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Quasi-partial *b*-metric spaces and some related fixed point theorems

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Abstract
In this paper, the quasi-partial *b*-metric space is defined and general fixed point theorems on this space are discussed with examples.
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1 Introduction

A generalization of the metric space can be obtained as a partial-metric space by replacing the condition $d(x,x) = 0$ with the condition $d(x,x) \leq d(x,y)$ for all x, y in the definition of the metric. In the year 1993, Czerwik [1] introduced the concept of a b-metric space as another generalization of the concept of metric space. Several authors have focused on fixed point theorems for a metric space, a partial-metric space, quasi-partial metric space and a partial b-metric space. For further information on the subject see [2–16].

The concept of a quasi-partial-metric space was introduced by Karapınar *et al.* [17]. He studied some fixed point theorems on these spaces whereas Shatanawi and Pitea [18] studied some coupled fixed point theorems on quasi-partial-metric spaces.

The aim of this paper is to introduce the concept of quasi-partial b-metric spaces which is a generalization of the concept of quasi-partial-metric spaces. The fixed point results are proved in setting of such spaces and some examples are given to verify the effectiveness of the main results.

2 Preliminaries

We begin the section with some basic definitions and concepts.

Definition 2.1 ([17]) A *quasi-partial metric* on a non-empty set X is a function $q : X \times X \to \mathbb{R}^+$, satisfying

(QPM_1) If $q(x,x) = q(x,y) = q(y,y)$, then $x = y$.
(QPM_2) $q(x,x) \leq q(x,y)$.
(QPM_3) $q(x,x) \leq q(y,x)$.
(QPM_4) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ for all $x, y, z \in X$.

A *quasi-partial-metric space* is a pair (X,q) such that X is a non-empty set and q is a *quasi-partial metric* on X.

Let q be a quasi-partial metric on the set X. Then

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y) \text{ is a metric on } X.$$

Lemma 2.1 ([17]) *For a quasi-partial metric q on X,*

$$p_q(x,y) = \frac{1}{2}\big[q(x,y) + q(y,x)\big] \quad \textit{for all } x,y \in X \textit{ is a partial metric on } X.$$

Lemmas 2.2 ([19–21])

(A) *A sequence $\{x_n\}$ is Cauchy in a partial-metric space (X,p) if and only if $\{x_n\}$ is Cauchy in the (corresponding) metric space (X,d_p).*

(B) *A partial-metric space (X,p) is complete if and only if the (corresponding) metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \quad \Leftrightarrow \quad p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 2.3 ([17]) *Let (X,q) be a quasi-partial metric space, let (X,p_q) be the corresponding partial-metric space, and let (X,d_{p_q}) be the corresponding metric space. Then the following statements are equivalent:*

(A) *The sequence $\{x_n\}$ is Cauchy in (X,q) and (X,q) is complete.*

(B) *The sequence $\{x_n\}$ is Cauchy in (X,p_q) and (X,p_q) is complete.*

(C) *The sequence $\{x_n\}$ is Cauchy in (X,d_{p_q}) and (X,d_{p_q}) is complete.*

Also,

$$\begin{aligned}
\lim_{n\to\infty} d_q(x,x_n) = 0 \quad &\Leftrightarrow \quad p_q(x,x) = \lim_{n\to\infty} p_q(x,x_n) = \lim_{n,m\to\infty} p_q(x_n,x_m) \\
&\Leftrightarrow \quad q(x,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n,m\to\infty} q(x_n,x_m) \\
&\qquad\qquad = \lim_{n\to\infty} q(x_n,x) = \lim_{n,m\to\infty} q(x_m,x_n).
\end{aligned}$$

Definition 2.2 ([17]) If $T : X \to X$ is any map on X, $O(x) = \{x, Tx, T^2x, \ldots\}$ is called the *orbit* of x. A mapping $G : X \to \mathbb{R}^+$ is *T-orbitally lower semi-continuous* at x if $\{x_n\}$ is a sequence in $O(x)$ and $\lim x_n = z$ implies $G(z) \leq \liminf G(x_n)$.

3 Quasi-partial *b*-metric space

We introduce the concept of quasi-partial b-metric space here.

Definition 3.1 A quasi-partial b-metric on a non-empty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $x,y,z \in X$:

(QPb_1) $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$,

(QPb_2) $qp_b(x,x) \leq qp_b(x,y)$,

(QPb_3) $qp_b(x,x) \leq qp_b(y,x)$,

(QPb_4) $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(y,z)] - qp_b(z,z)$.

A *quasi-partial b-metric space* is a pair (X,qp_b) such that X is a non-empty set and (X,qp_b) is a quasi partial b-metric on X. The number s is called the coefficient of (X,qp_b).

For a quasi-partial b-metric space (X, qp_b), the function $d_{qp_b} : X \times X \to \mathbb{R}^+$ defined by

$$d_{qp_b}(x,y) = qp_b(x,y) + qp_b(y,x) - qp_b(x,x) - qp_b(y,y) \text{ is a } b\text{-metric on } X.$$

Example 3.1 Let $X = [0,1]$.

Define $qp_b(x,y) = |x-y| + x$. Here

$$qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \quad \Rightarrow \quad x = y \quad \text{as } x = |x-y| + x = y \text{ gives } x = y.$$

Again, $qp_b(x,x) \leq qp_b(x,y)$ as $x \leq |x-y| + x$ and similarly, $qp_b(x,x) \leq qp_b(y,x)$ as $x \leq |y-x| + y$ for $0 < x < y$.

Also $qp_b(x,y) + qp_b(z,z) \leq s[qp_b(x,z) + qp_b(z,y)]$ as

$$|x-y| + x + z \leq s\bigl[|x-z| + x + |z-y| + z\bigr] \quad \text{for all } s \geq 1.$$

It can be observed that

$$|x-y| + x + z = |x-z+z-y| + x + z \leq |x-z| + |z-y| + x + z.$$

So (X, qp_b) is a quasi-partial b-metric space with $s \geq 1$.

Example 3.2 Let $X = [1, \infty)$.

Define $qp_b : X \times X \to \mathbb{R}^+$ as $qp_b(x,y) = \ln(xy)$. Then (X, qp_b) is a quasi-partial b-metric space.

Let $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow \ln(x^2) = \ln(xy) = \ln(y^2) \Rightarrow x = y$.

Let $x, y \in X$. Without loss of generality $x \leq y \Rightarrow \ln x \leq \ln y \Rightarrow 2\ln x \leq \ln x + \ln y \Rightarrow \ln(x^2) \leq \ln x + \ln y$.

Thus, $qp_b(x,x) \leq qp_b(x,y)$.

Similarly $qp_b(x,x) \leq qp_b(y,x)$.

For (QPb_4) we have

$$\begin{aligned} qp_b(x,y) &= \ln x + \ln y \\ &\leq s\ln x + s\ln y \quad \text{since } s \geq 1 \text{ and also } \ln x \geq 0 \text{ and } \ln y \geq 0 \\ &\leq s\ln x + s\ln y + 2\ln z(s-1) \quad \text{since } \ln z \geq 0 \text{ and } s-1 \geq 0 \\ &= s\bigl\{qp_b(x,z) + qp_b(z,y)\bigr\} - qp_b(z,z). \end{aligned}$$

Example 3.3 Let $X = [0, \frac{\pi}{4}]$ and define $qp_b : X \times X \to \mathbb{R}^+$ as

$$qp_b(x,y) = \sin x + \sin y.$$

Then (X, qp_b) is a quasi-partial b-metric space.

Lemma 3.4 *Let (X, qp_b) be a quasi-partial b-metric space. Then the following hold*:

(A) *If $qp_b(x,y) = 0$ then $x = y$.*

(B) *If $x \neq y$, then $qp_b(x,y) > 0$ and $qp_b(y,x) > 0$.*

The proof is similar to the case of quasi-partial-metric space [17].

Lemma 3.5 *Every quasi-partial space is a quasi-partial b-metric space. But the converse does not need to be true.*

Definition 3.2 Let (X, qp_b) be a quasi-partial b-metric. Then:

(i) A sequence $\{x_n\} \subset X$ *converges* to $x \in X$ if and only if

$$qp_b(x,x) = \lim_{n\to\infty} qp_b(x,x_n) = \lim_{n\to\infty} qp_b(x_n,x).$$

(ii) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m\to\infty} qp_b(x_n,x_m) \quad \text{and} \quad \lim_{n,m\to\infty} qp_b(x_m,x_n) \text{ exist (and are finite).}$$

(iii) The quasi-partial b-metric space (X, qp_b) is said to be *complete* if every Cauchy sequence $\{x_n\} \subset X$ converges with respect to τ_{qp_b} to a point $x \in X$ such that

$$qp_b(x,x) = \lim_{n,m\to\infty} qp_b(x_m,x_n) = \lim_{n,m\to\infty} qp_b(x_n,x_m).$$

(iv) A mapping $f : X \to X$ is said to be *continuous* at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Lemma 3.6 *Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.*

Proof Since (X, qp_b) is complete, every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_{qp_b} to a point $x \in X$ such that

$$qp_b(x,x) = \lim_{n,m\to\infty} qp_b(x_n,x_m) = \lim_{n,m\to\infty} qp_b(x_m,x_n). \tag{1}$$

Consider a Cauchy sequence $\{x_n\}$ in (X, d_{qp_b}). We will show that $\{x_n\}$ is Cauchy in (X, qp_b). Since $\{x_n\}$ is Cauchy in (X, d_{qp_b}), $\lim_{n,m\to\infty} d_{qp_b}(x_n,x_m)$ exists and is finite.

Also, $d_{qp_b}(x_n,x_m) = qp_b(x_n,x_m) + qp_b(x_m,x_n) - qp_b(x_n,x_n) - qp_b(x_m,x_m)$.

Clearly, $\lim_{n,m\to\infty} qp_b(x_n,x_m)$ and $\lim_{n,m\to\infty} qp_b(x_m,x_n)$ exist and are finite.

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, qp_b). Now, since (X, qp_b) is complete, the sequence $\{x_n\}$ converges with respect to τ_{qp_b} to a point $x \in X$ such that (1) holds.

For $\{x_n\}$ to be convergent in (X, d_{qp_b}) we will show that $d_{qp_b}(x,x) = \lim_{n\to\infty} d_{qp_b}(x,x_n)$.

If follows from the definition of d_{qp_b} that $d_{qp_b}(x,x) = 0$. Also,

$$\begin{aligned}\lim_{n\to\infty} d_{qp_b}(x,x_n) &= \lim_{n\to\infty} qp_b(x,x_n) + \lim_{n\to\infty} qp_b(x_n,x) - \lim_{n\to\infty} qp_b(x_n,x_n) - \lim_{n\to\infty} qp_b(x,x) \\ &= 0 \quad \text{by (1) and definition of convergence in } (X, qp_b).\end{aligned}$$

Hence, $d_{qp_b}(x,x) = \lim_{n\to\infty} d_{qp_b}(x,x_n)$. □

In [17] Karapınar *et al.* proved a fixed point theorem on quasi-partial-metric space. Motivated by this, we have generalized the results on a quasi-partial b-metric space.

4 The main results

Theorem 4.1 *Let (X, qp_b) be a quasi-partial b-metric space, and let $T : X \to X$. Then the following hold*:

(A) *There exists $\phi : X \to \mathbb{R}^+$ such that*

$$qp_b(x, Tx) \le \phi(x) - \phi(Tx) \quad \text{for all } x \in X \quad \text{if and only if}$$

$$\sum_{n=0}^{\infty} qp_b\big(T^n x, T^{n+1} x\big) \text{ converges for all } x \in X.$$

(B) *There exists $\phi : X \to \mathbb{R}^+$ such that*

$$qp_b(x, Tx) \le \phi(x) - \phi(Tx) \quad \text{for all } x \in O(x) \quad \text{if and only if}$$

$$\sum_{n=0}^{\infty} qp_b\big(T^n x, T^{n+1} x\big) \text{ converges for all } x \in O(x).$$

Proof (A) Let $x \in X$, and let

$$qp_b(x, Tx) \le \phi(x) - \phi(Tx).$$

Define the sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_0 = x \quad \text{and} \quad x_{n+1} = Tx_n = T^{n+1}x_0, \quad \text{for all } n = 0, 1, 2, \ldots.$$

Set $z_n(x) = \sum_{k=0}^{n} qp_b(x_k, x_{k+1}) = \sum_{k=0}^{n} qp_b(T^k x_0, T^{k+1} x_0)$. Then

$$\begin{aligned} z_n(x) &\le \sum_{k=0}^{n} \big[\phi\big(T^k x_0\big) - \phi\big(T^{k+1} x_0\big)\big] \\ &= \big[\phi(x_0) - \phi(Tx_0)\big] + \cdots + \big[\phi\big(T^n x_0\big) - \phi\big(T^{n+1} x_0\big)\big] \\ &= \big[\phi(x_0) - \phi\big(T^{n+1} x_0\big)\big] \le \phi(x_0) = \phi(x). \end{aligned} \tag{2}$$

Thus, (2) implies that $\{z_n(x)\}$ is bounded. Also $\{z_n(x)\}$ is non-decreasing and hence convergent. Therefore, $\sum_{n=0}^{\infty} qp_b(T^n x, T^{n+1} x)$ converges.

Conversely, define

$$\phi(x) = \sum_{n=0}^{\infty} qp_b\big(T^n x, T^{n+1} x\big) \quad \text{and} \quad z_n(x) = \sum_{k=0}^{n} qp_b\big(T^k x, T^{k+1} x\big).$$

Then

$$\phi(Tx) = \sum_{n=0}^{\infty} qp_b\big(T^{n+1} x, T^{n+2} x\big) \quad \text{and} \quad z_n(Tx) = \sum_{k=0}^{n} qp_b\big(T^{k+1} x, T^{k+2} x\big).$$

Using these definitions, we get

$$\begin{aligned} z_n(x) - z_n(Tx) &= \sum_{k=0}^{n} qp_b\big(T^k x, T^{k+1} x\big) - \sum_{k=0}^{n} qp_b\big(T^{k+1} x, T^{k+2} x\big) \\ &= qp_b(x, Tx) - qp_b\big(T^{n+1} x, T^{n+2} x\big). \end{aligned} \tag{3}$$

Since $\sum_{n=0}^{\infty} qp_b(T^n x, T^{n+1}x)$ converges for all $x \in X$,

$$\lim_{n\to\infty} z_n(x) = \phi(x) \quad \text{and} \quad \lim_{n\to\infty} qp_b\bigl(T^n x, T^{n+1}x\bigr) = 0.$$

Letting $n \to \infty$ in (3) gives $qp_b(x, Tx) = \phi(x) - \phi(Tx)$.

(B) It can easily be proved using part (A). □

Example 4.1 Let $X = [0,1]$. Define $qp_b(x,y) = |x - y| + |x|$.

Then $qp_b(x,y)$ satisfies all conditions of quasi-partial b-metric space. It is also quasi-partial metric. But for $x \neq y$, $qp_b(x,y) \neq qp_b(y,x)$ and $qp_b(x,x) \neq 0$ for $x \neq 0$. So qp_b is not a partial metric or a quasi-metric. Define $T : X \to X$ as $Tx = \frac{x}{3}$ for all $x \in X$. Then the series $\sum_{n=0}^{\infty} qp_b(T^n x, T^{n+1}x)$ is convergent. Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty} qp_b\bigl(T^n x, T^{n+1}x\bigr) &= \sum_{n=0}^{\infty} qp_b\left(\frac{x}{3^n}, \frac{x}{3^{n+1}}\right) = \sum_{n=0}^{\infty}\left|\frac{x}{3^n} - \frac{x}{3^{n+1}}\right| + \left|\frac{x}{3^n}\right| \\ &= \sum_{n=0}^{\infty}\left|\frac{2x}{3^{n+1}}\right| + \left|\frac{x}{3^n}\right| = \sum_{n=0}^{\infty}\frac{5x}{3^{n+1}} = \frac{5x}{3}\cdot\frac{1}{1-\frac{1}{3}} = \frac{5x}{2}.\end{aligned}$$

Then the conditions of Theorem 4.1 are satisfied for $\phi(x) = \frac{5x}{2}$. Indeed

$$qp_b(x, Tx) = qp_b\left(x, \frac{x}{3}\right) = \left|x - \frac{x}{3}\right| + |x| = \left|\frac{2x}{3}\right| + |x| = \frac{5x}{3} = \phi(x) - \phi(Tx).$$

The next result gives conditions for the existence of fixed points of operators on quasi-partial b-metric space.

Theorem 4.2 *Let (X, qp_b) and (Y, qp_b) be complete quasi-partial b-metric spaces. Let also $T : X \to X$, $R : X \to Y$, and $\phi : R(X) \to \mathbb{R}^+$. If there exist $x \in X$ and $c > 0$ such that*

$$\max\bigl\{qp_b(y, Ty), cqp_b(Ry, RTy)\bigr\} \le \phi(Ry) - \phi(RTy) \tag{4}$$

for all $y \in O(x)$, then the following hold:

(A) $\lim_{n\to\infty} T^n x = z$ *exists.*

(B) *$Tz = z$ if and only if $G(x) = qp_b(x, Tx)$ is T-orbitally lower semi-continuous at x.*

(C) $qp_b(x, T^n x) \le s^{n-1}\phi(Rx)$.

(D) *For $m > n$, $qp_b(T^n x, T^m x) \le s^{m-n}[\phi(RT^n x)]$.*

Proof (A) Let $x \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$x_0 = x \quad \text{and} \quad x_{n+1} = Tx_n = T^{n+1}x_0, \quad \text{for all } n = 0, 1, 2, \ldots.$$

We will show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Using (QPb_4), we get

$$\begin{aligned}qp_b(x_n, x_{n+2}) &\le s\bigl\{qp_b(x_n, x_{n+1}) + qp_b(x_{n+1}, x_{n+2})\bigr\} - qp_b(x_{n+1}, x_{n+1}) \\ &\le s\bigl\{qp_b(x_n, x_{n+1}) + qp_b(x_{n+1}, x_{n+2})\bigr\} \end{aligned} \tag{5}$$

and, similarly,

$$\begin{aligned} qp_b(x_n, x_{n+3}) &\le s\{qp_b(x_n, x_{n+2}) + qp_b(x_{n+2}, x_{n+3})\} - qp_b(x_{n+2}, x_{n+2}) \\ &\le s^2\{qp_b(x_n, x_{n+1}) + qp_b(x_{n+1}, x_{n+2})\} + s\{qp_b(x_{n+2}, x_{n+3})\}. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} qp_b(x_n, x_{n+4}) &\le s\{qp_b(x_n, x_{n+3}) + qp_b(x_{n+3}, x_{n+4})\} - qp_b(x_{n+3}, x_{n+3}) \\ &\le s^3\{qp_b(x_n, x_{n+1}) + qp_b(x_{n+1}, x_{n+2})\} + s^2\{qp_b(x_{n+2}, x_{n+3})\} \\ &\quad + s\{qp_b(x_{n+3}, x_{n+4})\}. \end{aligned}$$

On generalization, we get

$$\begin{aligned} qp_b(x_n, x_m) &\le s^{m-n-1}\{qp_b(x_n, x_{n+1}) + qp_b(x_{n+1}, x_{n+2})\} \\ &\quad + s^{m-n-2}\{qp_b(x_{n+2}, x_{n+3})\} + \cdots + s\{qp_b(x_{m-1}, x_m)\} \\ &\le s^{m-n-1}\{qp_b(T^n x, T^{n+1}x) + qp_b(T^{n+1}x, T^{n+2}x)\} \\ &\quad + s^{m-n-2}\{qp_b(T^{n+2}x, T^{n+3}x)\} + \cdots + s\{qp_b(T^{m-1}x, T^m x)\} \\ &= \sum_{k=n+1}^{m-1} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\} + s^{m-n-1}qp_b(x_n, x_{n+1}) \\ &= \sum_{k=n}^{m-1} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\} + s^{m-n-1}qp_b(x_n, x_{n+1}) - s^{m-n}qp_b(x_n, x_{n+1}) \\ &= \sum_{k=n}^{m-1} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\} - s^{m-n}qp_b(x_n, x_{n+1})\left[1 - \frac{1}{s}\right] \\ &\le \sum_{k=n}^{m-1} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\} \quad \text{for } m > n. \end{aligned} \tag{7}$$

Set $z_n(x) = \sum_{k=0}^{n} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\}$.

From (4) we have

$$\begin{aligned} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\} &\le s^{m-k}\max\{qp_b(T^k x, T^{k+1}x), cqp_b(RT^k x, RT^{k+1}x)\} \\ &\le s^{m-k}\{\phi(RT^k x) - \phi(RT^{k+1}x)\} \quad \text{for all } k = 0, 1, \ldots \end{aligned} \tag{8}$$

$$\begin{aligned} \Rightarrow \quad z_n(x) &\le \sum_{k=0}^{n} s^{m-k}\{\phi(RT^k x) - \phi(RT^{k+1}x)\} \\ &\le s^m\phi(Rx) - s^m\phi(RTx) + s^m\phi(RTx) - s^{m-1}\phi(RT^2x) + \cdots \\ &\quad + s^{m-n+1}\phi(RT^n x) - s^{m-n}\phi(RT^{n+1}x) \\ &= s^m\phi(Rx) - s^{m-n}\phi(RT^{n+1}x) \\ &\le s^m\phi(Rx). \end{aligned} \tag{9}$$

Thus, $\sum_{k=0}^{\infty} s^{m-k}\{qp_b(T^k x, T^{k+1}x)\}$ is convergent.

$$\Rightarrow \quad \sum_{n=0}^{\infty} s^{m-n}\bigl\{qp_b\bigl(T^n x, T^{n+1}x\bigr)\bigr\} \text{ is convergent.}$$

Taking the limit as $n, m \to \infty$ in (7), we get

$$\lim_{m,n\to\infty} qp_b(x_n, x_m) = \lim_{m,n\to\infty} \bigl(z_{m-1}(x) - z_{n-1}(x)\bigr) = 0. \tag{10}$$

Using similar arguments,

$$\lim_{m,n\to\infty} qp_b(x_m, x_n) = 0. \tag{11}$$

Thus the sequence $\{x_n\}$ is Cauchy in (X, qp_b). Since $(X, qp_b)$is complete, (X, d_{qp_b}) is also complete by Lemma 2.3, and hence $\lim_{n\to\infty} d_{qp_b}(T^n x, z) = 0$, $\lim_{n\to\infty} T^n x = z$.

Further, $\lim_{n\to\infty} qp_b(T^n x, T^{n+1}x) = 0$ and hence $\lim_{n\to\infty} qp_b(T^n x, T^{n+1}x) = qp_b(z, z) = 0$.

(B) Assume that $Tz = z$ and that x_n is a sequence in $O(x)$ with $x_n \to z$.

By Lemma 3.6,

$$\lim_{n\to\infty} d_{qp_b}(z, x_n) = 0 \quad \Leftrightarrow \quad qp_b(z, z) = \lim_{n\to\infty} qp_b(z, x_n) = \lim_{n,m\to\infty} qp_b(x_n, x_m). \tag{12}$$

Then $G(z) = qp_b(z, Tz) = qp_b(z, z) \leq \lim_{n\to\infty} \inf qp_b(x_n, Tx_n) = \lim_{n\to\infty} \inf G(x_n)$.

Thus G is T-orbitally lower semi-continuous at x.

Conversely, suppose that $x_n = T^n x \to z$ and that G is T-orbitally lower semi-continuous at x. Then

$$\begin{aligned} 0 \leq qp_b(z, Tz) = G(z) &\leq \lim_{n\to\infty} \inf G(x_n) = \lim_{n\to\infty} \inf qp_b\bigl(T^n x, T^{n+1}x\bigr) \\ &= \lim_{n\to\infty} \inf qp_b(x_n, x_{n+1}) = qp_b(z, z) = 0. \end{aligned} \tag{13}$$

By Lemma 3.4, we have $Tz = z$.

(C) We have, from (QPb_4) and (4),

$$\begin{aligned} qp_b\bigl(x, T^2x\bigr) &\leq s\bigl\{qp_b(x, Tx) + qp_b\bigl(Tx, T^2x\bigr)\bigr\} - qp_b(Tx, Tx) \\ &\leq s\bigl\{qp_b(x, Tx) + qp_b\bigl(Tx, T^2x\bigr)\bigr\}, \\ qp_b\bigl(x, T^3x\bigr) &\leq s\bigl\{qp_b\bigl(x, T^2x\bigr) + qp_b\bigl(T^2x, T^3x\bigr)\bigr\} - qp_b\bigl(T^2x, T^2x\bigr) \\ &\leq s\bigl[s\bigl\{qp_b(x, Tx) + qp_b\bigl(Tx, T^2x\bigr)\bigr\} + qp_b\bigl(T^2x, T^3x\bigr)\bigr] \\ &\leq s^2\bigl\{qp_b(x, Tx) + qp_b\bigl(Tx, T^2x\bigr)\bigr\} + s\bigl\{qp_b\bigl(T^2x, T^3x\bigr)\bigr\}. \end{aligned}$$

On generalization, we get

$$\begin{aligned} &qp_b\bigl(x, T^n x\bigr) \\ &\quad \leq s^{n-1}\bigl\{qp_b(x, Tx) + qp_b\bigl(Tx, T^2x\bigr)\bigr\} + s^{n-2}\bigl\{qp_b\bigl(T^2x, T^3x\bigr)\bigr\} + \cdots \\ &\qquad + s\bigl\{qp_b\bigl(T^{n-1}x, T^n x\bigr)\bigr\} \end{aligned}$$

$$
\begin{aligned}
&\le s^{n-1}\{qp_b(x,Tx)\} + s^{n-1}\{qp_b(Tx,T^2x)\} + s^{n-2}\{qp_b(T^2x,T^3x)\} + \cdots \\
&\quad + s\{qp_b(T^{n-1}x,T^nx)\} \\
&\le s^{n-1}\{\phi(Rx)-\phi(RTx)\} + s^{n-1}\{\phi(RTx)-\phi(RT^2x)\} \\
&\quad + s^{n-2}\{\phi(RT^2x)-\phi(RT^3x)\} + \cdots + s\{\phi(RT^{n-1}x)-\phi(RT^nx)\} \\
&\le s^{n-1}\phi(Rx) - s^{n-1}\phi(RT^2x) + s^{n-2}\phi(RT^2x) - s^{n-2}\phi(RT^3x) + \cdots \\
&\quad + s\phi(RT^{n-1}x) - s\phi(RT^nx) \\
&\le s^{n-1}\phi(Rx) - s\phi(RT^2x) - s\phi(RT^{n-1}x) - s\phi(RT^nx) \\
&\le s^{n-1}\phi(Rx).
\end{aligned} \tag{14}
$$

(D) From (7) we get

$$
qp_b(x_n,x_m) \le \sum_{k=n}^{m-1} s^{m-k}\{qp_b(T^kx,T^{k+1}x)\} \quad \text{for } m > n.
$$

Note that

$$
\begin{aligned}
&\sum_{k=n}^{m-1} s^{m-k} qp_b(T^kx,T^{k+1}x) \\
&\quad \le \sum_{k=n}^{m-1} s^{m-k}[\phi(RT^kx) - \phi(RT^{k+1}x)] \\
&\quad = s^{m-n}\phi(RT^nx) - s^{m-n}\phi(RT^{n+1}x) + s^{m-n-1}\phi(RT^{n+1}x) \\
&\quad\quad - s^{m-n-1}\phi(RT^{n+2}x) + \cdots + s\phi(RT^{m-1}x) - s\phi(RT^mx) \\
&\quad = s^{m-n}\phi(RT^nx) - s\phi(RT^{n+1}x) - s\phi(RT^{m-1}x) - s\phi(RT^mx) \\
&\quad \le s^{m-n}\phi(RT^nx).
\end{aligned} \tag{15}
$$

Here, $0 \le qp_b(x_n,x_m) = qp_b(T^nx,T^mx) \le s^{m-n}\phi(RT^nx)$ for $m > n$. □

Example 4.2 Let $X = Y = [0,1]$. Define $qp_b(x,y) = |x-y| + x$. Then qp_b is a quasi-partial b-metric with $s = 1$. Also define $T : X \to X$ as $T(x) = \frac{x}{3}$; $R : X \to Y$ as $R(x) = 3x$, and $\phi : R(X) \to \mathbb{R}^+$ as $\phi(x) = 3x$. Then for $c = 1$ and $x \in [0,1]$ we have

$$
\begin{aligned}
\max\{qp_b(y,Ty), cqp_b(Ry,RTy)\} &= \max\left\{qp_b\left(y,\frac{y}{3}\right), qp_b(3y,y)\right\} \\
&= \max\left\{\left|y-\frac{y}{3}\right| + y, |3y-y| + 3y\right\} \\
&= \max\left\{\frac{5y}{3}, 5y\right\} = 5y < 6y = \phi(3y) - \phi(y) \\
&= \phi(Ry) - \phi(RTy).
\end{aligned}
$$

We now prove that (A), (B), (C), and (D) of the above theorem hold:

(A) $\lim_{n\to\infty} T^nx = \lim_{n\to\infty} \frac{x}{3^n} = 0 = z$ (say).

So $\lim_{n\to\infty} T^n x = z$ exists.
(B) By (A) part above, $z = 0$.
Therefore $T(z) = T(0) = 0 = z$ holds trivially.
Hence whenever $G(x) = qp_b(x, Tx)$ is T-orbitally lower semi-continuous at x then $Tz = z$.
Conversely, let $Tz = z$ and we show that G is T-orbitally lower semi-continuous at x, *i.e.*,

$$G(z) \leq \liminf G(x_n) \quad \forall\{x_n\} \subseteq O(x), x_n \to z.$$

Let $\{x_n\} \subseteq O(x)$ be a sequence converging to z. Then

$$\begin{aligned} G(z) &= qp_b(z, Tz) = qp_b(z, z) = z \\ &= \frac{5z}{3} \text{ (as } z = 0) = \liminf \frac{5x_n}{3} \\ &= \liminf \frac{2x_n}{3} + x_n = \liminf \left| x_n - \frac{x_n}{3} \right| + x_n \\ &= \liminf qp_b\left(x_n, \frac{x_n}{3}\right) = \liminf qp_b(x_n, Tx_n) = \liminf G(x_n). \end{aligned}$$

Hence $G(z) = \liminf G(x_n)$.

$$\begin{aligned} \text{(C) } qp_b\left(x, T^n x\right) &= qp_b\left(x, \frac{x}{3^n}\right) = \left| x - \frac{x}{3^n} \right| + x = x\left(2 - \frac{1}{3^n}\right) < x(9) \quad \forall n \in N \\ &= \phi(3x) = s^{n-1}\phi(Rx) \quad \text{where } s = 1. \end{aligned}$$

(D) Let $m > n$ then

$$\begin{aligned} qp_b\left(T^n x, T^m x\right) &= qp_b\left(\frac{x}{3^n}, \frac{x}{3^m}\right) = \left| \frac{x}{3^n} - \frac{x}{3^m} \right| + \frac{x}{3^n} \\ &= \frac{x}{3^n}\left[2 - \frac{1}{3^{m-n}}\right] < \frac{x}{3^n}(9) \quad \forall n \in N \\ &= \phi\left(\frac{x}{3^{n-1}}\right) = \phi\left(3T^n x\right) = s^{m-n}\left[\phi\left(RT^n x\right)\right] \quad \text{where } s = 1. \end{aligned}$$

Corollary 4.3 *Let (X, qp_b) be a complete quasi-partial b-metric space. Let $T: X \to X$ and $\phi: X \to \mathbb{R}^+$. Suppose that there exists $x \in X$ such that*

$$qp_b(y, Ty) \leq \phi(y) - \phi(Ty) \quad \textit{for all } y \in O(x).$$

Then the following hold:
(A) $\lim_{n\to\infty} T^n x = z$ *exists*.
(B) $Tz = z$ *if and only if* $G(x) = qp_b(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x.
(C) $qp_b(x, T^n x) \leq s^{n-1}\phi(x)$.
(D) *For* $m > n$, $qp_b(T^n x, T^m x) \leq s^{m-n}\phi(T^n x)$.

Proof Take $Y = X$, $R = I$, and $c = 1$ in Theorem 4.2. □

Corollary 4.4 *Let (X, qp_b) be a complete quasi-partial b-metric space, and let $0 < k < 1$. Suppose that $T : X \to X$ and that there exists $x \in X$ such that*

$$qp_b(Ty, T^2y) \le kqp_b(y, Ty) \quad \text{for all } y \in O(x). \tag{16}$$

Then the following hold:

(A) $\lim_{n\to\infty} T^n x = z$ *exists.*

(B) $Tz = z$ *if and only if* $G(x) = qp_b(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x.

(C) $qp_b(x, T^n x) \le \frac{s^{n-1}}{1-k} qp_b(x, Tx)$.

Proof Set $\phi(y) = \frac{1}{1-k} qp_b(y, Ty)$ for $y \in O(x)$.

Let $y = T^n x$ in (16). Then

$$qp_b(T^{n+1}x, T^{n+2}x) \le kqp_b(T^n x, T^{n+1}x)$$

and

$$qp_b(T^n x, T^{n+1}x) - kqp_b(T^n x, T^{n+1}x) \le qp_b(T^n x, T^{n+1}x) - qp_b(T^{n+1}x, T^{n+2}x).$$

Thus, $qp_b(T^n x, T^{n+1}x) \le \frac{1}{1-k}[qp_b(T^n x, T^{n+1}x) - qp_b(T^{n+1}x, T^{n+2}x)]$ or $qp_b(y, Ty) \le [\phi(y) - \phi(Ty)]$.

(A)-(C) follow immediately from Corollary 4.3. □

Corollary 4.5 *Let (X, qp_b) be a complete quasi-partial b-metric space where qp_b is continuous. Let $T : X \to X$ and $\phi : X \to \mathbb{R}^+$ is continuous. Suppose that there exists $x \in X$ such that*

$$qp_b(y, Ty) \le \phi(y) - \phi(Ty) \quad \text{for all } y \in O(x).$$

Then the following hold:

(A) $\lim_{n\to\infty} T^n x = z$ *exists.*

(B) $q_p(z, z) \le s\phi(z)$.

Proof In Theorem 4.2(D) taking $m = n + 1$, $R = I$, $c = 1$, and $Y = X$,

$$qp_b(T^n x, T^{n+1}x) \le s[\phi(RT^n x)].$$

Now taking $\lim n \to \infty$

$$\lim_{n\to\infty} qp_b(T^n x, T^{n+1}x) \le \lim_{n\to\infty} s[\phi(T^n x)],$$

$$qp_b(z, z) \le s\phi(z).$$ □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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