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Solvability for a discrete fractional mixed type sum-difference equation boundary value problem in a Banach space

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Abstract

In this paper, by means of Darbo's fixed point theorem, we establish the existence of solutions to a nonlinear discrete fractional mixed type sum-difference equation boundary value problem in a Banach space. Additionally, as an application, we give an example to demonstrate the main result.

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Keywords: discrete fractional mixed type sum-difference equation; boundary value problem; Banach space; noncompactness measure; fixed point theorem

1 Introduction

Throughout this paper, we denote $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, \ldots, b\}$ for $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_1$. Moreover, for any Banach-valued function u on $\mathbb{N}_a$, we make the convention that $\sum_{s=k_1}^{k_2} u(s) = \theta$ if $k_1, k_2 \in \mathbb{N}_a$ with $k_1 > k_2$, where θ is the zero element of a given Banach space.

In this paper, we consider the following discrete fractional mixed type sum-difference equation boundary value problem in Banach space E:

$$\begin{cases} \Delta^\beta u(t) + f(t+\beta-1, u(t+\beta-1), (Tu)(t), (Su)(t)) = \theta, & t \in \mathbb{N}_0, \\ u(\beta-2) = \theta, \qquad \Delta^{\beta-1} u(\infty) = u_\infty, \end{cases} \tag{1.1}$$

where $\beta \in (1,2]$, Δ^β denotes the discrete Riemann-Liouville fractional difference of order β, $f : \mathbb{N}_{\beta-1} \times E \times E \times E \to E$ is a continuous function, θ is the zero element of E, $\Delta^{\beta-1}u(\infty) = \lim_{t\to+\infty} \Delta^{\beta-1}u(t) = u_\infty \in E$, and

$$(Tu)(t) = \sum_{s=0}^{t} k(t,s)u(s+\beta-1), \qquad (Su)(t) = \sum_{s=0}^{\infty} h(t,s)u(s+\beta-1),$$

where $k : D \to \mathbb{R}$, $D = \{(t,s) \in \mathbb{N}_0 \times \mathbb{N}_0 : s \le t\}$, $h : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$.

In the last years, discrete fractional calculus and fractional difference equations with various boundary conditions have been studied more intensively. For details, see [1–16]; we particularly should note that the recent monograph of Goodrich and Peterson [1] is an extremely useful textbook for readers to obtain the fundamental background in discrete fractional calculus. However, as far as we know, most of the recent papers, mainly based on the Krasnosel'skii fixed point theorem, are concerning about the existence of positive solutions for discrete fractional boundary value problems on a finite interval, and few papers can be found in the literature for discrete boundary value problems on an infinite interval [16].

Lv and Feng [16] initially introduced some basic conceptions and fundamental results on discrete fractional calculus for any Banach-valued function and also, using of the contraction mapping principle, investigated the existence and uniqueness of solutions for a class of fractional mixed type sum-difference equation boundary value problems on discrete infinite intervals in Banach spaces. This is the first attempt to study the discrete fractional difference equation boundary value problems in abstract spaces.

It is well known that the measure of noncompactness is a very powerful tool to deal with differential equations [17–20], difference equations [21–23], integration equations [24], and differential inclusions [25, 26]. So, in this paper, we l employ noncompact measures and Darbo's fixed point theorem to establish some conditions for the existence of solutions to problem (1.1). We point out that the main result is even new and efficient for integer order case of $\beta = 2$.

The outline for the remainder of this paper is as follows. In Section 2, we recall some useful preliminaries. In Section 3, we establish the existence result of problem (1.1), and finally we present in Section 4 an example illustrating the abstract theory.

2 Preliminaries

In this section, we begin by presenting here some necessary definitions for discrete fractional calculus, and more preliminary facts can be found, for example, in [1, 16, 27, 28].

Definition 2.1 For any t and ν, the falling factorial function is defined as

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

provided that the right-hand side is well defined. We make the convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 The νth discrete fractional sum of a function $f : \mathbb{N}_a \to E$ for $\nu > 0$ is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad t \in \mathbb{N}_{a+\nu}.$$

Also, we define the trivial sum $\Delta_a^{-0} f(t) = f(t)$, $t \in \mathbb{N}_a$.

Definition 2.3 The νth discrete Riemann-Liouville fractional difference of a function $f : \mathbb{N}_a \to E$ for $\nu > 0$ is defined by

$$\Delta_a^{\nu} f(t) = \Delta^n \Delta_a^{-(n-\nu)} f(t), \quad t \in \mathbb{N}_{a+n-\nu},$$

where n is the smallest integer greater than or equal to ν, and Δ^n is the nth-order forward difference operator. If $\nu = n \in \mathbb{N}_1$, then $\Delta_a^n f(t) = \Delta^n f(t)$.

We denote by $C(\mathbb{N}_a^b, E)$ the Banach space of all functions $\varpi : \mathbb{N}_a^b \to E$ with the usual supremum norm $\|\varpi\|_0 = \sup\{\|\varpi(t)\| : t \in \mathbb{N}_a^b\}$. Define the space

$$X = \left\{u : \mathbb{N}_{\beta-2} \to E \,\middle|\, \sup_{t\in\mathbb{N}_{\beta-2}} \frac{\|u(t)\|}{1 + t^{\underline{\beta-1}}} < +\infty\right\}$$

equipped with the norm

$$\|u\|_X = \sup_{t\in\mathbb{N}_{\beta-2}} \frac{\|u(t)\|}{1 + t^{\underline{\beta-1}}}.$$

Furthermore, by means of the linear functional analysis theory we can easily verify that $(X, \|\cdot\|_X)$ is a Banach space. It is worth reminding that here we use α, α_C, and α_X to denote the Kuratowski noncompactness measure of bounded sets in Banach spaces $E, C(\mathbb{N}_a^b, E)$, and X, respectively. For more details on the Kuratowski noncompactness measure, we refer the reader to [29, 30]. We state the following properties of the Kuratowski measure of noncompactness and the Darbo's fixed point theorem, which are needed for the sequel discussion.

Lemma 2.1 ([22]) *Let $A \subseteq C(\mathbb{N}_a^b, E)$ be bounded. Then*
 (i) $\alpha_C(A) = \alpha(A(\mathbb{N}_a^b))$;
 (ii) $\alpha(A(\mathbb{N}_a^b)) = \sup_{t\in\mathbb{N}_a^b} \alpha(A(t))$,
where $A(t) = \{\varpi(t) : \varpi \in A\}$ and $A(\mathbb{N}_a^b) = \bigcup_{t\in\mathbb{N}_a^b} A(t)$.

Lemma 2.2 ([30]) *Let D be a bounded, closed, and convex subset of a Banach space E. If an operator $\mathcal{A} : D \to D$ is a strict set contraction, then $\mathcal{A}$ has a fixed point in D.*

Remark 2.1 A bounded and continuous operator $\mathcal{A} : D \to E$ is called a strict set contraction if there is a constant $\lambda \in [0, 1)$ such that $\alpha(\mathcal{A}S) \le \lambda\alpha(S)$ for any bounded set $S \subset D$.

3 Main result

In this section, we establish the existence of solutions for problem (1.1) by using Darbo's fixed point theorem. For convenience and shortness of our presentation, for any $u \in X$, we denote

$$g_u(t) = f\bigl(t + \beta - 1, u(t + \beta - 1), (Tu)(t), (Su)(t)\bigr), \quad t \in \mathbb{N}_0,$$

for further discussion and list the following conditions:

(C_1)

$$\sup_{t\in\mathbb{N}_0} \sum_{s=0}^{t} \bigl|k(t,s)\bigr| < +\infty, \qquad \sup_{t\in\mathbb{N}_0} \sum_{s=0}^{\infty} \frac{|h(t,s)|[1 + (s + \beta - 1)^{\underline{\beta-1}}]}{1 + (t + \beta - 1)^{\underline{\beta-1}}} < +\infty;$$

(C_2) There exist nonnegative numbers q_i, $i \in \mathbb{N}_1^3$, and functions $p_1, p_2 : \mathbb{N}_{\beta-1} \to [0, \infty)$ with $p_1^* = \sum_{t=\beta-1}^{\infty} p_1(t)(1 + t^{\underline{\beta-1}}) < \Gamma(\beta)$ and $p_2^* = \sum_{t=\beta-1}^{\infty} p_2(t) < +\infty$ such that

$$\bigl\|f(t, u, v, w)\bigr\| \le p_1(t)\bigl(q_1\|u\| + q_2\|v\| + q_3\|w\|\bigr) + p_2(t)$$

for $(t, u, v, w) \in \mathbb{N}_{\beta-1} \times E \times E \times E$;

(C$_3$) For any positive number $r \in \mathbb{N}_{\beta-1}$, $f(t,u,v,w)$ is uniformly continuous on $\mathbb{N}_{\beta-1}^{r} \times B_E(\theta,r) \times B_E(\theta,r) \times B_E(\theta,r)$, where $B_E(\theta,r) = \{x \in E : \|x\| \le r\}$;

(C$_4$) There exist functions $l_i : \mathbb{N}_{\beta-1} \to [0,+\infty)$, $i \in \mathbb{N}_1^3$, such that

$$\alpha\big(f(t,V_1,V_2,V_3)\big) \le l_1(t)\alpha(V_1) + l_2(t)\alpha(V_2) + l_3(t)\alpha(V_3), \quad t \in \mathbb{N}_{\beta-1},$$

for all bounded sets $V_i \subset E$, $i \in \mathbb{N}_1^3$, and

$$\sum_{t=\beta-1}^{\infty} \big(1 + t^{\underline{\beta-1}}\big)\big[l_1(t) + k^* l_2(t) + h^* l_3(t)\big] < \Gamma(\beta).$$

Moreover, we set

$$k^* = \sup_{t\in\mathbb{N}_0} \sum_{s=0}^{t} \big|k(t,s)\big| \quad \text{and} \quad h^* = \sup_{t\in\mathbb{N}_0} \sum_{s=0}^{\infty} \frac{|h(t,s)|[1 + (s+\beta-1)^{\underline{\beta-1}}]}{1 + (t+\beta-1)^{\underline{\beta-1}}}$$

when (C$_1$) holds.

Next, we state and prove the following lemmas, which are necessary for the proof of the main result.

Lemma 3.1 *Assume that* (C$_1$) *and* (C$_2$) *hold. Then, for any* $u \in X$,

$$\sum_{t=0}^{\infty} \big\|g_u(t)\big\| \le p_1^*\big(q_1 + q_2 k^* + q_3 h^*\big)\|u\|_X + p_2^*. \tag{3.1}$$

Proof For any $u \in X$, $t \in \mathbb{N}_0$, using (C$_1$), (C$_2$), and the monotonicity of $t^{\underline{\beta-1}}$ for $t \in \mathbb{N}_{\beta-1}$ produces

$$\begin{aligned}
\big\|g_u(t)\big\| &= \big\|f\big(t+\beta-1, u(t+\beta-1), (Tu)(t), (Su)(t)\big)\big\| \\
&\le p_1(t+\beta-1)\big(q_1\big\|u(t+\beta-1)\big\| + q_2\big\|(Tu)(t)\big\| + q_3\big\|(Su)(t)\big\|\big) + p_2(t+\beta-1) \\
&\le p_1(t+\beta-1)\big[1 + (t+\beta-1)^{\underline{\beta-1}}\big]\left\{q_1 + q_2 \sum_{s=0}^{t} \frac{|k(t,s)|[1 + (s+\beta-1)^{\underline{\beta-1}}]}{1 + (t+\beta-1)^{\underline{\beta-1}}}\right. \\
&\quad \left. + q_3 \sum_{s=0}^{\infty} \frac{|h(t,s)|[1 + (s+\beta-1)^{\underline{\beta-1}}]}{1 + (t+\beta-1)^{\underline{\beta-1}}}\right\}\|u\|_X + p_2(t+\beta-1) \\
&\le p_1(t+\beta-1)\big[1 + (t+\beta-1)^{\underline{\beta-1}}\big]\big(q_1 + q_2 k^* + q_3 h^*\big)\|u\|_X + p_2(t+\beta-1).
\end{aligned} \tag{3.2}$$

By summing both sides of (3.2) we get (3.1). So, this proof is completed. □

Lemma 3.2 *Let* $h : \mathbb{N}_0 \to E$ *be given, and* $\beta \in (1,2]$. *The unique solution of*

$$\begin{cases} \Delta^{\beta} u(t) + h(t) = \theta, & t \in \mathbb{N}_0, \\ u(\beta-2) = \theta, & \Delta^{\beta-1} u(\infty) = u_\infty, \end{cases}$$

is

$$u(t)=\sum_{s=0}^{\infty}G(t,s)h(s)+\frac{u_\infty}{\Gamma(\beta)}t^{\underline{\beta-1}},\quad t\in\mathbb{N}_{\beta-2},$$

where

$$G(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\underline{\beta-1}}-(t-s-1)^{\underline{\beta-1}}, & s\in\mathbb{N}_0^{t-\beta},\\ t^{\underline{\beta-1}}, & s\in\mathbb{N}_{t-\beta+1}.\end{cases}$$

Remark 3.1 The proof of Lemma 3.2 is similar to that of Lemma 3.2 in [16]. Hence, we omit it here. Moreover, in view of the expression of $G(t,s)$, we can easily verify that $G(t,s)\geq 0$ and $\frac{G(t,s)}{1+t^{\underline{\beta-1}}}<\frac{1}{\Gamma(\beta)}$ for $(t,s)\in\mathbb{N}_{\beta-2}\times\mathbb{N}_0$.

For any $u\in X$, define the operator $\mathcal{F}$ by

$$(\mathcal{F}u)(t)=\sum_{s=0}^{\infty}G(t,s)g_u(s)+\frac{u_\infty}{\Gamma(\beta)}t^{\underline{\beta-1}},\quad t\in\mathbb{N}_{\beta-2},$$

and due to Lemma 3.1 and Remark 3.1, we have

$$\begin{aligned}\frac{\|(\mathcal{F}u)(t)\|}{1+t^{\underline{\alpha-1}}}&\leq\sum_{s=0}^{\infty}\frac{G(t,s)}{1+t^{\underline{\beta-1}}}\|g_u(s)\|+\frac{\|u_\infty\|t^{\underline{\beta-1}}}{\Gamma(\beta)(1+t^{\underline{\beta-1}})}\\&\leq\frac{1}{\Gamma(\beta)}\bigl\{p_1^*(q_1+q_2k^*+q_3h^*)\|u\|_X+p_2^*+\|u_\infty\|\bigr\},\end{aligned}$$

which implies that $\mathcal{F}:X\to X$ is well defined and bounded. Furthermore, from Lemma 3.2 we know that the existence of solutions for problem (1.1) is equivalent to that of fixed points of $\mathcal{F}$ in X.

Lemma 3.3 *Suppose that* (C_1), (C_2), *and* (C_3) *are satisfied. Then the operator* $\mathcal{F}:X\to X$ *is continuous.*

Proof Let $\{u_n\}_{n=1}^{\infty}\subset X$ and $u\in X$ be such that $\|u_n-u\|_X\to 0$ as $n\to\infty$. Then, $\{u_n\}_{n=1}^{\infty}$ is a bounded set of X, that is, there exists $M>0$ such that $\|u_n\|_X\leq M$ for $n\in\mathbb{N}_1$. By taking limit we also have that $\|u\|_X\leq M$.

In view of (C_2), for any $\epsilon>0$, there exists a positive integer L such that

$$\sum_{t=L+\beta}^{\infty}p_1(t)\bigl(1+t^{\underline{\beta-1}}\bigr)<\frac{\Gamma(\beta)}{6M(q_1+q_2k^*+q_3h^*)}\epsilon,\tag{3.3}$$

$$\sum_{t=L+\beta}^{\infty}p_2(t)<\frac{\Gamma(\beta)}{6}\epsilon.\tag{3.4}$$

On the other hand, condition (C_3) yields that there exists $N>0$ such that for any $n>N$ and $t\in\mathbb{N}_0^L$,

$$\bigl\|g_{u_n}(t)-g_u(t)\bigr\|\leq\frac{\Gamma(\beta)}{3(L+1)}\epsilon.\tag{3.5}$$

Therefore, for $t \in \mathbb{N}_{\beta-2}^{L+\beta-1}$ and $n > N$, by (C_1), (C_2), (3.3)-(3.5), and Remark 3.1 we obtain that

$$\begin{aligned}
&\frac{\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|}{1 + t^{\underline{\beta-1}}} \\
&\le \sum_{s=0}^{\infty} \frac{G(t,s)}{1 + t^{\underline{\beta-1}}} \| g_{u_n}(s) - g_u(s) \| \\
&< \frac{1}{\Gamma(\beta)} \left\{ \sum_{s=0}^{L} \| g_{u_n}(s) - g_u(s) \| + \sum_{s=L+1}^{\infty} \| g_{u_n}(s) - g_u(s) \| \right\} \\
&\le \frac{1}{\Gamma(\beta)} \sum_{s=0}^{L} \| g_{u_n}(s) - g_u(s) \| + \frac{1}{\Gamma(\beta)} \sum_{s=L+1}^{\infty} \{ \| g_{u_n}(s) \| + \| g_u(s) \| \} \\
&\le \frac{\epsilon}{3} + \frac{1}{\Gamma(\beta)} \sum_{s=L+1}^{\infty} \{ \| f(s+\beta-1, u_n(s+\beta-1), (Tu_n)(s), (Su_n)(s)) \| \\
&\quad + \| f(s+\beta-1, u(s+\beta-1), (Tu)(s), (Su)(s)) \| \} \\
&\le \frac{\epsilon}{3} + \frac{1}{\Gamma(\beta)} \sum_{s=L+1}^{\infty} \{ p_1(s+\beta-1)\big(q_1 \| u_n(s+\beta-1) \| + q_2 \| (Tu_n)(s) \| + q_3 \| (Su_n)(s) \| \big) \\
&\quad + p_1(s+\beta-1)\big(q_1 \| u(s+\beta-1) \| + q_2 \| (Tu)(s) \| + q_3 \| (Su)(s) \| \big) + 2p_2(s+\beta-1) \} \\
&\le \frac{\epsilon}{3} + \frac{2}{\Gamma(\beta)} \sum_{s=L+\beta}^{\infty} p_2(s) + \frac{1}{\Gamma(\beta)} \sum_{s=L+1}^{\infty} p_1(s+\beta-1)\big[1 + (s+\beta-1)^{\underline{\beta-1}}\big] \\
&\quad \times \left\{ q_1 + q_2 \sum_{\tau=0}^{s} \frac{|k(s,\tau)|[1 + (\tau+\beta-1)^{\underline{\beta-1}}]}{1 + (s+\beta-1)^{\underline{\beta-1}}} + q_3 \sum_{\tau=0}^{\infty} \frac{|h(s,\tau)|[1 + (\tau+\beta-1)^{\underline{\beta-1}}]}{1 + (s+\beta-1)^{\underline{\beta-1}}} \right\} \\
&\quad \times \big(\|u_n\|_X + \|u\|_X \big) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{2M(q_1 + q_2 k^* + q_3 h^*)}{\Gamma(\beta)} \sum_{s=L+\beta}^{\infty} p_1(s)\big(1 + s^{\underline{\beta-1}}\big) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Meanwhile, for $t \in \mathbb{N}_{L+\beta}$ and $n > N$, applying (3.3)-(3.5) again, we can easily verify that

$$\frac{\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|}{1 + t^{\underline{\beta-1}}} < \epsilon.$$

Then, we conclude that $\|\mathcal{F}u_n - \mathcal{F}u\|_X \le \epsilon$ for $n > N$. So $\mathcal{F}$ is continuous, and the proof is completed. □

Lemma 3.4 *Let B be a bounded subset of X. If (C_1) and (C_2) hold, then for any $\epsilon > 0$, there exists a positive number $N \in \mathbb{N}_{\beta-2}$ such that $\| \frac{(\mathcal{F}u)(t_2)}{1+t_2^{\underline{\beta-1}}} - \frac{(\mathcal{F}u)(t_1)}{1+t_1^{\underline{\beta-1}}} \| < \epsilon$ for each $u \in B$ and any $t_1, t_2 \in \mathbb{N}_N$.*

Proof In view of Lemma 3.1 and the boundedness of B, there exists $M > 0$ such that

$$\sum_{t=0}^{\infty} \| g_u(t) \| \le M \quad \text{for any } u \in B. \tag{3.6}$$

Note that

$$\begin{aligned}
&\left\| \frac{(\mathcal{F}u)(t_2)}{1+t_2^{\underline{\beta-1}}} - \frac{(\mathcal{F}u)(t_1)}{1+t_1^{\underline{\beta-1}}} \right\| \\
&\quad \le \frac{1}{\Gamma(\beta)} \left\| u_\infty + \sum_{s=0}^{\infty} g_u(s) \right\| \left| \frac{t_2^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}} - \frac{t_1^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}} \right| \\
&\qquad + \frac{1}{\Gamma(\beta)} \left\| \sum_{s=0}^{t_2-\beta} \frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}} g_u(s) - \sum_{s=0}^{t_1-\beta} \frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}} g_u(s) \right\|. \qquad (3.7)
\end{aligned}$$

Observing (3.7) together with $\lim_{t\to+\infty} \frac{t^{\underline{\beta-1}}}{1+t^{\underline{\beta-1}}} = 1$, we only need to show that for any $\epsilon > 0$, there exists sufficiently large positive number $N \in \mathbb{N}_{\beta-2}$ such that, for any $t_1, t_2 \in \mathbb{N}_N$,

$$\left\| \sum_{s=0}^{t_2-\beta} \frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}} g_u(s) - \sum_{s=0}^{t_1-\beta} \frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}} g_u(s) \right\| < \epsilon.$$

Relation (3.6) yields that there exists a positive number $L \in \mathbb{N}_0$ such that

$$\sum_{t=L+1}^{\infty} \left\| g_u(t) \right\| \le \frac{\epsilon}{3} \quad \text{uniformly with respect to } u \in B. \qquad (3.8)$$

On the other hand, from the monotonicity of $t^{\underline{\beta-1}}$ we can declare that $\lim_{t\to+\infty} \frac{(t-L-1)^{\underline{\beta-1}}}{1+t^{\underline{\beta-1}}} = 1$.
In fact, for any $t \in \mathbb{N}_{L+\beta+1}$, $L \in \mathbb{N}_0$, we have

$$\begin{aligned}
\frac{(t-L-1)^{\underline{\beta-1}}}{t^{\underline{\beta-1}}} &= \frac{\Gamma(t-L)\Gamma(t-\beta+2)}{\Gamma(t+1)\Gamma(t-\beta+1-L)} \\
&= \frac{(t+1-\beta)^{\underline{L}}}{t^{\underline{L}}} \\
&= \frac{\prod_{j=0}^{L-1}(t-j-\beta+1)}{\prod_{j=0}^{L-1}(t-j)} \\
&= \prod_{j=0}^{L-1} \left(1 + \frac{1-\beta}{t-j}\right),
\end{aligned}$$

which implies that

$$\lim_{t\to+\infty} \frac{(t-L-1)^{\underline{\beta-1}}}{1+t^{\underline{\beta-1}}} = \lim_{t\to+\infty} \frac{\frac{(t-L-1)^{\underline{\beta-1}}}{t^{\underline{\beta-1}}}}{\frac{1}{t^{\underline{\beta-1}}}+1} = \frac{1}{0+1} = 1.$$

So, there exists $N \in \mathbb{N}_{L+\beta+1}$ such that for any $t_1, t_2 \in \mathbb{N}_N$ and $s \in \mathbb{N}_0^L$,

$$\begin{aligned}
&\left| \frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}} - \frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}} \right| \\
&\quad \le \left(1 - \frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}}\right) + \left(1 - \frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}}\right)
\end{aligned}$$

$$\begin{aligned}
&\le \left(1-\frac{(t_2-L-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}}\right)+\left(1-\frac{(t_1-L-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}}\right)\\
&< \frac{\epsilon}{3M}. \qquad (3.9)
\end{aligned}$$

Now taking $t_1, t_2 \in \mathbb{N}_N$, by (3.8) and (3.9) we get

$$\begin{aligned}
&\left\|\sum_{s=0}^{t_2-\beta}\frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}}g_u(s)-\sum_{s=0}^{t_1-\beta}\frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}}g_u(s)\right\|\\
&\quad\le \sum_{s=0}^{L}\left|\frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}}-\frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}}\right|\left\|g_u(s)\right\|\\
&\qquad+\sum_{s=L+1}^{t_2-\beta}\frac{(t_2-s-1)^{\underline{\beta-1}}}{1+t_2^{\underline{\beta-1}}}\left\|g_u(s)\right\|+\sum_{s=L+1}^{t_1-\beta}\frac{(t_1-s-1)^{\underline{\beta-1}}}{1+t_1^{\underline{\beta-1}}}\left\|g_u(s)\right\|\\
&\quad< \frac{\epsilon}{3M}\sum_{s=0}^{\infty}\left\|g_u(s)\right\|+2\sum_{s=L+1}^{\infty}\left\|g_u(s)\right\|\\
&\quad\le \epsilon.
\end{aligned}$$

Therefore, the proof is completed. □

Lemma 3.5 *Let B be a bounded subset of X. If* (C_1) *and* (C_2) *hold, then*

$$\alpha_X(\mathcal{F}B)=\sup_{t\in\mathbb{N}_{\beta-2}}\alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right).$$

Proof First, we claim that $\alpha_X(\mathcal{F}B)\le\sup_{t\in\mathbb{N}_{\beta-2}}\alpha(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}})$.

In view of Lemma 3.4, we know that for any $\epsilon>0$, there exists a positive number $N\in\mathbb{N}_{\beta-2}$ such that, for $t_1, t_2\in\mathbb{N}_N$,

$$\left\|\frac{(\mathcal{F}u)(t_2)}{1+t_2^{\underline{\beta-1}}}-\frac{(\mathcal{F}u)(t_1)}{1+t_1^{\underline{\beta-1}}}\right\|<\epsilon \quad \text{uniformly with respect to } u\in B. \qquad (3.10)$$

Denote by $\mathcal{F}B|_{\mathbb{N}_{\beta-2}^N}$ the restriction of $\mathcal{F}B$ on $\mathbb{N}_{\beta-2}^N$. By Lemma 2.1 we obtain

$$\alpha_X(\mathcal{F}B|_{\mathbb{N}_{\beta-2}^N})=\sup_{t\in\mathbb{N}_{\beta-2}^N}\alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right)\le\sup_{t\in\mathbb{N}_{\beta-2}}\alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right).$$

So, there exists a partition of B such that $B=\bigcup_{i=1}^n B_i$, $\mathcal{F}B|_{\mathbb{N}_{\beta-2}^N}=\bigcup_{i=1}^n\mathcal{F}B_i|_{\mathbb{N}_{\beta-2}^N}$ and

$$\operatorname{diam}_X(\mathcal{F}B_i|_{\mathbb{N}_{\beta-2}^N})<\sup_{t\in\mathbb{N}_{\beta-2}}\alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right)+\epsilon,\quad i\in\mathbb{N}_1^n, \qquad (3.11)$$

where $\operatorname{diam}_X(\cdot)$ denotes the diameter of a bounded subset of X. Moreover, for all $\mathcal{F}u_1,\mathcal{F}u_2\in\mathcal{F}B_i$, $i\in\mathbb{N}_1^n$, and $t\in\mathbb{N}_N$, (3.10) and (3.11) imply that

$$\begin{aligned}
&\left\|\frac{(\mathcal{F}u_1)(t)}{1+t^{\underline{\beta-1}}}-\frac{(\mathcal{F}u_2)(t)}{1+t^{\underline{\beta-1}}}\right\| \\
&\quad\leq \left\|\frac{(\mathcal{F}u_1)(t)}{1+t^{\underline{\beta-1}}}-\frac{(\mathcal{F}u_1)(N)}{1+N^{\underline{\beta-1}}}\right\| + \left\|\frac{(\mathcal{F}u_1)(N)}{1+N^{\underline{\beta-1}}}-\frac{(\mathcal{F}u_2)(N)}{1+N^{\underline{\beta-1}}}\right\| + \left\|\frac{(\mathcal{F}u_2)(N)}{1+N^{\underline{\beta-1}}}-\frac{(\mathcal{F}u_2)(t)}{1+t^{\underline{\beta-1}}}\right\| \\
&\quad< \epsilon + \sup_{t\in\mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right) + \epsilon + \epsilon \\
&\quad= \sup_{t\in\mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right) + 3\epsilon. \qquad (3.12)
\end{aligned}$$

Hence, it follows from (3.11) and (3.12) that

$$\operatorname{diam}_X(\mathcal{F}B_i) \leq \sup_{t\in\mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right) + 3\epsilon.$$

Since $\mathcal{F}B = \bigcup_{i=1}^{n} \mathcal{F}B_i$, we get that $\alpha_X(\mathcal{F}B) \leq \sup_{t\in\mathbb{N}_{\beta-2}} \alpha(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}) + 3\epsilon$, which by the arbitrariness of ϵ implies that

$$\alpha_X(\mathcal{F}B) \leq \sup_{t\in\mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right).$$

Next, we show that $\sup_{t\in\mathbb{N}_{\beta-2}} \alpha(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\alpha-1}}}) \leq \alpha_X(\mathcal{F}B)$. For any given $\epsilon > 0$, there exists a partition $\mathcal{F}B = \bigcup_{i=1}^{n} \mathcal{F}B_i$ such that $\operatorname{diam}_X(\mathcal{F}B_i) < \alpha_X(\mathcal{F}B) + \epsilon$. Therefore, for any $t \in \mathbb{N}_{\beta-2}$ and $u_1, u_2 \in B_i$, we obtain

$$\left\|\frac{(\mathcal{F}u_2)(t)}{1+t^{\underline{\beta-1}}}-\frac{(\mathcal{F}u_1)(t)}{1+t^{\underline{\beta-1}}}\right\| \leq \left\|\mathcal{F}u_2-\mathcal{F}u_1\right\|_X < \alpha_X(\mathcal{F}B) + \epsilon.$$

In accordance with $(\mathcal{F}B)(t) = \bigcup_{i=1}^{n} \mathcal{F}B_i(t)$, we get $\alpha(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}) \leq \alpha_X(\mathcal{F}B) + \epsilon$. Because ϵ is arbitrary, we have

$$\sup_{t\in\mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}B)(t)}{1+t^{\underline{\beta-1}}}\right) \leq \alpha_X(\mathcal{F}B).$$

Consequently, the proof of this lemma is complete. □

With all auxiliary results in hand, now we state the main result.

Theorem 3.1 *If* (C_1)-(C_4) *hold, then problem* (1.1) *has at least one solution u in X.*

Proof Choose

$$R > \frac{\|u_\infty\| + p_2^*}{\Gamma(\beta) - p_1^*(q_1 + q_2k^* + q_3h^*)}$$

and let

$$B = \bigl\{u \in X : \|u\|_X \leq R\bigr\}.$$

Then, for any $u \in B$, by Lemma 3.1 and Remark 3.1 we have

$$\begin{aligned}\frac{\|(\mathcal{F}u)(t)\|}{1+t^{\underline{\alpha-1}}} &\le \sum_{s=0}^{\infty} \frac{G(t,s)}{1+t^{\underline{\beta-1}}} \|g_u(s)\| + \frac{\|u_\infty\| t^{\underline{\beta-1}}}{\Gamma(\beta)(1+t^{\underline{\beta-1}})} \\ &\le \frac{1}{\Gamma(\beta)} \bigl\{ p_1^*(q_1 + q_2 k^* + q_3 h^*) \|u\|_X + p_2^* + \|u_\infty\| \bigr\} \le R,\end{aligned}$$

which implies that $\mathcal{F} : B \to B$.

Set $D = \overline{\mathrm{co}}(\mathcal{F}B)$. Obviously, D is a bounded, convex, and closed subset of B. In the sequel, we show that the operator $\mathcal{F} : D \to D$ is a strict contraction.

Observing that $\mathcal{F}D \subset \mathcal{F}B \subset D$, together with Lemma 3.3, we know that $\mathcal{F} : D \to D$ is bounded and continuous. Finally, we show that there exists a constant $\lambda \in [0,1)$ such that $\alpha_X(\mathcal{F}V) \le \lambda \alpha_X(V)$ for any $V \subset D$. Moreover, in view of Lemma 3.5, we only need to verify

$$\sup_{t \in \mathbb{N}_{\beta-2}} \alpha\left(\frac{(\mathcal{F}V)(t)}{1+t^{\underline{\beta-1}}} \right) \le \lambda \alpha_X(V). \tag{3.13}$$

For a positive integer $n > t - \beta$, define

$$(\mathcal{F}_n u)(t) = \frac{1}{\Gamma(\beta)} \sum_{s=0}^{t-\beta} \bigl[t^{\underline{\beta-1}} - (t-s-1)^{\underline{\beta-1}} \bigr] g_u(s) + \frac{1}{\Gamma(\beta)} \sum_{s=t-\beta+1}^{n} t^{\underline{\beta-1}} g_u(s) + \frac{u_\infty}{\Gamma(\beta)} t^{\underline{\beta-1}}.$$

Then from (C_1) and (C_2), for any $u \in V$, we have

$$\begin{aligned}&\left\| \frac{(\mathcal{F}_n u)(t)}{1+t^{\underline{\beta-1}}} - \frac{(\mathcal{F}u)(t)}{1+t^{\underline{\beta-1}}} \right\| \\ &\quad\le \frac{1}{\Gamma(\beta)} \sum_{s=n+1}^{\infty} \|g_u(s)\| \\ &\quad\le \frac{1}{\Gamma(\beta)} \sum_{s=n+1}^{\infty} \bigl\{ p_1(s+\beta+1) \bigl[1 + (s+\beta-1)^{\underline{\beta-1}}\bigr] (q_1 + q_2 k^* + q_3 h^*) R + p_2(s+\beta-1) \bigr\},\end{aligned}$$

which implies that

$$d_{\mathrm{H}}\left(\frac{(\mathcal{F}_n V)(t)}{1+t^{\underline{\beta-1}}}, \frac{(\mathcal{F}V)(t)}{1+t^{\underline{\beta-1}}} \right) \to 0 \quad \text{as } n \to \infty, t \in \mathbb{N}_{\beta-2},$$

where $d_{\mathrm{H}}(\cdot,\cdot)$ denotes the Hausdorff metric in space E. So, by the properties of noncompactness measure we obtain

$$\lim_{n\to\infty} \alpha\left(\frac{(\mathcal{F}_n V)(t)}{1+t^{\underline{\beta-1}}} \right) = \alpha\left(\frac{(\mathcal{F}V)(t)}{1+t^{\underline{\beta-1}}} \right), \quad t \in \mathbb{N}_{\beta-2}. \tag{3.14}$$

Now we estimate

$$\alpha\left(\frac{(\mathcal{F}_n V)(t)}{1+t^{\underline{\beta-1}}} \right) \quad (n > t - \beta).$$

In view of (C_1) and (C_2), we know that $\{g_u(t) : u \in V\}$ is bounded. Relying on Lemma 2.1, Lemma 3.5, and (C_4), we have

$$\begin{aligned}
\alpha\left(\frac{(\mathcal{F}_n V)(t)}{1+t^{\underline{\beta-1}}}\right) &\le \frac{1}{\Gamma(\beta)}\alpha\left(\left\{\sum_{s=0}^{t-\beta} g_u(s) : u \in V\right\}\right) + \frac{1}{\Gamma(\beta)}\alpha\left(\left\{\sum_{s=t-\beta+1}^{n} g_u(s) : u \in V\right\}\right) \\
&\le \frac{1}{\Gamma(\beta)}\sum_{s=0}^{n}\alpha\bigl(f\bigl(s+\beta-1, V(s+\beta-1), (TV)(s), (SV)(s)\bigr)\bigr) \\
&\le \frac{1}{\Gamma(\beta)}\sum_{s=0}^{n}\bigl[l_1(s+\beta-1)\alpha\bigl(V(s+\beta-1)\bigr) + l_2(s+\beta-1)\alpha\bigl((TV)(s)\bigr) \\
&\quad + l_3(s+\beta-1)\alpha\bigl((SV)(s)\bigr)\bigr] \\
&\le \frac{1}{\Gamma(\beta)}\sum_{s=0}^{n}\bigl[1+(s+\beta-1)^{\underline{\beta-1}}\bigr]\Biggl\{l_1(s+\beta-1) \\
&\quad + l_2(s+\beta-1)\sum_{\tau=0}^{s}\frac{|k(s,\tau)|[1+(\tau+\beta-1)^{\underline{\beta-1}}]}{[1+(s+\beta-1)^{\underline{\beta-1}}]} \\
&\quad + l_3(s+\beta-1)\sum_{\tau=0}^{\infty}\frac{|h(s,\tau)|[1+(\tau+\beta-1)^{\underline{\beta-1}}]}{[1+(s+\beta-1)^{\underline{\beta-1}}]}\Biggr\}\alpha_X(V) \\
&\le \frac{1}{\Gamma(\beta)}\sum_{s=\beta-1}^{n+\beta-1}\bigl[1+s^{\underline{\beta-1}}\bigr]\bigl[l_1(s)+l_2(s)k^* + l_3(s)h^*\bigr]\alpha_X(V).
\end{aligned}$$

By (3.14) we immediately get

$$\alpha\left(\frac{(\mathcal{F}V)(t)}{1+t^{\underline{\beta-1}}}\right) \le \lambda\alpha_X(V) \quad \text{with } \lambda = \frac{1}{\Gamma(\beta)}\sum_{s=\beta-1}^{n+\beta-1}\bigl[1+s^{\underline{\beta-1}}\bigr]\bigl[l_1(s)+l_2(s)k^*+l_3(s)h^*\bigr].$$

So (3.13) holds with $\lambda \in [0,1)$, and from Lemma 2.2 we immediately obtain that problem (1.1) has at least one solution in D. Hence, the proof is completed. □

4 An example

Example 4.1 Consider the following infinite system of scalar discrete fractional difference equations:

$$\begin{cases}
\Delta^{3/2}u_n(t) + \frac{3^{-(t+1)}}{n[1+(t+1/2)^{\underline{1/2}}]^2}\sin[u_n(t+1/2)] + \frac{2^{-t}}{2n^2[1+(t+1/2)^{\underline{1/2}}]} \\
\quad \times \{1 + u_{2n+1}(t+1/2) + \sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}u_{3n}(s+1/2) \\
\quad + \sum_{s=0}^{\infty}\frac{\cos(t^2 s)}{(s+2)^{\underline{2}}[1+(s+1/2)^{\underline{1/2}}]}u_{n+1}(s+1/2)\}^{1/2} = 0, \quad t \in \mathbb{N}_0, \\
u_n(-1/2) = 0, \qquad \Delta^{1/2}u_n(\infty) = \frac{1}{n!}, \quad n \in \mathbb{N}_1.
\end{cases} \tag{4.1}$$

Conclusion System (4.1) has at least one solution $\{u_n(t)\}_{n=1}^{\infty}$ such that $u_n(t) \to 0$ as $n \to \infty$ for $t \in \mathbb{N}_{-1/2}$.

Proof Let $E = c_0 = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0\}$. Evidently, $(E, \|\cdot\|)$ is a Banach space with the norm $\|u\| = \sup_n |u_n|$ for any $u \in E$. Then infinite system (4.1) can be regarded as a boundary value problem of the form (1.1) in the Banach space E. In this case, $\beta = 3/2$, $\theta = (0, 0, \ldots, 0, \ldots) \in E$, $u_\infty = (1, 1/2!, \ldots, 1/n!, \ldots) \in E$,

$$k(t,s) = \frac{1}{(t+s+2)^{\underline{2}}}, \qquad h(t,s) = \frac{\cos(t^2 s)}{(s+2)^{\underline{2}}[1+(s+1/2)^{\underline{1/2}}]},$$

and $f = (f_1, f_2, \ldots, f_n, \ldots)$ with

$$f_n(t,u,v,w) = \frac{3^{-t-1/2}}{n(1+t^{\underline{1/2}})^2} \sin u_n + \frac{2^{-t+1/2}}{2n^2(1+t^{\underline{1/2}})} (1 + u_{2n+1} + v_{3n} + w_{n+1})^{1/2}, \tag{4.2}$$

where $t \in \mathbb{N}_{1/2}$ and $u = (u_1, u_2, \ldots, u_n, \ldots)$, $v = (v_1, v_2, \ldots, v_n, \ldots)$, $w = (w_1, w_2, \ldots, w_n, \ldots) \in E$. From the expression of f_n we easily to see that $f : \mathbb{N}_{1/2} \times E \times E \times E \to E$ is continuous. Furthermore, we can easily verify that

$$\begin{aligned}
k^* &= \sup_{t \in \mathbb{N}_0} \sum_{s=0}^{t} \frac{1}{(t+s+2)^{\underline{2}}} = \sup_{t \in \mathbb{N}_0} \frac{1}{2(t+1)} = \frac{1}{2} < \infty, \\
h^* &= \sup_{t \in \mathbb{N}_0} \frac{1}{1+(t+1/2)^{\underline{1/2}}} \sum_{s=0}^{\infty} \frac{|\cos(t^2 s)|[1+(s+1/2)^{\underline{1/2}}]}{[1+(s+1/2)^{\underline{1/2}}](s+2)^{\underline{2}}} \\
&\le \sup_{t \in \mathbb{N}_0} \frac{1}{1+(t+1/2)^{\underline{1/2}}} \sum_{s=0}^{\infty} \frac{1}{(s+2)^{\underline{2}}} \\
&\le \frac{1}{1+\Gamma(3/2)} < \frac{3}{4} < \infty.
\end{aligned}$$

So, condition (C_1) is satisfied. On the other hand, using the simple inequality

$$(1+z)^\gamma \le 1 + \gamma z \quad \text{for } z \in [0, +\infty), \gamma \in (0,1),$$

we see from (4.2) that

$$\begin{aligned}
\left|f_n(t,u,v,w)\right| &\le \frac{3^{-t-1/2}}{n(1+t^{\underline{1/2}})^2} |\sin u_n| + \frac{2^{-t+1/2}}{2n^2(1+t^{\underline{1/2}})} \bigl(1 + \|u\| + \|v\| + \|w\|\bigr)^{1/2} \\
&\le \frac{2^{-t-1/2}}{2n^2(1+t^{\underline{1/2}})} \bigl(\|u\| + \|v\| + \|w\|\bigr) + \frac{2^{-t+1/2}}{2n^2(1+t^{\underline{1/2}})} + \frac{3^{-t-1/2}}{n(1+t^{\underline{1/2}})^2}, \quad n \in \mathbb{N}_1,
\end{aligned}$$

and, therefore,

$$\bigl\|f(t,u,v,w)\bigr\| \le p_1(t)\bigl(\|u\| + \|v\| + \|w\|\bigr) + p_2(t),$$

where

$$p_1(t) = \frac{2^{-t-1/2}}{2(1+t^{\underline{1/2}})}, \qquad p_2(t) = \frac{2^{-t+1/2}}{2(1+t^{\underline{1/2}})} + \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})^2}, \qquad q_1 = q_2 = q_3 = 1,$$

which implies

$$p_1^* = 1/2 < \Gamma(3/2), \qquad p_2^* < 3/2.$$

So condition (C_2) is satisfied. We can also verify that (C_3) holds by (4.2). Finally, we check condition (C_4). Let $f = f^{(1)} + f^{(2)}$, where

$$f_n^{(1)}(t,u,v,w) = \frac{3^{-t-1/2}}{n(1+t^{\underline{1/2}})^2}\sin u_n,$$
$$f_n^{(2)}(t,u,v,w) = \frac{2^{-t+1/2}}{2n^2(1+t^{\underline{1/2}})}(1+u_{2n+1}+v_{3n}+w_{n+1})^{1/2}.$$

Then we obtain that for any bounded sets $V_i \subset E$, $i \in \mathbb{N}_1^3$, $\alpha(f^2(t,V_1,V_2,V_3)) = 0$, $t \in \mathbb{N}_{1/2}$. In fact, since V_i, $i \in \mathbb{N}_1^3$, are bounded, there exists $r > 0$ such that $V_i \subset B_E(\theta, r)$, $i \in \mathbb{N}_1^3$. Let $\{u^{(m)}\}_{m=1}^\infty \in V_1$, $\{v^{(m)}\}_{m=1}^\infty \in V_2$, $\{w^{(m)}\}_{m=1}^\infty \in V_3$. Then for any fixed $t \in \mathbb{N}_{1/2}$, we have

$$\bigl|f_n^{(2)}\bigl(t,u^{(m)},v^{(m)},w^{(m)}\bigr)\bigr| \le \frac{3r}{4n^2[1+\Gamma(3/2)]} + \frac{1}{2n^2[1+\Gamma(3/2)]}, \quad n,m \in \mathbb{N}_1. \tag{4.3}$$

Therefore, $\{f_n^{(2)}(t,u^{(m)},v^{(m)},w^{(m)})\}$ is bounded, and so, by the diagonal method we can choose a subsequence $\{m_k\} \subset \{m\}$ such that

$$f_n^{(2)}\bigl(t,u^{(m_k)},v^{(m_k)},w^{(m_k)}\bigr) \to z_n \quad \text{as } k \to \infty, n \in \mathbb{N}_1. \tag{4.4}$$

Now, (4.3) and (4.4) imply

$$|z_n| \le \frac{3r}{4n^2[1+\Gamma(3/2)]} + \frac{1}{2n^2[1+\Gamma(3/2)]}, \quad n,m \in \mathbb{N}_1. \tag{4.5}$$

So $z = (z_1, z_2, \ldots, z_n, \ldots) \in c_0 = E$, and it is easy to see from (4.3)-(4.5) that

$$\bigl\|f^{(2)}\bigl(t,u^{(m_k)},v^{(m_k)},w^{(m_k)}\bigr) - z\bigr\| = \sup_n \bigl|f_n^{(2)}\bigl(t,u^{(m_k)},v^{(m_k)},w^{(m_k)}\bigr) - z_n\bigr| \to 0 \quad \text{as } k \to \infty.$$

Thus, we have proved that $f^2(t,V_1,V_2,V_3)$ is relatively compact in E and

$$\alpha\bigl(f^2(t,V_1,V_2,V_3)\bigr) = 0. \tag{4.6}$$

On the other hand, for any $t \in \mathbb{N}_{1/2}$, $u,\overline{u} \in V_1$, $v,\overline{v} \in V_2$, $w,\overline{w} \in V_3$, we have

$$\bigl|f_n^{(1)}(t,u,v,w) - f_n^{(1)}(t,\overline{u},\overline{v},\overline{w})\bigr| \le \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})^2}|u_n - \overline{u}_n|,$$

and, therefore,

$$\alpha\bigl(f^{(1)}(t,V_1,V_2,V_3)\bigr) \le \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})^2}\alpha(V_1). \tag{4.7}$$

From (4.6) and (4.7) we have

$$\alpha\bigl(f(t,V_1,V_2,V_3)\bigr) \le \alpha\bigl(f^{(1)}(t,V_1,V_2,V_3)\bigr) \le \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})^2}\alpha(V_1).$$

In view of $\sum_{t=1/2}^{\infty} \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})} < 1/2 < \Gamma(3/2)$, we get that condition (C_4) holds with $l_1(t) = \frac{3^{-t-1/2}}{(1+t^{\underline{1/2}})^2}$ and $l_2(t) = l_3(t) = 0$. So by Theorem 3.1 our conclusion follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Goodrich, CS, Peterson, AC: Discrete Fractional Calculus. Springer, Berlin (2015)
2. Atıcı, FM, Eloe, PW: Two-point boundary value problems for finite fractional difference equations. J. Differ. Equ. Appl. **17**(4), 445-456 (2011)
3. Goodrich, CS: Solutions to a discrete right-focal fractional boundary value problem. Int. J. Difference Equ. **5**(2), 195-216 (2010)
4. Goodrich, CS: Existence of a positive solution to a system of discrete fractional boundary value problems. Appl. Math. Comput. **217**(9), 4740-4753 (2011)
5. Goodrich, CS: On a discrete fractional three-point boundary value problem. J. Differ. Equ. Appl. **18**(3), 397-415 (2012)
6. Goodrich, CS: On discrete sequential fractional boundary value problems. J. Math. Anal. Appl. **385**(1), 111-124 (2012)
7. Goodrich, CS: On semipositone discrete fractional boundary value problems with non-local boundary conditions. J. Differ. Equ. Appl. **19**(11), 1758-1780 (2013)
8. Dahal, R, Duncan, D, Goodrich, CS: Systems of semipositone discrete fractional boundary value problems. J. Differ. Equ. Appl. **20**(3), 473-491 (2014)
9. Holm, M: Solutions to a discrete, nonlinear, $(N-1,1)$ fractional boundary value problem. Int. J. Dyn. Syst. Differ. Equ. **3**(1-2), 267-287 (2011)
10. Pan, Y, Han, Z, Sun, S, Hou, C: The existence of solutions to a class of boundary value problems with fractional difference equations. Adv. Differ. Equ. **2013**, 275 (2013)
11. Ferreira, RAC: Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one. J. Differ. Equ. Appl. **19**(5), 712-718 (2013)
12. Chen, F, Zhou, Y: Existence and Ulam stability of solutions for discrete fractional boundary value problem. Discrete Dyn. Nat. Soc. **2013**, Article ID 459161 (2013)
13. Lv, W: Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator. Adv. Differ. Equ. **2012**, Article ID 163 (2012)
14. Lv, W: Solvability for discrete fractional boundary value problems with a p-Laplacian operator. Discrete Dyn. Nat. Soc. **2013**, Article ID 679290 (2013)
15. Lv, W: Solvability for a discrete fractional three-point boundary value problem at resonance. Abstr. Appl. Anal. **2014**, Article ID 601092 (2014)
16. Lv, W, Feng, J: Nonlinear discrete fractional mixed type sum-difference equation boundary value problems in Banach spaces. Adv. Differ. Equ. **2014**, Article ID 184 (2014)
17. O'Regan, D: Measures of noncompactness, Darbo maps and differential equations in abstract spaces. Acta Math. Hung. **69**(3), 233-261 (1995)
18. Agarwal, RP, Benchohra, M, Seba, D: On the application of measure of noncompactness to the existence of solutions for fractional differential equations. Results Math. **55**(3), 221-230 (2009)
19. Xue, X: Nonlocal nonlinear differential equations with a measure of noncompactness in Banach spaces. Nonlinear Anal. **70**(7), 2593-2601 (2009)
20. Zhang, X, Feng, M, Ge, W: Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces. Nonlinear Anal. **70**(2), 584-597 (2009)
21. Agarwal, RP, O'Regan, D: Difference equations in Banach spaces. J. Aust. Math. Soc. Ser. A **64**(2), 277-284 (1998)
22. Agarwal, RP, O'Regan, D: A fixed-point approach for nonlinear discrete boundary value problems. Comput. Math. Appl. **36**(10-12), 115-121 (1998)
23. González, C, Jiménez-Melado, A: Set-contractive mappings and difference equations in Banach spaces. Comput. Math. Appl. **45**(6-9), 1235-1243 (2003)
24. Aghajani, A, Banaś, J, Jalilian, Y: Existence of solutions for a class of nonlinear Volterra singular integral equations. Comput. Math. Appl. **62**(3), 1215-1227 (2011)
25. Papageorgiou, NS: Existence of solutions for hyperbolic differential inclusions in Banach spaces. Arch. Math. **28**(3-4), 205-213 (1992)
26. Cardinali, T, Rubbioni, P: On the existence of mild solutions of semilinear evolution differential inclusions. J. Math. Anal. Appl. **308**(2), 620-635 (2005)
27. Atıcı, FM, Eloe, PW: A transform method in discrete fractional calculus. Int. J. Difference Equ. **2**(2), 165-176 (2007)
28. Holm, M: Sum and difference compositions in discrete fractional calculus. CUBO **13**(3), 153-184 (2011)
29. Lakshmikantham, V, Leela, S: Nonlinear Differential Equations in Abstract Spaces. Pergamon Press, Oxford (1981)
30. Guo, D, Lakshmikantham, V, Liu, X: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic, Dordrecht (1996)