq t \leq c,$$
$$\frac{d^2}{dt^2}\left(c^{p-q}t^q - \frac{\alpha}{2}t^2\right) \geq q(q-1)c^{p-2} - \alpha \geq 0, \quad \text{if } t \geq c.$$

So, $f \in \mathcal{K}_1^c([0,\infty))$. If $C_1 \geq C_2$, then (9) gives

$$\Phi\left(X^p\right) - \Phi(X)^p + \gamma_1 1_K \geq C_1 \geq C_2 \geq d\Psi\left(Y^q\right) - d\Psi(Y)^q + \gamma_2 1_K,$$

where $\gamma_1 \geq 0$ is defined similar to β_1 with max instead of min and $\gamma_2 \leq 0$ is defined similar to β_2 with min instead of max.

Remark 2 Let the assumptions of Theorem 3 be satisfied. If $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$ and $C_1 \geq C_2$, we obtain the following extension of (9):

$$\begin{aligned} C_1 + \beta_1 1_K &\geq \Phi\left(f(X)\right) - f\left(\Phi(X)\right) + \beta_1 1_K \geq C_1 \geq C_2 \\ &\geq \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right) + \beta_2 1_K \geq C_2 + \beta_2 1_K. \end{aligned} \tag{14}$$

In the dual case, if $f \in \overset{\bullet}{\mathcal{K}}{}_2^c((a,b))$ and $C_1 \leq C_2$, then the reverse inequalities are valid in (14).

Applying Theorem 3 we obtain a version of the converse of Levinson's inequality with more operators.

Corollary 4 *Let X_i, Φ_i $(i = 1,\dots,k_1)$, Y_j, Ψ_j $(j = 1,\dots,k_2)$, m, M, n, N, D_1, D_2 be as in Corollary 2. Let m_x, M_x $(m_x \leq M_x)$, and n_y, N_y $(n_y \leq N_y)$ be bounds of the operators $X = \sum_{i=1}^{k_1} \Phi_i(X_i)$ and $Y = \sum_{i=1}^{k_2} \Psi_i(Y_i)$, respectively. If $f \in \mathcal{K}_1^c((a,b))$ and $D_1 \geq D_2$, then*

$$\sum_{i=1}^{k_1} \Phi_i\bigl(f(X_i)\bigr) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) + \beta_1 1_K$$
$$\geq D_1 \geq D_2 \geq \sum_{i=1}^{k_2} \Psi_i\bigl(f(Y_i)\bigr) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) + \beta_2 1_K, \tag{15}$$

where β_1 and β_2 are as in Theorem 3.

If $f \in \mathcal{K}_2^c((a,b))$ and $D_1 \leq D_2$ holds, then the reverse inequalities are valid in (15) *with β_1 and β_2 as in Theorem* 3 *in the dual case.*

Proof We use the same technique as in the proof of Corollary 2. We omit the details. □

Remark 3 Applying Corollary 4 to positive linear mappings $\Phi_i, \Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ determined by $\Phi_i : B \mapsto p_i B$, $i = 1,\dots,k_1$, and $\Psi_i : B \mapsto q_i B$, $i = 1,\dots,k_2$, we obtain the following obvious result with convex combinations of the operators X_i, $i = 1,\dots,k_1$, and Y_j, $j = 1,\dots,k_2$:

Let X_i, Y_j be operators as in Corollary 4, such that $a < m_x \leq M_x \leq c \leq m_y \leq M_y < b$ for some $a, b, c \in \mathbb{R}$. Let $(p_1,\dots,p_{k_1})$ be a k_1-tuple and $(q_1,\dots,q_{k_2})$ be a k_2-tuple of positive scalars such that $\sum_{i=1}^{k_1} p_i = 1$ and $\sum_{j=1}^{k_2} q_j = 1$.

If $f \in \mathcal{K}_1^c((a,b))$ and $P \leq Q$, then

$$\sum_{i=1}^{k_1} p_i f(X_i) - f(\bar{X}) + \beta_1 1_K \leq P \leq Q \leq \sum_{j=1}^{k_2} q_j f(Y_j) - f(\bar{Y}) + \beta_2 1_K,$$

where β_1 and β_2 are as in Theorem 3,

$$P := \frac{\alpha}{2} \sum_{i=1}^{k_1} p_i (X_i - \bar{X})^2, \qquad Q := \frac{\alpha}{2} \sum_{j=1}^{k_2} q_j (Y_j - \bar{Y})^2,$$

and $\bar{X} := \sum_{i=1}^{k_1} p_i X_i$, $\bar{Y} := \sum_{j=1}^{k_1} q_j Y_j$ denote the weighted arithmetic means of the operators.

3 Refined Levinson's operator inequality

In this section we obtain a refinement of Levison's operator inequality (7) given in Section 2 under weaker conditions.

The absolute value of $B \in \mathcal{B}(H)$ is defined by $|B| = (B^*B)^{1/2}$.

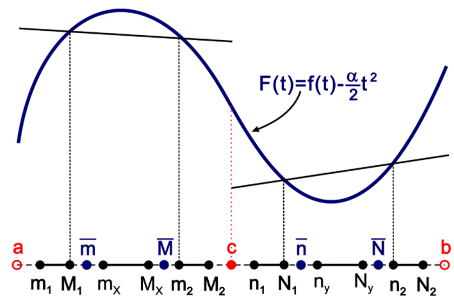
For convenience, we introduce the abbreviations $\bar{\Delta}$ and δ as follows:

- $\bar{\Delta} \equiv \bar{\Delta}_B(m,M) := \frac{1}{2} 1_K - \frac{1}{M-m} |B - \frac{m+M}{2} 1_K|$,

where $B \in \mathcal{B}_h(H)$ is a self-adjoint operator, Φ is a normalized positive linear mapping and m, M $(m < M)$ are some scalars such that spectra $\mathrm{Sp}(X) \subseteq [m,M]$. Since $m1_K \leq B \leq M1_K$, we have $-\frac{M-m}{2} 1_K \leq B - \frac{m+M}{2} 1_K \leq \frac{M-m}{2} 1_K$ and $0 \leq |\Phi(B) - \frac{m+M}{2} 1_K| \leq \frac{M-m}{2} 1_K$. It follows $\bar{\Delta} \geq 0$.

- $\delta \equiv \delta_{f,\alpha}(m,M) := 2f(\frac{m+M}{2}) - f(m) - f(M) + \frac{\alpha}{4}(M-m)^2$,

Figure 4 Spectra conditions in a refined Levinson's inequality for two operators and a 3-convex function.



where $f:[m,M]\to\mathbb{R}$ is a continuous function and $\alpha\in\mathbb{R}$. Obviously, if $F(t)=f(t)-\frac{\alpha}{2}t^2$ is *concave* (resp. *convex*) then $\delta\geq 0$ (resp. $\delta\leq 0$).

First, we give refined Levinson's operator inequality for two pairs of operators.

Theorem 5 *Let $\Phi,\Psi:\mathcal{B}(H)\oplus\mathcal{B}(H)\to\mathcal{B}(K)$ be normalized mappings such that $\Phi(\operatorname{diag}(B_1,B_2))=\Phi_1(B_1)+\Phi_2(B_2)$ and $\Psi(\operatorname{diag}(B_1,B_2))=\Psi_1(B_1)+\Psi_2(B_2)$, where Φ_1, Φ_2, Ψ_1, Ψ_2 are positive linear mappings.*

Let $X=\operatorname{diag}(X_1,X_2)$, $Y=\operatorname{diag}(Y_1,Y_2)$, where $X_1,X_2,Y_1,Y_2\in\mathcal{B}_h(H)$ are self-adjoint operators with spectra $\operatorname{Sp}(X_1)\subseteq[m_1,M_1]$, $\operatorname{Sp}(X_2)\subseteq[m_2,M_2]$, $\operatorname{Sp}(Y_1)\subseteq[n_1,N_1]$, $\operatorname{Sp}(Y_2)\subseteq[n_2,N_2]$ $M_1<m_2$, $N_1<n_2$. Let $a<m_1\leq M_1\leq m_x\leq M_x\leq m_2\leq M_2\leq c\leq n_1\leq N_1\leq n_y\leq N_y\leq n_2\leq N_2<b$, where m_x, M_x and n_y, N_y are bounds of $\Phi(X)$ and $\Psi(Y)$, respectively. (See Figure 4.)

If $f\in\mathcal{K}_1^c((a,b))$ and $C_1\leq C_2$ (see (4)), then

$$\begin{aligned}\Phi\big(f(X)\big)-f\big(\Phi(X)\big)&\leq\Phi\big(f(X)\big)-f\big(\Phi(X)\big)+\delta_1\bar{X}\leq C_1\\&\leq C_2\leq\Psi\big(f(Y)\big)-f\big(\Psi(Y)\big)+\delta_2\bar{Y}\leq\Psi\big(f(Y)\big)-f\big(\Psi(Y)\big),\end{aligned}\tag{16}$$

where $\delta_1=\delta_{f,\alpha}(\bar{m},\bar{M})\geq 0$, $\bar{X}=\bar{\Delta}_{\Phi(X)}(\bar{m},\bar{M})\geq 0$ for arbitrary numbers $\bar{m}\in[M_1,m_x]$, $\bar{M}\in[M_x,m_2]$, $\bar{m}<\bar{M}$ and $\delta_2=\delta_{f,\alpha}(\bar{n},\bar{N})\leq 0$, $\bar{Y}=\bar{\Delta}_{\Psi(Y)}(\bar{n},\bar{N})\geq 0$ for arbitrary numbers $\bar{n}\in[N_1,n_y]$, $\bar{N}\in[N_y,n_2]$, $\bar{n}<\bar{N}$.

But, if $f\in\mathcal{K}_2^c((a,b))$ and $C_1\geq C_2$ holds, then the reverse inequalities are valid in (16), with $\delta_1\leq 0$ and $\delta_2\geq 0$.

Proof We will give the proof for $f\in\mathcal{K}_1^c((a,b))$. Since $F(t)=f(t)-\frac{\alpha}{2}t^2$ is concave on $[m_1,c]\subset(a,c]$ for some constant α, the refined Jensen's operator inequality for a concave function implies (see [13], Theorem 3)

$$\begin{aligned}&F\big(\Phi(X)\big)\geq\Phi\big(F(X)\big)+\widetilde{\delta}_1\bar{X}\geq\Phi\big(F(X)\big)\\&\quad\Rightarrow\quad C_1\geq\Phi\big(f(X)\big)-f\big(\Phi(X)\big)+\delta_1\bar{X}\geq\Phi\big(f(X)\big)-f\big(\Phi(X)\big),\end{aligned}\tag{17}$$

since $0\leq\widetilde{\delta}_1=2F(\frac{\bar{m}+\bar{M}}{2})-F(\bar{m})-F(\bar{M})=\delta_{f,\alpha}(\bar{m},\bar{M})=\delta_1$ and

$$\bar{X}=\frac{1}{2}1_K-\frac{1}{\bar{M}-\bar{m}}\left|\Phi_1(X_1)+\Phi_2(X_2)-\frac{\bar{m}+\bar{M}}{2}1_K\right|=\bar{\Delta}_{\Phi(X)}(\bar{m},\bar{M}).$$

Similarly, since F is convex on $[c,N_2] \subset [c,b)$ for some constant α, the refined Jensen's operator inequality for a convex function implies (see [13], Theorem 3)

$$F\big(\Psi(Y)\big) \leq \Psi\big(F(Y)\big) - \widetilde{\delta}_2 \bar{Y} \geq \Psi\big(F(Y)\big)$$
$$\Rightarrow \quad C_2 \leq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \delta_2 \bar{Y} \leq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big), \tag{18}$$

since $0 \leq \widetilde{\delta}_2 = F(\bar{N}) + F(\bar{N}) - 2F(\frac{\bar{n}+\bar{N}}{2}) = -\delta_{f,\alpha}(\bar{n},\bar{N}) = -\delta_2$ and

$$\bar{Y} = \frac{1}{2}1_K - \frac{1}{\bar{N}-\bar{n}}\left|\Psi_1(Y_1) + \Psi_2(Y_2) - \frac{\bar{n}+\bar{N}}{2}1_K\right| = \bar{\Delta}_{\Psi(Y)}(\bar{n},\bar{N}).$$

Combining inequalities (17) and (18) we obtain the desired inequality (16). □

Example 2 Let $\Phi_i, \Psi_i, X_i, Y_i, m_i, M_i \geq 0$, $n_i, N_i \geq 0$, $i = 1,2$, Φ, Ψ, X, Y, m_x, M_x, n_y, N_y be as in Theorem 5.

We will use the same technique as in Example 1 and we will apply Theorem 5 putting $f(t) = t^p$ on $(0,c]$, $f(t) = dt^q$ on $[c,\infty)$, where $c > 0$ and $d = c^{p-q}$.

(i) If $p \in [0,1]$, $q \in (-\infty,0] \cup [1,\infty)$, and $\alpha = 0$, then $f \in \mathcal{K}_1^c([0,\infty))$. So, (16) gives

$$\Phi\big(X^p\big) - \Phi(X)^p + \delta_1 \bar{X} \leq 0 \leq d\Psi\big(Y^q\big) - d\Psi(Y)^q + \delta_2 \bar{Y},$$

where

$$\delta_1 = 2^{1-p}(\bar{m}+\bar{M})^p - \bar{m}^p - \bar{M}^p \geq 0, \qquad \bar{X} = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\Phi(X) - \frac{\bar{M}+\bar{m}}{2}1_K\right| \geq 0,$$
$$\delta_2 = d\big(2^{1-q}(\bar{n}+\bar{N})^q - \bar{n}^q - \bar{N}^q\big) \leq 0, \qquad \bar{Y} = \frac{1}{2}1_K - \frac{1}{\bar{N}-\bar{n}}\left|\Psi(Y) - \frac{\bar{N}+\bar{n}}{2}1_K\right| \geq 0.$$

(ii) If $p,q \in [0,1] \cup [2,\infty)$, $p^2 - p \leq q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then $f \in \mathcal{K}_1^c([0,\infty))$. If

$$C_1 := \frac{\alpha}{2}\big[\Phi\big(X^2\big) - \Phi(X)^2\big] \leq C_2 := \frac{\alpha}{2}\big[\Psi\big(Y^2\big) - \Psi(Y)^2\big],$$

then (16) gives

$$\Phi\big(X^p\big) - \Phi(X)^p + \delta_1 \bar{X} \leq C_1 \leq C_2 \leq d\Psi\big(Y^q\big) - d\Psi(Y)^q + \delta_2 \bar{Y},$$

where

$$\delta_1 = 2^{1-p}(\bar{m}+\bar{M})^p - \bar{m}^p - \bar{M}^p + \alpha(\bar{M}-\bar{m})^2/4 \geq 0,$$
$$\delta_2 = d\big(2^{1-q}(\bar{n}+\bar{N})^q - \bar{n}^q - \bar{N}^q\big) + \alpha(\bar{N}-\bar{n})^2/4 \leq 0,$$

and $\bar{X}, \bar{Y} \geq 0$ as in the case (i).

(iii) If $p,q \in (-\infty,0] \cup [1,2]$, $p^2 - p \geq q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then $f \in \mathcal{K}_2^c([0,\infty))$. If $C_1 \geq C_2$ (> 0), then (16) gives

$$\Phi(X^p) - \Phi(X)^p + \delta_1 \bar{X} \geq C_1 \geq C_2 \geq d\Psi(Y^q) - d\Psi(Y)^q + \delta_2 \bar{Y},$$

where $\delta_1 \geq 0$, $\delta_2 \leq 0$, and $\bar{X}, \bar{Y} \geq 0$ as in the case (ii).

The first and the last inequality in (16) are obvious, so we omit them.

Levinson's operator inequality (7) holds with the weaker condition: $f \in \mathcal{K}_1^c(I)$ and with spectra conditions (see [3], Theorem 5). Next, applying Theorem 5 we obtain a refinement of this inequality. The proof is the same as for Corollary 2 and we omit the details.

Corollary 6 *Let* $(\Phi_1,\dots,\Phi_{k_1})$ *be a unital* k_1*-tuple and* $(\Psi_1,\dots,\Psi_{k_2})$ *be a unital* k_2*-tuple of positive linear mappings* $\Phi_i, \Psi_j : \mathcal{B}(H) \to \mathcal{B}(K)$. *Let* $(X_1,\dots,X_{k_1})$ *be a* k_1*-tuple and* $(Y_1,\dots,Y_{k_2})$ *be a* k_2*-tuple of self-adjoint operators* X_i *and* $Y_j \in \mathcal{B}_h(H)$ *with spectra contained in* $[m_i, M_i]$ *and* $[n_j, N_j]$, *respectively, such that*

$$a < m_i \leq M_i \leq c \leq n_j \leq N_j < b, \quad i = 1,\dots,k_1, j = 1,\dots,k_2,$$
$$(m_x, M_x) \cap [m_i, M_i] = \emptyset, \quad i = 1,\dots,k_1, \qquad (m_y, M_y) \cap [n_j, N_j] = \emptyset, \quad j = 1,\dots,k_2,$$
$$m < M, \qquad n < N,$$

where m_x, M_x *and* n_y, N_y *are bounds of* $X = \sum_{i=1}^{k_1} \Phi_i(X_i)$ *and* $Y = \sum_{i=1}^{k_2} \Psi_i(Y_i)$, *respectively, and*

$$m := \max\{M_i | M_i \leq m_x, i = 1,\dots,k_1\}, \qquad M := \min\{m_i | m_i \geq M_x, i = 1,\dots,k_1\},$$
$$n := \max\{N_i | N_i \leq n_y, i = 1,\dots,k_2\}, \qquad N := \min\{n_i | n_i \geq N_y, i = 1,\dots,k_2\}.$$

If $f \in \mathcal{K}_1^c((a,b))$ *and* $D_1 \leq D_2$ *(see* (8)*), then*

$$\begin{aligned}
&\sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) \\
&\quad \leq \sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) + \delta_1 \bar{X} \leq D_1 \\
&\quad \leq D_2 \leq \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) \\
&\quad \leq \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) + \delta_2 \bar{Y},
\end{aligned} \tag{19}$$

where $\delta_1 = \delta_{f,\alpha}(\bar{m}, \bar{M}) \geq 0$, $\bar{X} = \bar{\Delta}_X(\bar{m}, \bar{M}) \geq 0$ *for arbitrary numbers* $\bar{m} \in [m, m_x]$, $\bar{M} \in [M_x, M]$, $\bar{m} < \bar{M}$ *and* $\delta_2 = \delta_{f,\alpha}(\bar{n}, \bar{N}) \leq 0$, $\bar{Y} = \bar{\Delta}_Y(\bar{n}, \bar{N}) \geq 0$ *for arbitrary numbers* $\bar{n} \in [n, n_y]$, $\bar{N} \in [N_y, N]$, $\bar{n} < \bar{N}$.

But, if $f \in \mathcal{K}_2^c((a,b))$ *and* $D_1 \geq D_2$ *holds, then the reverse inequalities are valid in* (19), *with* $\delta_1 \leq 0$ *and* $\delta_2 \geq 0$.

4 Refined converse of Levinson's operator inequality

In this section we obtain a refined converse of Levison's operator inequality (15) given in Section 2.

For convenience, we introduce the abbreviation

$$\widetilde{\Delta} \equiv \widetilde{\Delta}_{\Phi,B}(m,M) := \Phi\left(\frac{1}{2}1_H - \frac{1}{M-m}\left|B - \frac{m+M}{2}1_H\right|\right),$$

where $B \in \mathcal{B}_h(H)$ is a self-adjoint operator, Φ is a normalized positive linear mapping and m, M ($m < M$) are some scalars such that spectra $\mathrm{Sp}(X) \subseteq [m,M]$. Obviously, $\widetilde{\Delta} \geq 0$.

First, we give a refinement of (9) for two pairs of operators.

Theorem 7 *Let $\Phi, \Psi : \mathcal{B}(H) \oplus \mathcal{B}(H) \to \mathcal{B}(K)$ be normalized mappings such that $\Phi(\mathrm{diag}(B_1,B_2)) = \Phi_1(B_1) + \Phi_2(B_2)$ and $\Psi(\mathrm{diag}(B_1,B_2)) = \Psi_1(B_1) + \Psi_2(B_2)$, where Φ_1, Φ_2, Ψ_1, Ψ_2 are positive linear mappings. Let $X = \mathrm{diag}(X_1,X_2)$, $Y = \mathrm{diag}(Y_1,Y_2)$, where $X_1, X_2, Y_1, Y_2 \in \mathcal{B}_h(H)$ are self-adjoint operators with spectra $\mathrm{Sp}(X_1), \mathrm{Sp}(X_2) \subseteq [m,M]$, $\mathrm{Sp}(Y_1), \mathrm{Sp}(Y_2) \subseteq [n,N]$, such that $a < m \leq M \leq c \leq n \leq N < b$. Let m_x, M_x and n_y, N_y be bounds of the operators $\Phi(X)$ and $\Psi(Y)$, respectively (see Figure 3). If $f \in \mathcal{K}_1^c((a,b))$ and $C_1 \geq C_2$ (see (4)), then*

$$\begin{aligned}
&\Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K \\
&\quad \geq \Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K - \delta_1 \widetilde{X} \geq C_1 \\
&\quad \geq C_2 \geq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K - \delta_2 \widetilde{Y} \geq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K,
\end{aligned} \tag{20}$$

where β_1, β_2 are defined as in Theorem 3, $\delta_1 = \delta_{f,\alpha}(m,M) \geq 0$, $\widetilde{X} = \widetilde{\Delta}_{\Phi,X}(m,M) \geq 0$, $\delta_2 = \delta_{f,\alpha}(n,N) \leq 0$, and $\widetilde{Y} = \widetilde{\Delta}_{\Psi,Y}(n,N) \geq 0$.

If $f \in \mathcal{K}_2^c((a,b))$ and $C_1 \leq C_2$ holds, then the reverse inequalities are valid in (20), with $\delta_1 \leq 0$ and $\delta_2 \geq 0$ and β_1 and β_2 as in Theorem 3 in the dual case.

Proof We will give the proof for $f \in \mathcal{K}_1^c((a,b))$. Since $F(t) = f(t) - \frac{\alpha}{2}t^2$ is concave on $[m,c] \subset (a,c]$ for some constant α, the refined converse of Jensen's inequality for a concave function implies (see [14], Theorem 8)

$$\begin{aligned}
&\Phi\big(F(X)\big) - F\big(\Phi(X)\big) \geq \min_{m_x \leq t \leq M_x} \left\{ f^{\mathrm{line}}_{\alpha,[m,M]}(t) - f(t) - \frac{\alpha}{2}t^2 \right\} 1_K - \widetilde{\widetilde{\delta}}_1 \widetilde{X} \\
&\quad \Rightarrow \quad \Phi\big(f(X)\big) - \frac{\alpha}{2}\Phi\big(X^2\big) - f\big(\Phi(X)\big) + \frac{\alpha}{2}\Phi(X)^2 + \beta_1 1_K + \widetilde{\widetilde{\delta}}_1 \widetilde{X} \geq 0 \\
&\quad \Rightarrow \quad \Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K - \widetilde{\delta}_1 \widetilde{X} \geq C_1,
\end{aligned} \tag{21}$$

since $0 \geq \widetilde{\widetilde{\delta}}_1 = F(m) + F(M) - 2F(\frac{m+M}{2}) = -\delta_{f,\alpha}(m,M) = -\widetilde{\delta}_1$ and

$$\begin{aligned}
\widetilde{X} &= \frac{1}{2}1_K - \frac{1}{M-m}\left\{ \Phi_1\left(\left|X_1 - \frac{m+M}{2}1_H\right|\right) + \Phi_2\left(\left|X_2 - \frac{m+M}{2}1_H\right|\right) \right\} \\
&= \frac{1}{2}1_K - \frac{1}{M-m}\Phi\left(\left|X - \frac{m+M}{2}1_H\right|\right) \\
&= \Phi\left(\frac{1}{2}1_H - \frac{1}{M-m}\left|X - \frac{m+M}{2}1_H\right|\right) = \widetilde{\Delta}_{\Phi,X}(m,M).
\end{aligned}$$

Similarly, since F is convex on $[c,N_2] \subset [c,b)$ for some constant α, the refined converse of Jensen's inequality for a convex function implies (see [14], Theorem 8)

$$\begin{aligned}
&\Psi\big(F(Y)\big) - F\big(\Psi(Y)\big) \le \max_{n_y \le t \le N_y} \left\{ f^{\text{line}}_{\alpha,[n,N]}(t) - f(t) - \frac{\alpha}{2}t^2 \right\} - \widetilde{\widetilde{\delta}}_2 \widetilde{Y} \\
&\quad\Rightarrow \quad \Psi\big(f(Y)\big) - \frac{\alpha}{2}\Psi\big(Y^2\big) - f\big(\Psi(Y)\big) + \frac{\alpha}{2}\Psi(Y)^2 + \beta_2 1_K + \widetilde{\widetilde{\delta}}_2 \widetilde{Y} \le 0 \\
&\quad\Rightarrow \quad C_2 \ge \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K - \widetilde{\delta}_2 \widetilde{Y}, \qquad (22)
\end{aligned}$$

since $0 \le \widetilde{\widetilde{\delta}}_2 = F(n) + F(N) - 2F(\frac{n+N}{2}) = -\delta_{f,\alpha}(n,N) = -\widetilde{\delta}_2$ and

$$\widetilde{Y} = \Psi\left(\frac{1}{2}1_H - \frac{1}{N-n}\left| Y - \frac{n+N}{2}1_H \right|\right) = \widetilde{\Delta}_{\Psi,Y}(n,N).$$

Combining inequalities (21) and (22) we obtain the desired inequality (20). □

Example 3 Let Φ_i, Ψ_i, X_i, Y_i, $i = 1,2$, $m,M \ge 0$, $n,N \ge 0$, Φ, Ψ, X, Y, m_x, M_x, n_y, N_y be as in Theorem 7.

We will apply Theorem 7 putting $f(t) = t^p$ on $(0,c]$, $f(t) = dt^q$ on $[c,\infty)$, where $c > 0$ and $d = c^{p-q}$.

(i) If $p \in (-\infty,0] \cup [1,\infty)$, $q \in [0,1]$, and $\alpha = 0$, then reverse of (20) gives

$$\Phi\big(X^p\big) - \Phi(X)^p + \beta_1^\circ 1_K - \delta_1 \widetilde{X} \le 0 \le d\Psi\big(Y^q\big) - d\Psi(Y)^q + \beta_2^\circ 1_K - \delta_2 \widetilde{Y},$$

where β_1°, β_2° are as in Example 1(i), and

$$\begin{aligned}
&\delta_1 = 2^{1-p}(m+M)^p - m^p - M^p \ge 0, \qquad \widetilde{X} = \frac{1}{2}1_K - \frac{1}{M-m}\Phi\left(\left| X - \frac{M+m}{2}1_H \right|\right), \\
&\delta_2 = d\big(2^{1-q}(n+N)^q/2 - n^q - N^q\big) \le 0, \qquad \widetilde{Y} = \frac{1}{2}1_K - \frac{1}{N-n}\Psi\left(\left| Y - \frac{N+n}{2}1_H \right|\right).
\end{aligned}$$

(ii) If $p,q \in (-\infty,0] \cup [1,2]$, $p^2 - p \ge q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then $f \in \mathcal{K}_2^c([0,\infty))$. If

$$(0 <) \quad C_1 := \frac{\alpha}{2}\big[\Phi\big(X^2\big) - \Phi(X)^2\big] \le C_2 := \frac{\alpha}{2}\big[\Psi\big(Y^2\big) - \Psi(Y)^2\big],$$

then the reverse of (20) gives

$$\Phi\big(X^p\big) - \Phi(X)^p + \beta_1 1_K - \delta_1 \widetilde{X} \le C_1 \le C_2 \le d\Psi\big(Y^q\big) - d\Psi(Y)^q + \beta_2 1_K - \delta_2 \widetilde{Y},$$

where β_1, β_2 are as in Example 1(ii),

$$\begin{aligned}
&\delta_1 = 2^{1-p}(m+M)^p - m^p - M^p + \alpha(M-m)^2/4 \ge 0, \\
&\delta_2 = d\big(2^{1-q}(n+N)^q - n^q - N^q\big) + \alpha(N-n)^2/4 \le 0
\end{aligned}$$

and $\widetilde{X}$, $\widetilde{Y}$ are as in the case (i).

(iii) If $p,q \in [0,1] \cup [2,\infty)$, $p^2 - p \le q^2 - q$, and $\alpha = c^{p-2}(p^2 - p + q^2 - q)/2$, then $f \in \mathcal{K}_1^c([0,\infty))$. If $C_1 \ge C_2$, then (20) gives

$$\Phi(X^p) - \Phi(X)^p + \gamma_1 1_K - \delta_1 \widetilde{X} \ge C_1 \ge C_2 \ge d\Psi(Y^q) - d\Psi(Y)^q + \gamma_2 1_K - \delta_2 \widetilde{Y},$$

where $\gamma_1 \ge 0$ is defined similar to β_1 with max instead of min and $\gamma_2 \le 0$ is defined similar to β_2 with min instead of max, and $\delta_1 \le 0$, $\delta_2 \ge 0$, $\widetilde{X}$, $\widetilde{Y}$ are as in the case (ii).

The first and the last inequality in (20) are obvious, so we omit them.

Remark 4 Let the assumptions of Theorem 5 be satisfied. If $f \in \dot{\mathcal{K}}_1^c([m_1,N_2])$ and $C_1 \ge C_2$, we obtain the following extension of (16):

$$\begin{aligned} C_1 + \beta_1 1_K &\ge \Phi(f(X)) - f(\Phi(X)) + \beta_1 1_K \\ &\ge \Phi(f(X)) - f(\Phi(X)) + \beta_1 1_K - \delta_1 \bar{X} \ge C_1 \\ &\ge C_2 \ge \Psi(f(Y)) - f(\Psi(Y)) + \beta_2 1_K - \delta_2 \bar{Y} \\ &\ge \Psi(f(Y)) - f(\Psi(Y)) + \beta_2 1_K \ge C_2 + \beta_2 1_K. \end{aligned} \tag{23}$$

But, if $f \in \dot{\mathcal{K}}_2^c((a,b))$ and $C_1 \le C_2$, then the reverse inequalities are valid in (23).

Applying Theorem 7 we obtain a refinement of (15). We omit the proof.

Corollary 8 *Let $(\Phi_1,\dots,\Phi_{k_1})$ be a unital k_1-tuple and $(\Psi_1,\dots,\Psi_{k_2})$ be a unital k_2-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathcal{B}(H) \to \mathcal{B}(K)$. Let $(X_1,\dots,X_{k_1})$ be a k_1-tuple and $(Y_1,\dots,Y_{k_2})$ be a k_2-tuple of self-adjoint operators X_i and $Y_j \in \mathcal{B}_h(H)$ with spectra contained in $[m,M]$ and $[n,N]$, respectively, such that $a < m \le M \le c \le n \le N < b$. Let m_x, M_x and n_y, N_y be bounds of $X = \sum_{i=1}^{k_1} \Phi_i(X_i)$ and $Y = \sum_{i=1}^{k_2} \Psi_i(Y_i)$, respectively.*

If $f \in \mathcal{K}_1^c((a,b))$ and $D_1 \ge D_2$ (see (8)), then

$$\begin{aligned} &\sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) + \beta_1 1_K \\ &\quad \ge \sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) - \delta_1 \widetilde{X} + \beta_1 1_K \ge D_1 \\ &\quad \ge D_2 \ge \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) + \beta_2 1_K - \delta_2 \widetilde{Y} \\ &\quad \ge \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) + \beta_2 1_K, \end{aligned} \tag{24}$$

where β_1 and β_2 are defined as in Theorem 3, $\delta_1 = \delta_{f,\alpha}(m,M) \ge 0$, $\widetilde{X} = \sum_{i=1}^{k_1} \widetilde{\Delta}_{\Phi_i,X_i}(m,M) \ge 0$, $\delta_2 = \delta_{f,\alpha}(n,N) \le 0$, and $\widetilde{Y} = \sum_{i=1}^{k_2} \widetilde{\Delta}_{\Psi_i,y_i}(n,N) \ge 0$.

If $f \in \mathcal{K}_2^c((a,b))$ and $D_1 \le D_2$ holds, then the reverse inequalities are valid in (24), with $\delta_1 \le 0$ and $\delta_2 \ge 0$ and β_1 and β_2 as in Theorem 3 in the dual case.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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