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Existence of unbounded solutions of boundary value problems for singular differential systems on whole line

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Abstract

Motivated by (Kyoung and Yong-Hoon in Sci. China Math. 53:967-984, 2010) and (Chen and Zhang in Sci. China Math. 54:959-972, 2011), this paper is concerned with a boundary value problem of singular second-order differential systems with quasi-Laplacian operators on the whole line. By constructing a completely continuous nonlinear operator and using a fixed point theorem, sufficient conditions guaranteeing the existence of at least one unbounded solution are established. The methods used are standard, however, their exposition in the framework of such a kind of problems is new and skillful. Three concrete examples are given to illustrate the main theorem.

MSC: 34B10; 34B15; 35B10

Keywords: singular second-order differential system with quasi-Laplacian operators on the whole line; boundary value problem; unbounded solution; fixed point theorem

1 Introduction

As is well known, various problems arising in heat conduction [1, 2], chemical engineering [3], underground water flow [4], thermo-elasticity [5], and plasma physics [6] can be reduced to the nonlocal problems with integral boundary conditions.

Boundary value problems for second-order differential equations with integral boundary conditions constitute a very interesting and important class of problems. They include as special cases two-, three-, multi-point and nonlocal boundary value problems [7–13] as special cases. For such problems and comments on their importance, we refer the reader to [14–19] and [20], and the references therein.

The theory of boundary value problems on the whole line for differential equations or integral equations arises in different areas on applied mathematics and physics. Since an infinite interval is noncompact, the study of boundary value problems on the whole line is more complicated, especially for boundary value problems with integral boundary conditions on the whole line, not many work was done in the literature (see [21–30] and the references therein). Furthermore, most of the results above are in the scalar case.

Differential equations governed by nonlinear differential operators have been widely studied. In this setting the most investigated operator is the classical p-Laplacian, that is, $\Phi_p(x) = |x|^{p-2}x$ with $p > 1$, which, in recent years, has been generalized to other types of differential operators that preserve the monotonicity of the p-Laplacian but are not homogeneous. These more general operators, which are usually referred to as Φ-Laplacians (or quasi-Laplacians or quasi-Laplacian operators), are involved in some models, *e.g.* in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$\left[\Phi(x')\right]' = f(t, x, x'), \quad t \in (-\infty, +\infty),$$

where $\Phi : R \to R$ is an increasing homeomorphism such that $\Phi(0) = 0$. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, *e.g.* reaction diffusion equations with non-constant diffusivity and porous media equations. This leads to consider nonlinear differential operators of the type $[a(t, x, x')\Phi(x')]'$, where a is a positive continuous function. For a comprehensive bibliography on this subject, see *e.g.* [8, 31] and [11, 32].

The systems of second-order ordinary differential equations arise from many fields in physics and chemistry. For example in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases and in concentration in chemical or biological problems; see [33–36] and the references therein.

In [12, 13], the authors studied the existence, nonexistence, and multiplicity of positive solutions of two-point boundary value problems on finite intervals for second-order ordinary differential p-Laplacian systems with parameters. To get the solutions, the upper and lower solution method, the fundamental properties of the fixed point index, and the fixed point index theorem were used.

The asymptotic theory of ordinary differential equations is an area in which there is great activity among a large number of investigators. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole line (half line). The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole line (half line).

In [37], authors studied the solvability of the resonant second-order boundary value problems with the one-dimensional p-Laplacian at resonance on a half line

$$\begin{aligned} &\left(c(t)\phi_p\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right), \quad 0 < t < \infty, \\ &x(0) = \sum_{i=1}^{n} \mu_i x(\xi_i), \qquad \lim_{t\to+\infty} c(t)\phi_p\left(x'(t)\right) = 0 \end{aligned}$$

and

$$\begin{aligned} &\left(c(t)\phi_p\left(x'(t)\right)\right)' + g(t)h\left(t, x(t), x'(t)\right) = 0, \quad 0 < t < +\infty, \\ &x(0) = \int_0^{+\infty} g(s)x(s)\,ds, \qquad \lim_{t\to+\infty} c(t)\phi_p\left(x'(t)\right) = 0, \end{aligned}$$

with multi-point and integral boundary conditions, respectively, where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\sum_{s=1}^{m} \mu_i x(\xi_i) = 1$, $\int_0^{+\infty} g(s)\,ds = 1$. The arguments are based upon an extension of Mawhin's continuation theorem due to Ge and Ren [38]. In [39–42], authors studied the existence of solutions or positive solutions of boundary value problems of differential equations with p-Laplacian on half lines.

In recent paper [29], the author considered the existence of solutions of the following boundary value problem for a second-order singular differential equation on the whole line:

$$\begin{aligned} &\big[\Phi\big(\rho(t)a\big(t,x(t),x'(t)\big)x'(t)\big)\big]' + f\big(t,x(t),x'(t)\big) = 0, \quad t \in R, \\ &\lim_{t\to-\infty} \rho(t)a\big(t,x(t),x'(t)\big)x'(t) - \int_{-\infty}^{+\infty} \alpha(s)x(s)\,ds = \int_{-\infty}^{+\infty} g\big(s,x(s),x'(s)\big)\,ds, \\ &\lim_{t\to+\infty} \rho(t)a\big(t,x(t),x'(t)\big)x'(t) + \int_{-\infty}^{+\infty} \beta(s)x'(s)\,ds = \int_{-\infty}^{+\infty} h\big(s,x(s),x'(s)\big)\,ds, \end{aligned}$$

where $\rho \in C^0(R,[0,+\infty))$ with $\rho(t) > 0$ for all $t \neq 0$ satisfies

$$\int_{-\infty}^{0} \frac{1}{\rho(s)}\,ds = +\infty, \qquad \int_{0}^{+\infty} \frac{1}{\rho(s)}\,ds = +\infty,$$

$a : R \times R \times R \to (0,+\infty)$ is continuous, and there exist constants $m > 0$, $M > 0$ such that

$$m \le a\left(t, \big(1+\tau(t)\big)x, \frac{y}{\rho(t)}\right) \le M, \quad t \in R, x \in R, y \in R$$

and for each $r > 0$, $|x|, |y| \le r$ imply that $a(t,(1+\tau(t))x, \frac{y}{\rho(t)}) \to a_{\pm\infty}$ uniformly as $t \to \pm\infty$, where $\tau(t) = |\int_0^t \frac{ds}{\rho(s)}|$, $\alpha, \beta : R \to [0,+\infty)$ are continuous functions satisfying

$$\begin{aligned} &\int_{-\infty}^{+\infty} \alpha(s)\,ds > 0, \qquad \int_{0}^{+\infty} \alpha(s)\int_0^s \frac{dr}{\rho(r)}\,ds < +\infty, \\ &\int_{-\infty}^{0} \alpha(s)\int_s^0 \frac{dr}{\rho(r)}\,ds < +\infty, \qquad \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s)}\,ds < +\infty, \end{aligned}$$

f, g, h defined on R^3 are nonnegative Carathéodory functions, $\Phi \in C^1(R)$ is continuous and strictly increasing on R, $\Phi(0) = 0$ and its inverse function denoted by Φ^{-1} is continuous too, moreover, for Φ^{-1} there exist constants $L > 0$ and $L_n > 0$ such that $\Phi^{-1}(x_1x_2) \le L\Phi^{-1}(x_1)\Phi^{-1}(x_2)$ and

$$\Phi^{-1}(x_1 + \cdots + x_n) \le L_n\big[\Phi^{-1}(x_1) + \cdots + \Phi^{-1}(x_n)\big], \quad x_i \ge 0\ (i = 1,2,\ldots,n).$$

In [43], the authors investigated the existence and multiplicity of nonnegative solutions for the following integral boundary value problem on the whole line:

$$\begin{aligned} &\big(p(t)x'(t)\big)' + \lambda q(t)f\big(t,x(t),x'(t)\big) = 0, \quad t \in R, \\ &a_1 \lim_{t\to-\infty} x(t) - b_1 \lim_{t\to-\infty} p(t)x'(t) = \int_{-\infty}^{\infty} \psi(s)g_1\big(s,x(s),x'(s)\big)\,ds, \\ &a_2 \lim_{t\to+\infty} x(t) + b_2 \lim_{t\to+\infty} p(t)x'(t) = \int_{-\infty}^{\infty} \psi(s)g_2\big(s,x(s),x'(s)\big)\,ds, \end{aligned}$$

where $\lambda > 0$ is a parameter, $f, g_1, g_2 \in C(R \times [0,\infty) \times R, [0,\infty))$, $q, \psi \in C(R,(0,\infty))$, and $p \in C(R,(0,\infty)) \cap C^1(R)$. Here, the values of $\int_{-\infty}^{+\infty} g_i(s,x(s),x'(s))\,ds$ $(i = 1,2)$, $\int_{-\infty}^{+\infty} \frac{ds}{p(s)}$, and $\sup_{s\in R}\psi(s)$ are finite and $a_1+a_2>0$, $b_i>0$ $(i=1,2)$ satisfying $D=a_2b_1+a_1b_2+a_1a_2\int_{-\infty}^{+\infty}\frac{ds}{p(s)}>0$.

On the one hand, to the best of our knowledge, there have been no papers concerned with the existence of unbounded solutions of boundary value problems of singular second-order differential systems with quasi-Laplacian operators [32] on the whole lines.

On the other hand, in all above mentioned papers, the boundary conditions are posed at the two end points 0 and $+\infty$ (or $-\infty$ and $+\infty$) and the solutions obtained are defined on $[0,+\infty)$ (or R). An interesting question occurs: when one subjects the boundary conditions on one end point $-\infty$ and a intermediate point 0, how could one get solutions defined on R of a boundary value problem of differential equations on the whole line?

Motivated by [12, 13, 29] and the reason mentioned above, we consider the following boundary value problem for the singular second-order differential system on the whole line with quasi-Laplacian operators:

$$\begin{aligned}&\bigl[\Phi\bigl(\rho(t)a\bigl(t,x(t),x'(t)\bigr)x'(t)\bigr)\bigr]'+f\bigl(t,y(t),y'(t)\bigr)=0,\quad t\in R,\\&\bigl[\Psi\bigl(\varrho(t)b\bigl(t,y(t),y'(t)\bigr)y'(t)\bigr)\bigr]'+g\bigl(t,x(t),x'(t)\bigr)=0,\quad t\in R,\end{aligned}\tag{1.1}$$

subject to the integral boundary conditions

$$\begin{aligned}&x(\xi)=\int_{-\infty}^{+\infty}\phi\bigl(s,y(s),y'(s)\bigr)\,ds,\\&\lim_{t\to-\infty}\rho(t)x'(t)=\int_{-\infty}^{+\infty}\varphi\bigl(s,y(s),y'(s)\bigr)\,ds,\\&y(\eta)=\int_{-\infty}^{+\infty}\chi\bigl(s,x(s),x'(s)\bigr)\,ds,\\&\lim_{t\to-\infty}\varrho(t)y'(t)=\int_{-\infty}^{+\infty}\psi\bigl(s,x(s),x'(s)\bigr)\,ds,\end{aligned}\tag{1.2}$$

where

(a) $\rho,\varrho\in C^0(R,(0,\infty))$ are continuous on R and satisfy

$$\begin{aligned}&\int_{-\infty}^{0}\frac{1}{\rho(s)}\,ds=+\infty,\qquad\int_{0}^{+\infty}\frac{1}{\rho(s)}\,ds=+\infty,\\&\int_{-\infty}^{0}\frac{1}{\varrho(s)}\,ds=+\infty,\qquad\int_{0}^{+\infty}\frac{1}{\varrho(s)}\,ds=+\infty;\end{aligned}$$

(b) $a,b:R\times R\times R\to(0,+\infty)$ are continuous and satisfy

$$\begin{aligned}&\lim_{t\to\pm\infty}a\biggl(t,\bigl(1+\tau(t)\bigr)u,\frac{v}{\rho(t)}\biggr)=a_\pm>0,\\&\lim_{t\to\pm\infty}b\biggl(t,\bigl(1+\sigma(t)\bigr)u,\frac{v}{\varrho(t)}\biggr)=b_\pm>0\end{aligned}$$

uniformly for u, v in each bounded interval, there exist constants $m_i>0$, $M_i>0$ such that

$$\begin{aligned}&m_1\le a\biggl(t,\bigl(1+\tau(t)\bigr)u,\frac{v}{\rho(t)}\biggr)\le M_1,\\&m_2\le b\biggl(t,\bigl(1+\sigma(t)\bigr)u,\frac{v}{\varrho(t)}\biggr)\le M_2,\quad t\in R,u,v\in R,\end{aligned}$$

and both

$$(u,v) \to a\left(t, \big(1+\tau(t)\big)u, \frac{v}{\rho(t)}\right) \quad \text{and} \quad (u,v) \to b\left(t, \big(1+\sigma(t)\big)u, \frac{v}{\varrho(t)}\right)$$

are uniformly continuous for $t \in R$, where $\tau(t) = |\int_\xi^t \frac{1}{\rho(s)}\,ds|$ and $\sigma(t) = |\int_\eta^t \frac{1}{\varrho(s)}\,ds|$;

(c) Φ, Ψ are quasi-Laplacian operators (Definition 2.1 in Section 2), the inverse operators of Φ, Ψ are denoted by Φ^{-1} and Ψ^{-1}, respectively, the supporting functions (Definition 2.1 in Section 2) of Φ and Φ^{-1} are denoted by ω_1 and ν_1, respectively, the supporting functions of Ψ and Ψ^{-1} by ω_2 and ν_2;

(d) f defined on R^3 is a σ-Carathéodory function, g defined on R^3 is a τ-Carathéodory functions (Definitions 2.2 and 2.3 in Section 2);

(e) ϕ, φ defined on R^3 are σ-Carathéodory functions, χ, ψ defined on R^3 τ-Carathéodory functions;

(f) $\xi, \eta \in R$ are fixed constants.

The purpose of this paper is to establish sufficient conditions for the existence of at least one unbounded solution of BVP (1.1)-(1.2).

This paper may be the first one to establish existence results for such a kind of problems. Compared to previous results, our work has the following new features.

Firstly, our study is on singular nonlinear differential systems (f, g, ϕ, φ, ρ, ϱ, χ, and ψ may be singular). The nonnegative functions ρ, ϱ satisfy the assumption (a), however, the assumptions $\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} < +\infty$ and $\int_{-\infty}^{+\infty} \frac{du}{\varrho(u)} < +\infty$ are made in [1, 8, 24–27, 29–31].

Secondly, this paper generalizes the boundary value problems on finite intervals discussed in [12, 13] to ones on the whole lines, the main tools used in this paper is the well-known Schauder fixed point theorem (not the upper and lower solution method, the fundamental properties of the fixed point index, and the fixed point index theorem used in [12, 13]).

Thirdly, a completely continuous operator is constructed, and a special Banach space has been developed to overcome the difficulties due to the singularity and to the application of the fixed point theorem.

Fourthly, we generalize the boundary value problems of differential equations on finite interval discussed in [12, 13] to one in whole lines. By comparing with [29], the nonlinear differential operators $[\Phi(\rho(t)a(t,x(t),x'(t))x'(t))]'$ and $[\Psi(\varrho(t)b(t,y(t),y'(t))y'(t))]'$ are more general. In [22], the authors studied the boundary value problem

$$\ddot{x} = f(t,x,\dot{x}), \qquad x(-\infty) = x(+\infty), \qquad \dot{x}(-\infty) = \dot{x}(+\infty).$$

Under adequate hypotheses and using the Bohnenblust-Karlin fixed point theorem for multivalued mappings, the existence of solutions was established. However, the Banach space

$$X := \left\{x \in C^2(R) : (\exists)x(\pm\infty), (\exists)\dot{x}(\pm\infty)\right\}$$

was used in [22]. In our paper, the Banach space

$$X = \left\{x : x, x' \in C^0(R), \lim_{t\to\pm\infty} \frac{x(t)}{1+\tau(t)} \text{ and } \lim_{t\to\pm\infty} \rho(t)x'(t) \text{ exist}\right\}$$

is used; see Claim 2.1 and Claim 2.2 in Section 2.

Finally, we discuss the boundary value problem with integral boundary conditions, that is, system (1.1) including three-point, multi-point and nonlocal boundary value problems as special cases and the quasi-Laplacian terms $[\Phi(\rho(t)a(t,x(t))x'(t))]'$ and $[\Psi(\varrho(t)b(t,y(t))y'(t))]'$ are involved. In (1.1)-(1.2), the boundary conditions are posed at the points ξ, η, and $-\infty$ and the solution obtained are defined on R.

By an unbounded solution of BVP (1.1)-(1.2) we mean a pair of functions $(x,y) \in C^1(R)$ such that

$$\begin{aligned}&\bigl[\Phi\bigl(\rho a x'\bigr)\bigr]' : t \to \bigl[\Phi\bigl(\rho(t)a\bigl(t,x(t),x'(t)\bigr)x'(t)\bigr)\bigr]',\\&\bigl[\Psi\bigl(\varrho b y'\bigr)\bigr]' : t \to \bigl[\Psi\bigl(\varrho(t)b\bigl(t,y(t),y'(t)\bigr)y'(t)\bigr)\bigr]'\end{aligned}$$

belong to $L^1(R)$, and x, y satisfy the prescribed asymptotic behavior, *i.e.*, the following limits:

$$\lim_{t\to\pm\infty}\frac{x(t)}{1+\tau(t)},\qquad \lim_{t\to\pm\infty}\frac{y(t)}{1+\sigma(t)},\qquad \lim_{t\to\pm\infty}\rho(t)x'(t),\qquad \lim_{t\to\pm\infty}\varrho(t)y'(t)$$

exist and all equations in (1.1)-(1.2) are satisfied.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. Three examples to illustrate the main theorem are given in Section 4.

2 Preliminary results

In this section, we present some background definitions in Banach spaces and state an important fixed point theorem. The preliminary results are given too.

Definition 2.1 [32] An odd homeomorphism Φ of the real line R onto itself is called a quasi-Laplacian operator if there exists a homeomorphism ω of $[0,+\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \geq 0$ we have

$$\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2). \tag{2.1}$$

ω is called the supporting function of Φ.

Remark 2.1 Note that any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in R,$$

is a quasi-Laplacian operator, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geq 0$.

Remark 2.2 It is clear that a quasi-Laplacian operator Φ and any corresponding supporting function ω are increasing functions vanishing at zero; moreover, their inverses Φ^{-1} and ν, respectively, are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2) \tag{2.2}$$

for all $w_1, w_2 \geq 0$, and ν is called the supporting function of Φ^{-1}.

Remark 2.3 It is well known that $\Phi(s) = |s|^{p-2}s$ with $p > 1$ is called p-Laplacian. One sees that a quasi-Laplacian operator contains a p-Laplacian as a special case.

Definition 2.2 $G : R \times R \times R \to R$ is called a τ-Carathéodory function if it satisfies:

(i) $t \to G(t, (1+\tau(t))u, \frac{1}{\rho(t)}v)$ is measurable for any $u, v \in R$;

(ii) $(u, v) \to G(t, (1+\tau(t))u, \frac{1}{\rho(t)}v)$ is continuous for a.e. $t \in R$;

(iii) for each $r > 0$, there exists a nonnegative function $\phi_r \in L^1(R)$ such that $|u|, |v| \le r$ implies

$$\left|G\left(t, \left(1+\tau(t)\right)u, \frac{1}{\rho(t)}v\right)\right| \le \phi_r(t), \quad \text{a.e. } t \in R.$$

Definition 2.3 $H : R \times R \times R \to R$ is called a σ-Carathédory function if it satisfies:

(i) $t \to H(t, (1+\sigma(t))u, \frac{1}{\varrho(t)}v)$ is measurable for any $u, v \in R$;

(ii) $(u, v) \to H(t, (1+\sigma(t))u, \frac{1}{\varrho(t)}v)$ is continuous for a.e. $t \in R$;

(iii) for each $r > 0$, there exists a nonnegative function $\phi_r \in L^1(R)$ such that $|u|, |v| \le r$ implies

$$\left|H\left(t, \left(1+\sigma(t)\right)u, \frac{1}{\varrho(t)}v\right)\right| \le \phi_r(t), \quad \text{a.e. } t \in R.$$

Let $C^0(R)$ be the set of all continuous functions on R. Define

$$X = \left\{x : R \to R : x, x' \in C^0(R), \lim_{t\to\pm\infty} \frac{x(t)}{1+\tau(t)} \text{ and } \lim_{t\to\pm\infty} \rho(t)x'(t) \text{ exist}\right\}.$$

For $x \in X$, define

$$\|x\| = \|x\|_X = \max\left\{\sup_{t\in R} \frac{|x(t)|}{1+\tau(t)}, \sup_{t\in R} \rho(t)\left|x'(t)\right|\right\}, \quad x \in X.$$

Claim 2.1 *X is a Banach space with the norm $\|\cdot\|$ defined.*

Proof In fact, we see easily that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then it follows that

$$\begin{aligned} &x_u, x_u' \in C^0(R), \quad \lim_{t\to\pm\infty} \frac{x_u(t)}{1+\tau(t)}, \qquad \lim_{t\to\pm\infty} \rho(t)x_u'(t) \quad \text{exist}, u \in N, \\ &\|x_u - x_v\| = \max\left\{\sup_{t\in R} \frac{|x_u(t) - x_v(t)|}{1+\tau(t)}, \sup_{t\in R} \rho(t)\left|x_u'(t) - x_v'(t)\right|\right\} \to 0, \quad u, v \to +\infty. \end{aligned} \tag{2.3}$$

Thus there exist two functions x_0 and y_0 defined on R such that

$$\lim_{u\to+\infty} \frac{x_u(t)}{1+\tau(t)} = x_0(t), \qquad \lim_{u\to+\infty} \rho(t)x_u'(t) = y_0(t), \quad t \in R. \tag{2.4}$$

Denote $z_0(t) = (1+\tau(t))x_0(t)$ and $w_0(t) = \frac{y_0(t)}{\rho(t)}$ for $t \in R$. This means that the functions $z_0 : R \to R$ are well defined.

Step 1. Prove that $z_0, w_0 \in C^0(R)$.

For any $t_0 \in R$, we find that

$$\begin{aligned}
\left|z_0(t) - z_0(t_0)\right| &= \left|\left(1 + \tau(t)\right)x_0(t) - \left(1 + \tau(t_0)\right)x_0(t_0)\right| \\
&\leq \left|\left(1 + \tau(t)\right)x_0(t) - x_u(t)\right| + \left|x_u(t) - x_u(t_0)\right| \\
&\quad + \left|x_u(t_0) - \left(1 + \tau(t_0)\right)x_0(t_0)\right| \\
&= \left(1 + \tau(t)\right)\left|\frac{x_u(t)}{1 + \tau(t)} - x_0(t)\right| + \left|x_u(t) - x_u(t_0)\right| \\
&\quad + \left(1 + \tau(t_0)\right)\left|\frac{x_u(t_0)}{1 + \tau(t_0)} - x_0(t_0)\right|.
\end{aligned}$$

From (2.3) and (2.4) we see that $\lim_{t\to t_0} z_0(t) = z_0(t_0)$. Then z_0 is continuous at $t = t_0$. So $z_0 \in C^0(R)$. Similarly we can prove that $w_0 \in C^0(R)$.

Step 2. Prove that the limits $\lim_{t\to\pm\infty} \frac{z_0(t)}{1+\tau(t)}$ and $\lim_{t\to\pm\infty} \rho(t)w_0(t)$ exist.

From (2.3), we get

$$\begin{aligned}
&\left|\lim_{t\to\pm\infty} \frac{x_u(t)}{1 + \tau(t)} - \lim_{t\to\pm\infty} \frac{x_v(t)}{1 + \tau(t)}\right| \to 0, \quad u, v \to +\infty, \\
&\left|\lim_{t\to\pm\infty} \rho(t)x_u'(t) - \lim_{t\to\pm\infty} \rho(t)x_v'(t)\right| \to 0, \quad u, v \to +\infty.
\end{aligned}$$

It follows that $\{\lim_{t\to\pm\infty} \frac{x_u(t)}{1+\tau(t)}\}$ and $\{\lim_{t\to\pm\infty} \rho(t)x_u'(t)\}$ are Cauchy sequences. So both $\lim_{u\to+\infty} \lim_{t\to\pm\infty} \frac{x_u(t)}{1+\tau(t)}$ and $\lim_{u\to+\infty} \lim_{t\to\pm\infty} \rho(t)x_u'(t)$ exist. Then (2.3) implies that

$$\lim_{t\to\pm\infty} \frac{z_0(t)}{1 + \tau(t)} = \lim_{t\to\pm\infty} x_0(t) = \lim_{t\to\pm\infty} \lim_{u\to+\infty} \frac{x_u(t)}{1 + \tau(t)} = \lim_{u\to+\infty} \lim_{t\to\pm\infty} \frac{x_u(t)}{1 + \tau(t)}$$

exists. Similarly we can prove that $\lim_{t\to\pm\infty} \rho(t)w_0(t)$ exists.

Step 3. Prove that $w_0(t) = z_0'(t)$ for all $t \in R$.

For $t \in R$, there exists a constant $c_u \in R$ such that

$$\begin{aligned}
\left|x_u(t) - c_u - \int_0^t w_0(s)\,ds\right| &= \left|\int_0^t \left[x_u'(s) - w_0(s)\right]ds\right| \\
&\leq \left|\int_0^t \frac{1}{\rho(s)}\left|\rho(s)x_u'(s) - \rho(s)w_0(s)\right|ds\right| \\
&\leq \left|\int_0^t \frac{ds}{\rho(s)}\right| \sup_{t\in R}\left|\rho(t)x_u'(t) - y_0(t)\right| \to 0 \quad \text{as } u \to +\infty.
\end{aligned}$$

So $\lim_{u\to+\infty} x_u(t) - c_u = \int_0^t w_0(s)\,ds$. Then there exists $c_0 \in R$ such that $(1 + \tau(t))x_0(t) - c_0 = \int_0^t w_0(s)\,ds$. So $w_0'(t) = z_0(t)$.

Step 4. Prove that $x_u \to x_0$ as $u \to +\infty$ in X.

We have by (2.3) and (2.4)

$$\|x_u - x_0\| = \max\left\{\sup_{t\in R} \frac{|x_u(t) - x_0(t)|}{1 + \tau(t)}, \sup_{t\in R} \rho(t)\left|x_u'(t) - x_0'(t)\right|\right\} \to 0 \quad \text{as } u \to +\infty.$$

From the above discussion, we see that $x_0 \in X$ with $x_u \to x_0$ as $u \to +\infty$. It follows that X is a Banach space. □

Claim 2.2 *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied*:

(i) *both* $\{t \to \frac{x(t)}{1+\tau(t)} : x \in M\}$ *and* $\{t \to \rho(t)x'(t) : x \in M\}$ *are uniformly bounded*;

(ii) *both* $\{t \to \frac{x(t)}{1+\tau(t)} : x \in M\}$ *and* $\{t \to \rho(t)x'(t) : x \in M\}$ *are equicontinuous in any subinterval* $[a,b] \subset R$;

(iii) *both* $\{t \to \frac{x(t)}{1+\tau(t)} : x \in M\}$ *and* $\{t \to \rho(t)x'(t) : x \in M\}$ *are equiconvergent as* $s \to \pm\infty$.

Proof $\Leftarrow$ From Claim 2.1, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is, for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by (i)-(iii), there exist constants $A > 0$, $\delta > 0$, an integer $N > 0$, we have

$$\begin{aligned}
&\left|\frac{x(t_1)}{1+\tau(t_1)} - \frac{x(t_2)}{1+\tau(t_2)}\right|, \left|\rho(t_1)x'(t_1) - \rho(t_2)x'(t_2)\right| \le \frac{\epsilon}{3}, \\
&\quad t_1, t_2 \le -N \text{ or } t_1, t_2 \ge N, x \in M, \\
&\|x\| = \max\left\{\sup_{t\in R} \frac{|x(t)|}{1+\tau(t)}, \sup_{t\in R} \rho(t)\left|x'(t)\right|\right\} \le A, \quad x \in M, \\
&\left|\frac{x(t_1)}{1+\tau(t_1)} - \frac{x(t_2)}{1+\tau(t_2)}\right|, \left|\rho(t_1)x'(t_1) - \rho(t_2)x'(t_2)\right| \le \frac{\epsilon}{3}, \\
&\quad t_1, t_2 \in [-N,N], |t_1 - t_2| < \delta, x \in M.
\end{aligned}$$

Define $X|_{[-N,N]} = \{x|_{[-N,N]} : x \in X\}$. For $x \in X|_{[-N,N]}$, define

$$\|x\|_N = \max\left\{\sup_{t\in[-N,N]} \frac{|x(t)|}{1+\tau(t)}, \sup_{t\in[-N,N]} \rho(t)\left|x'(t)\right|\right\}.$$

Similarly to Claim 2.1, we can prove that $X_{[-N,N]}$ is a Banach space with the norm $\|\cdot\|_N$.

Let $M|_{[-N,N]} = \{t \to x(t), t \in [-N,N] : x \in M\}$. Then $M|_{[-N,N]}$ is a subset of $X|_{[-N,N]}$. By Ascoli-Arzela theorem, we can know that $M|_{[-N,N]}$ is relatively compact in $X|_{[-N,N]}$. Thus, there exist $x_1, x_2, \ldots, x_k \in M$ such that, for any $x \in M$, we find that there exists some $i = 1, 2, \ldots, k$ such that

$$\|x - x_i\|_N = \max\left\{\sup_{t\in[-N,N]} \frac{|x(t) - x_i(t)|}{1+\tau(t)}, \sup_{t\in[-N,N]} \rho(t)\left|x'(t) - x_i'(t)\right|\right\} \le \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we find that

$$\begin{aligned}
\|x - x_i\|_X &= \max\left\{\sup_{t\in R} \frac{|x(t) - x_i(t)|}{1+\tau(t)}, \sup_{t\in R} \rho(t)\left|x'(t) - x_i'(t)\right|\right\} \\
&\le \max\left\{\sup_{t\le -N} \frac{|x(t) - x_i(t)|}{1+\tau(t)}, \sup_{t\le -N} \rho(t)\left|x'(t) - x_i'(t)\right|,\right. \\
&\quad \sup_{|t|\le N} \frac{|x(t) - x_i(t)|}{1+\tau(t)}, \sup_{|t|\le N} \rho(t)\left|x'(t) - x_i'(t)\right|, \\
&\quad \left.\sup_{t\ge N} \frac{|x(t) - x_i(t)|}{1+\tau(t)}, \sup_{t\ge N} \rho(t)\left|x'(t) - x_i'(t)\right|\right\}
\end{aligned}$$

$$\le \max\left\{\sup_{t\le -N}\frac{|x(t)-x_i(t)|}{1+\tau(t)}, \sup_{t\le -N}\rho(t)\left|x'(t)-x_i'(t)\right|, \frac{\epsilon}{3}, \sup_{t\ge N}\frac{|x(t)-x_i(t)|}{1+\tau(t)}, \sup_{t\ge N}\rho(t)\left|x'(t)-x_i'(t)\right|\right\}.$$

We find that

$$\begin{aligned}\sup_{t\le -N}\frac{|x(t)-x_i(t)|}{1+\tau(t)} &\le \sup_{t\le -N}\left|\frac{x(t)}{1+\tau(t)}-\frac{x(-N)}{1+\tau(-N)}\right| \\ &\quad+\left|\frac{x(-N)}{1+\tau(-N)}-\frac{x_i(-N)}{1+\tau(-N)}\right| \\ &\quad+\sup_{t\le -N}\left|\frac{x_i(-N)}{1+\tau(-N)}-\frac{x_i(t)}{1+\tau(t)}\right| \\ &\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.\end{aligned}$$

Similarly we can prove that $\sup_{t\le -N}\rho(t)|x'(t)-x_i'(t)|\le\epsilon$, $\sup_{t\ge N}\frac{|x(t)-x_i(t)|}{1+\tau(t)}\le\epsilon$, $\sup_{t\ge N}\rho(t)|x'(t)-x_i'(t)|\le\epsilon$.

So, for any $\epsilon>0$, M has a finite ϵ-net $\{U_{x_1},U_{x_2},\dots,U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

$\Rightarrow$ Assume that M is relatively compact, then for any $\epsilon>0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1},U_{x_2},\dots,U_{x_k}\}$ with $x_i\subset M$. Then for any $x\in M$, there exists U_{x_i} such that $x\in U_{x_i}$ and

$$\|x\|\le\|x-x_i\|+\|x_i\|\le\epsilon+\max\left\{\|x_i\|:i=1,2,\dots,k\right\}.$$

It follows that $\{\|x\|:x\in M\}$ is uniformly bounded. Then (i) holds.

Let $[-N,N]$ be any subinterval in R. Then there exists $\delta>0$ such that

$$\left|\frac{x_i(t_1)}{1+\tau(t_1)}-\frac{x_i(t_2)}{1+\tau(t_2)}\right|<\epsilon$$

for all $t_1,t_2\in[-N,N]$ with $|t_1-t_2|<\delta$ and $i=1,2,\dots,k$. For $x\in M$, there exists an i such that $x\in U_{x_i}$. Then we have for $t_1,t_2\in[-N,N]$ with $|t_1-t_2|<\delta$ that

$$\begin{aligned}\left|\frac{x(t_1)}{1+\tau(t_1)}-\frac{x(t_2)}{1+\tau(t_2)}\right| &\le \left|\frac{x(t_1)}{1+\tau(t_1)}-\frac{x_i(t_1)}{1+\tau(t_1)}\right|+\left|\frac{x_i(t_1)}{1+\tau(t_1)}-\frac{x_i(t_2)}{1+\tau(t_2)}\right| \\ &\quad+\left|\frac{x_i(t_2)}{1+\tau(t_2)}-\frac{x(t_2)}{1+\tau(t_2)}\right| \\ &\le 3\epsilon.\end{aligned}$$

$\{t\to\frac{x(t)}{1+\tau(t)}:x\in M\}$ is equicontinuous in $[-N,N]$. Similarly we can prove that $\{t\to\rho(t)x'(t):x\in M\}$ is equicontinuous in $[-N,N]$. It follows that (ii) holds.

Now we prove that (iii) holds. It is easily seen that there exists $N>0$ such that

$$\left|\frac{x_i(t_1)}{1+\tau(t_1)}-\frac{x_i(t_2)}{1+\tau(t_2)}\right|<\epsilon$$

for all $t_1, t_2 \le -N$, $i = 1, 2, \dots, k$. For $x \in M$, there exists i such that $x \in U_{x_i}$. So

$$\begin{aligned}\left|\frac{x(t_1)}{1+\tau(t_1)} - \frac{x(t_2)}{1+\tau(t_2)}\right| &\le \left|\frac{x(t_1)}{1+\tau(t_1)} - \frac{x_i(t_1)}{1+\tau(t_1)}\right| + \left|\frac{x_i(t_1)}{1+\tau(t_1)} - \frac{x_i(t_2)}{1+\tau(t_2)}\right| \\ &\quad + \left|\frac{x_i(t_2)}{1+\tau(t_2)} - \frac{x(t_2)}{1+\tau(t_2)}\right| \\ &\le 3\epsilon, \quad t_1, t_2 \le -N.\end{aligned}$$

Then $\lim_{t\to-\infty} \frac{x(t)}{1+\tau(t)}$ exists. Similarly we can prove that $\lim_{t\to+\infty} \frac{x(t)}{1+\tau(t)}$, $\lim_{t\to-\infty} \rho(t)x'(t)$, and $\lim_{t\to+\infty} \rho(t)x'(t)$ exist. Hence (iii) holds. Consequently, the claim is proved. □

Define

$$Y = \left\{y : R \to R : y, y' \in C^0(R), \lim_{t\to\pm\infty} \frac{y(t)}{1+\sigma(t)} \text{ and } \lim_{t\to\pm\infty} \varrho(t)y'(t) \text{ exist}\right\}.$$

For $y \in Y$, define the norm of y by

$$\|y\| = \|y\|_Y = \max\left\{\sup_{t\in R} \frac{|y(t)|}{1+\sigma(t)}, \sup_{t\in R} \varrho(t)\left|y'(t)\right|\right\}.$$

One can prove that Y is a Banach space with the norm $\|y\|$ for $y \in Y$.

Define $E = X \times Y$ with the norm

$$\left\|(x, y)\right\| = \max\left\{\|x\|, \|y\|\right\} \quad \text{for } (x, y) \in E.$$

One can prove that E is a Banach space.

Define the linear operator T by $(T(x, y))(t) = ((T_1(x, y))(t), (T_2(x, y))(t))$ with

$$\begin{aligned}\left(T_1(x, y)\right)(t) &= \int_{-\infty}^{+\infty} \phi\left(s, y(s), y'(s)\right) ds \\ &\quad + \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w, y(w), y'(w))\, dw) - \int_{-\infty}^{s} f(w, y(w), y'(w))\, dw)}{\rho(s)a(s, x(s), x'(s))} ds, \\ &\quad (x, y) \in E, \\ \left(T_2(x, y)\right)(t) &= \int_{-\infty}^{+\infty} \chi\left(s, x(s), x'(s)\right) ds \\ &\quad + \int_{\eta}^{t} \frac{\Psi^{-1}(\Psi(b_- \int_{-\infty}^{+\infty} \psi(w, x(w), x'(w))\, dw) - \int_{-\infty}^{s} g(w, x(w), x'(w))\, dw)}{\varrho(s)b(s, y(s), y'(s))} ds, \\ &\quad (x, y) \in E.\end{aligned}$$

Lemma 2.1 *Suppose that* (a)-(f) *hold. Then* $T : E \to E$ *is well defined,* $(x, y) \in E$ *is a solution of BVP* (1.1)-(1.2) *if and only if* $(x, y) \in E$ *is a fixed point of* T *and is completely continuous.*

Proof We divide the proof into three steps:

Step 1. Prove that $T : E \to E$ is well defined.

For $(x,y) \in E$, we know that there is constant $r > 0$ such that $\|x\| \le r$. By (d) and (e), we know that there is $\phi_r \in L^1(R)$ such that

$$\begin{aligned}
\bigl|f\bigl(t,y(t),y'(t)\bigr)\bigr| &= \left|f\left(t,\bigl(1+\sigma(t)\bigr)\frac{y(t)}{1+\sigma(t)},\frac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t),\\
\bigl|g\bigl(t,x(t),x'(t)\bigr)\bigr| &= \left|g\left(t,\bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right)\right| \le \phi_r(t),\\
\bigl|\phi\bigl(t,y(t),y'(t)\bigr)\bigr| &= \left|\phi\left(t,\bigl(1+\sigma(t)\bigr)\frac{y(t)}{1+\sigma(t)},\frac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t),\\
\bigl|\varphi\bigl(t,y(t),y'(t)\bigr)\bigr| &= \left|\varphi\left(t,\bigl(1+\sigma(t)\bigr)\frac{y(t)}{1+\sigma(t)},\frac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t),\\
\bigl|\chi\bigl(t,x(t),x'(t)\bigr)\bigr| &= \left|\chi\left(t,\bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right)\right| \le \phi_r(t),\\
\bigl|\psi\bigl(t,x(t),x'(t)\bigr)\bigr| &= \left|\psi\left(t,\bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right)\right| \le \phi_r(t).
\end{aligned} \tag{2.5}$$

By (b), we get

$$\begin{aligned}
a\bigl(t,x(t),x'(t)\bigr) &= \left|a\left(t,\bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right)\right| \in [m_1,M_1],\\
b\bigl(t,y(t),y'(t)\bigr) &= \left|b\left(t,\bigl(1+\sigma(t)\bigr)\frac{y(t)}{1+\sigma(t)},\frac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \in [m_2,M_2],\\
\lim_{t\to\pm\infty} a\bigl(t,x(t),x'(t)\bigr) &= \lim_{t\to\pm\infty} a\left(t,\bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right) = a_\pm,\\
\lim_{t\to\pm\infty} b\bigl(t,y(t),y'(t)\bigr) &= \lim_{t\to\pm\infty} b\left(t,\bigl(1+\sigma(t)\bigr)\frac{y(t)}{1+\sigma(t)},\frac{1}{\varrho(t)}\varrho(t)y'(t)\right) = b_\pm.
\end{aligned} \tag{2.6}$$

Hence the following integrals are convergent:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \phi\bigl(s,y(s),y'(s)\bigr)\,ds, \int_{-\infty}^{+\infty} \chi\bigl(s,x(s),x'(s)\bigr)\,ds,\\
&\int_{-\infty}^{+\infty} \varphi\bigl(w,y(w),y'(w)\bigr)\,dw, \int_{-\infty}^{+\infty} \psi\bigl(w,y(w),y'(w)\bigr)\,dw,\\
&\int_{-\infty}^{s} f\bigl(w,y(w),y'(w)\bigr)\,dw, \int_{-\infty}^{s} g\bigl(w,x(w),x'(w)\bigr)\,dw.
\end{aligned}$$

So $(T_1(x,y)) \in C^0(R)$ and we get by using L'Hôpital's rule

$$\begin{aligned}
&\lim_{t\to\pm\infty} \frac{(T_1(x,y))(t)}{1+\tau(t)}\\
&\quad = \lim_{t\to\pm\infty} \frac{1}{1+\tau(t)} \int_{-\infty}^{+\infty} \phi\bigl(s,y(s),y'(s)\bigr)\,ds + \lim_{t\to\pm\infty} \frac{1}{1+\tau(t)}\\
&\qquad \times \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{s} f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds\\
&\quad = \lim_{t\to\pm\infty} \frac{1}{1+\tau(t)}
\end{aligned}$$

$$\times \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{s} f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds$$
$$= \pm \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{t} f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}$$
$$= \begin{cases} -\int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw, \\ \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\,dw)}{a_+}. \end{cases}$$

One sees that

$$\rho(t)\big(T(x,y)\big)'(t) = \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{t} f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}.$$

Similarly we can prove that $t \to \rho(t)(T_1(x,y))'(t)$ is continuous on R and

$$\lim_{t\to\pm} \rho(t)\big(T_1(x,y)\big)'(t) = \begin{cases} \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw, \\ \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\,dw)}{a_+}. \end{cases}$$

Hence $T_1(x,y) \in X$. Similarly we can show that $T_2(x,y) \in Y$. Then $T : E \to E$ is well defined.

Step 2. It is easy to show that $(x,y) \in E$ is a solution of BVP (1.1)-(1.2) if and only if $(x,y) \in E$ is a fixed point of T. We omit the details.

Step 3. Prove that T is completely continuous.

Firstly, we prove that T is continuous, *i.e.*, both T_1 and T_2 are continuous. Let $(x_n,y_n) \in E$ ($n = 0,1,2,\ldots$) and $(x_n,y_n) \to (x_0,y_0)$ as $n \to \infty$. We have

$$\begin{aligned} &\lim_{n\to\infty} \sup_{t\in R} \frac{1}{1+\tau(t)} \big|x_n(t) - x_0(t)\big| = 0, \\ &\lim_{n\to\infty} \sup_{t\in R} \rho(t)\big|x_n'(t) - x_0'(t)\big| = 0, \\ &\lim_{n\to\infty} \sup_{t\in R} \frac{1}{1+\sigma(t)} \big|y_n(t) - y_0(t)\big| = 0, \\ &\lim_{n\to\infty} \sup_{t\in R} \varrho(t)\big|y_n'(t) - y_0'(t)\big| = 0. \end{aligned} \tag{2.7}$$

We will prove that

$$\big(T_1(x_n,y_n), T_2(x_n,y_n)\big) \to \big(T_1(x_0,y_0), T_2(x_0,y_0)\big), \quad n \to \infty. \tag{2.8}$$

It is easy to see that there exists $r > 0$ such that

$$\frac{|x_n(t)|}{1+\tau(t)}, \sup_{t\in R} \rho(t)\big|x_n'(t)\big|, \sup_{t\in R} \frac{|y_n(t)|}{1+\sigma(t)}, \sup_{t\in R} \varrho(t)\big|y_n'(t)\big| \le r, \quad t \in R.$$

By (b), (d), and (e), we know that there is $\phi_r \in L^1(R)$ such that (2.3) and (2.4) hold with $x = x_n$.

Denote

$$\overline{M} = \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(w)\,dw\right) + \int_{-\infty}^{+\infty} \phi_r(w)\,dw\right).$$

Now we get by using (2.3) and (2.4)

$$\begin{aligned}
&\rho(t)\big|\big(T_1(x_n,y_n)\big)'(t)-\big(T_1(x_0,y_0)\big)'(t)\big| \\
&\quad=\bigg|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_n(w),y_n'(w))\,dw)-\int_{-\infty}^{t}f(w,y_n(w),y_n'(w))\,dw)}{a(t,x_n(t),x_n'(t))} \\
&\qquad-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_0(w),y_0'(w))\,dw)-\int_{-\infty}^{t}f(w,y_0(w),y_0'(w))\,dw)}{a(t,x_0(t),x_0'(t))}\bigg| \\
&\quad\le\frac{1}{m_1}\bigg|\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_n(w),y_n'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_n(w),y_n'(w)\big)\,dw\bigg) \\
&\qquad-\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)\bigg| \\
&\qquad+\bigg|\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)\bigg| \\
&\qquad\times\bigg|\frac{1}{a(t,x_n(t),x_n'(t))}-\frac{1}{a(t,x_0(t),x_0'(t))}\bigg| \\
&\quad\le\frac{1}{m_1}\bigg|\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_n(w),y_n'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_n(w),y_n'(w)\big)\,dw\bigg) \\
&\qquad-\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)\bigg| \\
&\qquad+\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\phi_r(w)\,dw\bigg)+\int_{-\infty}^{+\infty}\phi_r(w)\,dw\bigg) \\
&\qquad\times\frac{1}{m_1^2}\big|a\big(t,x_n(t),x_n'(t)\big)-a\big(t,x_0(t),x_0'(t)\big)\big| \\
&\quad\le\frac{1}{m_1}\bigg|\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_n(w),y_n'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_n(w),y_n'(w)\big)\,dw\bigg) \\
&\qquad-\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)\bigg| \\
&\qquad+\frac{\overline{M}}{m_1^2}\big|a\big(t,x_n(t),x_n'(t)\big)-a\big(t,x_0(t),x_0'(t)\big)\big|.
\end{aligned}$$

For each $\epsilon>0$, we will prove that there exists $N>0$ such that

$$\sup_{t\in R}\rho(t)\big|\big(T_1(x_n,y_n)\big)'(t)-\big(T_1(x_0,y_0)\big)'(t)\big|<\epsilon,\quad n>N. \tag{2.9}$$

From (b), there exists $\delta>0$ such that

$$\bigg|a\bigg(t,\frac{1}{1+\tau(t)}u_1,\frac{1}{\rho(t)}v_1\bigg)-a\bigg(t,\frac{1}{1+\tau(t)}u_2,\frac{1}{\rho(t)}v_2\bigg)\bigg|<\frac{m_1^2\epsilon}{2\overline{M}}$$

holds for all u_1, u_2, v_1, v_2 satisfying $|u_1-u_2|<\delta$ and $|v_1-v_2|<\delta$. From (2.6), there exists $N_1>0$ such that

$$\frac{1}{1+\tau(t)}\big|x_n(t)-x_0(t)\big|<\delta,\qquad\rho(t)\big|x_n'(t)-x_0'(t)\big|<\delta,\quad n>N_1,t\in R.$$

Hence for $n > N_1$, we have

$$\begin{aligned}
&\left|a\bigl(t,x_n(t),x_n'(t)\bigr) - a\bigl(t,x_0(t),x_0'(t)\bigr)\right| \\
&\quad = \left|a\left(t,\frac{(1+\tau(t))x_n(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x_n'(t)\right)\right. \\
&\qquad \left. - a\left(t,\frac{(1+\tau(t))x_0(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x_0'(t)\right)\right| \\
&\quad < \frac{m_1^2\epsilon}{2\overline{M}}, \quad t \in R.
\end{aligned} \tag{2.10}$$

Since Φ^{-1} is uniformly continuous on $[-\overline{M},\overline{M}]$, there exists $\delta_1 > 0$ such that

$$\left|\Phi^{-1}(w_1) - \Phi^{-1}(w_2)\right| < \frac{m_1\epsilon}{2}, \quad |w_1 - w_2| < \delta_1, w_1, w_2 \in [-\overline{M},\overline{M}]. \tag{2.11}$$

We can prove that there exists $N_2 > 0$ such that

$$\left|\int_{-\infty}^{+\infty} \left|f\bigl(w,y_n(w),y_n'(w)\bigr) - f\bigl(w,y_0(w),y_0'(w)\bigr)\right| dw\right| < \frac{\delta_1}{2}, \quad n > N_2. \tag{2.12}$$

In fact, by

$$\int_{-\infty}^{+\infty} \left|f\bigl(w,y_n(w),y_n'(w)\bigr) - f\bigl(w,y_0(w),y_0'(w)\bigr)\right| dw \le 2\int_{-\infty}^{+\infty} \phi_r(w)\,dw < +\infty,$$

there exists $\overline{M}_1 > 0$ such that

$$\begin{aligned}
&\int_{-\infty}^{-\overline{M}_1} \left|f\bigl(w,y_n(w),y_n'(w)\bigr) - f\bigl(w,y_0(w),y_0'(w)\bigr)\right| dw < \frac{\delta_1}{6}, \\
&\int_{\overline{M}_1}^{+\infty} \left|f\bigl(w,y_n(w),y_n'(w)\bigr) - f\bigl(w,y_0(w),y_0'(w)\bigr)\right| dw < \frac{\delta_1}{6}.
\end{aligned}$$

By the Lebesgue dominant convergence theorem, we have

$$\lim_{n\to\infty} \int_{-\overline{M}_1}^{\overline{M}_1} f\bigl(w,y_n(w),y_n'(w)\bigr)\,dw = \int_{-\overline{M}_1}^{\overline{M}_1} f\bigl(w,y_0(w),y_0'(w)\bigr)\,dw,$$

then there exists $N_2 > 0$ such that for $n > N_2$ we have

$$\left|\int_{-\overline{M}_1}^{\overline{M}_1} f\bigl(w,y_n(w),y_n'(w)\bigr)\,dw - \int_{-\overline{M}_1}^{\overline{M}_1} f\bigl(w,y_0(w),y_0'(w)\bigr)\,dw\right| < \frac{\delta_1}{6}, \quad n > N_3.$$

Hence

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} f\bigl(w,y_n(w),y_n'(w)\bigr)\,dw - \int_{-\infty}^{+\infty} f\bigl(w,y_0(w),y_0'(w)\bigr)\,dw\right| \\
&\quad \le \int_{-\overline{M}_1}^{\overline{M}_1} \left|f\bigl(w,y_n(w),y_n'(w)\bigr) - f\bigl(w,y_0(w),y_0'(w)\bigr)\right| dw
\end{aligned}$$

$$+\int_{-\infty}^{-\overline{M}_1}\left|f\left(w,y_n(w),y_n'(w)\right)-f\left(w,y_0(w),y_0'(w)\right)\right|dw$$
$$+\int_{\overline{M}_1}^{+\infty}\left|f\left(w,y_n(w),y_n'(w)\right)-f\left(w,y_0(w),y_0'(w)\right)\right|dw$$
$$<3\frac{\delta_1}{6}=\frac{\delta_1}{2},\quad n>N_2.$$

Since Φ is uniformly continuous on $[-\overline{M},\overline{M}]$, then there exists $\delta_2>0$ such that

$$\left|\Phi(w_1)-\Phi(w_2)\right|<\frac{\delta_1}{2},\quad |w_1-w_2|<\delta_2, w_1,w_2\in[-\overline{M},\overline{M}]. \tag{2.13}$$

Similarly we find that there exists $N_3>0$ such that

$$\left|\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right|<\frac{\delta_2}{a_-},\quad n>N_3.$$

It follows that

$$\left|a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw-a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right|<\delta_2,\quad n>N_3.$$

One sees that

$$\left|a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw\right|\le\overline{M},\quad n=0,1,2,\ldots.$$

Then for $n>N_3$, (2.11) implies that

$$\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw\right)-\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right)\right|<\frac{\delta_1}{2}.$$

Together with (2.10), we get

$$\begin{aligned}
&\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw\right)-\int_{-\infty}^{t}f\left(w,y_n(w),y_n'(w)\right)dw\right.\\
&\qquad\left.-\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right)-\int_{-\infty}^{t}f\left(w,y_0(w),y_0'(w)\right)dw\right)\right|\\
&\quad\le\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw\right)-\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right)\right|\\
&\qquad+\int_{-\infty}^{t}\left|f\left(w,y_n(w),y_n'(w)\right)-f\left(w,y_0(w),y_0'(w)\right)\right|dw\\
&\quad\le\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_n(w),y_n'(w)\right)dw\right)-\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\left(w,y_0(w),y_0'(w)\right)dw\right)\right|\\
&\qquad+\int_{-\infty}^{+\infty}\left|f\left(w,y_n(w),y_n'(w)\right)-f\left(w,y_0(w),y_0'(w)\right)\right|dw\\
&\quad<\frac{\delta_1}{2}+\frac{\delta_1}{2}=\delta_1,\quad n>\max\{N_2,N_3\}.
\end{aligned}$$

Together with (2.9), we have

$$\begin{aligned}
&\Bigg|\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_n(w),y_n'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_n(w),y_n'(w)\big)\,dw\bigg)\\
&\qquad-\Phi^{-1}\bigg(\Phi\bigg(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)-\int_{-\infty}^{t}f\big(w,y_0(w),y_0'(w)\big)\,dw\bigg)\Bigg|\\
&\quad<\frac{m_1\epsilon}{2},\quad n>\max\{N_2,N_3\}.
\end{aligned}\tag{2.14}$$

Thus (2.8) and (2.12) imply that

$$\begin{aligned}
&\rho(t)\big|\big(T_1(x_n,y_n)\big)'(t)-\big(T_1(x_0,y_0)\big)'(t)\big|\\
&\quad<\frac{1}{m_1}\frac{m_1\epsilon}{2}+\frac{\overline{M}}{m_1^2}\frac{m_1^2\epsilon}{2\overline{M}}=\epsilon,\quad n>\max\{N_1,N_2,N_3\},t\in R.
\end{aligned}$$

Then (2.7) holds.

Similarly we can prove that there exists $N_4>0$ such that

$$\bigg|\int_{-\infty}^{+\infty}\phi\big(s,y_n(s),y_n'(s)\big)\,ds-\int_{-\infty}^{+\infty}\phi\big(s,y_0(s),y_0'(s)\big)\,ds\bigg|<\frac{\epsilon}{2},\quad n>N_1.\tag{2.15}$$

So for $n>N_4$ we get

$$\begin{aligned}
&\frac{1}{1+\tau(t)}\big|\big(T_1(x_n,y_n)\big)(t)-\big(T_1(x_0,y_0)\big)(t)\big|\\
&\quad\le\frac{1}{1+\tau(t)}\bigg|\int_{-\infty}^{+\infty}\phi\big(s,y_n(s),y_n'(s)\big)\,ds-\int_{-\infty}^{+\infty}\phi\big(s,y_0(s),y_0'(s)\big)\,ds\bigg|+\frac{1}{1+\tau(t)}\\
&\qquad\times\bigg|\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_n(w),y_n'(w))\,dw)-\int_{-\infty}^{s}f(w,y_n(w),y_n'(w))\,dw)}{\rho(s)a(s,x_n(s),x_n'(s))}\,ds\\
&\qquad-\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_0(w),y_0'(w))\,dw)-\int_{-\infty}^{s}f(w,y_0(w),y_0'(w))\,dw)}{\rho(s)a(s,x_0(s),x_0'(s))}\,ds\bigg|\\
&\quad\le\frac{\epsilon}{2}+\frac{1}{1+\tau(t)}\\
&\qquad\times\bigg|\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_n(w),y_n'(w))\,dw)-\int_{-\infty}^{s}f(w,y_n(w),y_n'(w))\,dw)}{\rho(s)a(s,x_n(s),x_n'(s))}\,ds\\
&\qquad-\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_0(w),y_0'(w))\,dw)-\int_{-\infty}^{s}f(w,y_0(w),y_0'(w))\,dw)}{\rho(s)a(s,x_0(s),x_0'(s))}\,ds\bigg|.
\end{aligned}$$

We know from (2.8) and (2.12) that

$$\begin{aligned}
&\frac{1}{1+\tau(t)}\bigg|\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_n(w),y_n'(w))\,dw)-\int_{-\infty}^{s}f(w,y_n(w),y_n'(w))\,dw)}{\rho(s)a(s,x_n(s),x_n'(s))}\,ds\\
&\qquad-\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_0(w),y_0'(w))\,dw)-\int_{-\infty}^{s}f(w,y_0(w),y_0'(w))\,dw)}{\rho(s)a(s,x_0(s),x_0'(s))}\,ds\bigg|\\
&\quad\le\frac{1}{1+\tau(t)}\bigg|\int_{\xi}^{t}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y_n(w),y_n'(w))\,dw)-\int_{-\infty}^{s}f(w,y_n(w),y_n'(w))\,dw)}{\rho(s)a(s,x_n(s),x_n'(s))}\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w, y_0(w), y_0'(w))\,dw) - \int_{-\infty}^{s} f(w, y_0(w), y_0'(w))\,dw)}{\rho(s) a(s, x_n(s), x_n'(s))}\,ds \biggr| \\
&+ \frac{1}{1+\tau(t)} \\
&\times \biggl| \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w, y_0(w), y_0'(w))\,dw) - \int_{-\infty}^{s} f(w, y_0(w), y_0'(w))\,dw)}{\rho(s) a(s, x_n(s), x_n'(s))}\,ds \\
&- \int_{\xi}^{t} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w, y_0(w), y_0'(w))\,dw) - \int_{-\infty}^{s} f(w, y_0(w), y_0'(w))\,dw)}{\rho(s) a(s, x_0(s), x_0'(s))}\,ds \biggr| \\
\le{}& \frac{1}{m_1} \frac{1}{1+\tau(t)} \biggl| \int_{\xi}^{t} \frac{1}{\rho(s)} \biggl| \Phi^{-1}\biggl(\Phi\biggl(a_- \int_{-\infty}^{+\infty} \varphi\bigl(w, y_n(w), y_n'(w)\bigr)\,dw \biggr) \\
&- \int_{-\infty}^{s} f\bigl(w, y_n(w), y_n'(w)\bigr)\,dw \biggr) - \Phi^{-1}\biggl(\Phi\biggl(a_- \int_{-\infty}^{+\infty} \varphi\bigl(w, y_0(w), y_0'(w)\bigr)\,dw \biggr) \\
&- \int_{-\infty}^{s} f\bigl(w, y_0(w), y_0'(w)\bigr)\,dw \biggr) \biggr| \,ds \biggr| \\
&+ \frac{1}{1+\tau(t)} \biggl| \int_{\xi}^{t} \frac{\overline{M}}{\rho(s)} \biggl| \frac{1}{a(s, x_n(s), x_n'(s))} - \frac{1}{a(s, x_0(s), x_0'(s))} \biggr| \,ds \biggr| \\
\le{}& \frac{1}{m_1} \frac{1}{1+\tau(t)} \biggl| \int_{\xi}^{t} \frac{1}{\rho(s)} \frac{m_1 \epsilon}{2}\,ds \biggr| \\
&+ \frac{1}{m_1^2} \frac{1}{1+\tau(t)} \biggl| \int_{\xi}^{t} \frac{\overline{M}}{\rho(s)} \bigl| a\bigl(s, x_n(s), x_n'(s)\bigr) - a\bigl(s, x_0(s), x_0'(s)\bigr) \bigr| \,ds \biggr| \\
<{}& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad n > \max\{N_1, N_2, N_3\}.
\end{aligned}
$$

So for $n > \max\{N_1, N_2, N_3, N_4\}$, we get

$$
\frac{1}{1+\tau(t)} \bigl| \bigl(T_1(x_n, y_n)\bigr)(t) - \bigl(T_1(x_0, y_0)\bigr)(t) \bigr| < \frac{3\epsilon}{2}, \quad t \in R.
$$

Hence

$$
\lim_{n\to\infty} \sup_{t\in R} \frac{1}{1+\tau(t)} \bigl| \bigl(T_1(x_n, y_n)\bigr)(t) - \bigl(T_1(x_0, y_0)\bigr)(t) \bigr| = 0.
$$

From the above discussion, we get $T_1(x_n, y_n) \to T_1(x_0, y_0)$ as $n \to \infty$.

Similarly we can prove that $T_2(x_n, y_n) \to T_2(x_0, y_0)$ as $n \to \infty$. Then (2.6) is proved. So T is continuous.

Secondly we prove that T maps bounded sets into relatively compact sets. Let $\Omega \in E$ be a bounded set. We will prove that

(i) $T(\Omega)$ is bounded in E;

(ii) $\{t \to \frac{(T_1(x,y))(t)}{1+\tau(t)} : (x, y) \in \Omega\}$, $\{t \to \frac{(T_2(x,y))(t)}{1+\sigma(t)} : (x, y) \in \Omega\}$, $\{t \to \rho(t)(T_1(x, y))'(t) : (x, y) \in \Omega\}$, and $\{t \to \varrho(t)(T_2(x, y))'(t) : (x, y) \in \Omega\}$ are equicontinuous on each sub-closed interval $[a, b]$ of R;

(iii) $\{t \to \frac{(T_1(x,y))(t)}{1+\tau(t)} : (x, y) \in \Omega\}$, $\{t \to \frac{(T_2(x,y))(t)}{1+\sigma(t)} : (x, y) \in \Omega\}$, $\{t \to \rho(t)(T_1(x, y))'(t) : (x, y) \in \Omega\}$, and $\{t \to \varrho(t)(T_2(x, y))'(t) : (x, y) \in \Omega\}$ are equiconvergent at $t = \pm\infty$.

Since $\Omega \in E$ is bounded, then there exists $r > 0$ such that $\|(x,y)\| \leq r$ for all $(x,y) \in \Omega$. By (d) and (e), we know that there $\phi_r \in L^1(R)$ and for all $(x,y) \in \Omega$ such that (2.3) and (2.4) hold.

(i) Prove that $T(\Omega)$ is bounded in E.

It is easy to show for all $(x,y) \in \Omega$ that

$$\begin{aligned}
&\frac{1}{1+\tau(t)}\big|\big(T_1(x,y)\big)(t)\big| \\
&\quad\leq \frac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\big|\phi\big(s,y(s),y'(s)\big)\big|\,ds + \frac{1}{1+\tau(t)} \\
&\qquad\times\left|\int_\xi^t \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}|\varphi(w,y(w),y'(w))|\,dw) + \int_{-\infty}^{s}|f(w,y(w),y'(w))|\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds\right| \\
&\quad\leq \int_{-\infty}^{+\infty}\phi_r(w)\,dw + \frac{1}{1+\tau(t)}\left|\int_\xi^t \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}\phi_r(w)\,dw) + \int_{-\infty}^{+\infty}\phi_r(w)\,dw)}{\rho(s)m_1}\,ds\right| \\
&\quad\leq \int_{-\infty}^{+\infty}\phi_r(w)\,dw + \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}\phi_r(w)\,dw) + \int_{-\infty}^{+\infty}\phi_r(w)\,dw)}{m_1}
\end{aligned}$$

and

$$\begin{aligned}
&\rho(t)\big|\big(T_1(x,y)\big)(t)\big| \\
&\quad\leq \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}|\varphi(w,y(w),y'(w))|\,dw) + \int_{-\infty}^{t}|f(w,y(w),y'(w))|\,dw)}{a(t,x(t),x'(t))} \\
&\quad\leq \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}\phi_r(w)\,dw) + \int_{-\infty}^{+\infty}\phi_r(w)\,dw)}{m_1}.
\end{aligned}$$

Hence $T_1(\Omega)$ is bounded.

Similarly we can show that $T_2(\Omega)$ is bounded. Thus $T(\Omega)$ is bounded.

(ii) Prove that

$$\begin{aligned}
&\left\{t \to \frac{(T_1(x,y))(t)}{1+\tau(t)} : (x,y) \in \Omega\right\}, \qquad \left\{t \to \frac{(T_2(x,y))(t)}{1+\sigma(t)} : (x,y) \in \Omega\right\}, \\
&\big\{t \to \rho(t)\big(T_1(x,y)\big)'(t) : (x,y) \in \Omega\big\}, \qquad \big\{t \to \varrho(t)\big(T_2(x,y)\big)'(t) : (x,y) \in \Omega\big\}
\end{aligned}$$

are equicontinuous on each sub-closed interval $[a,b]$ of R.

For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $(x,y) \in \Omega$, we have

$$\begin{aligned}
&\left|\frac{1}{1+\tau(t_1)}\big(T_1(x,y)\big)(t_1) - \frac{1}{1+\tau(t_2)}\big(T_1(x,y)\big)(t_2)\right| \\
&\quad\leq \left|\frac{1}{1+\tau(t_1)} - \frac{1}{1+\tau(t_2)}\right|\int_{-\infty}^{+\infty}\big|\phi\big(w,y(w),y'(w)\big)\big|\,dw + \left|\frac{1}{1+\tau(t_1)}\right. \\
&\qquad\times\int_\xi^{t_1}\frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds \\
&\qquad-\frac{1}{1+\tau(t_2)} \\
&\qquad\times\left.\int_\xi^{t_2}\frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \left|\tau(t_1)-\tau(t_2)\right| \int_{-\infty}^{+\infty} \phi_r(w)\,dw + \left|\frac{1}{1+\tau(t_1)} - \frac{1}{1+\tau(t_2)}\right| \\
&\quad \times \left|\int_{\xi}^{t_1} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} |\varphi(w,y(w),y'(w))|\,dw) + \int_{-\infty}^{+\infty} |f(w,y(w),y'(w))|\,dw)}{\rho(s)m_1}\,ds\right| \\
&\quad + \frac{1}{1+\tau(t_2)} \\
&\quad \times \int_{t_1}^{t_2} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} |\varphi(w,y(w),y'(w))|\,dw) + \int_{-\infty}^{+\infty} |f(w,y(w),y'(w))|\,dw)}{\rho(s)m_1}\,ds \\
&\le \left|\tau(t_1)-\tau(t_2)\right| \int_{-\infty}^{+\infty} \phi_r(w)\,dw \\
&\quad + \left|\tau(t_1)-\tau(t_2)\right| \int_a^b \frac{ds}{\rho(s)} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \phi_r(w)\,dw) + \int_{-\infty}^{+\infty} \phi_r(w)\,dw)}{m_1} \\
&\quad + \int_{t_1}^{t_2} \frac{ds}{\rho(s)} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \phi_r(w)\,dw) + \int_{-\infty}^{+\infty} \phi_r(w)\,dw)}{m_1} \\
&\to 0 \quad \text{uniformly as } t_1 \to t_2.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\left|\rho(t_1)\big(T_1(x,y)\big)'(t_1) - \rho(t_2)\big(T_1(x,y)\big)'(t_2)\right| \\
&\quad \le \left|\frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{t_1} f(w,y(w),y'(w))\,dw)}{a(t_1,x(t_1),x'(t_1))}\right. \\
&\qquad \left. - \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\,dw) - \int_{-\infty}^{t_2} f(w,y(w),y'(w))\,dw)}{a(t_1,x(t_1),x'(t_2))}\right| \\
&\qquad + \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(w)\,dw\right) + \int_{-\infty}^{+\infty} \phi - r(w)\,dw\right) \\
&\qquad \times \left|\frac{1}{a(t_1,x(t_1),x'(t_1))} - \frac{1}{a(t_2,x(t_2),x'(t_2))}\right| \\
&\quad \le \frac{1}{m_1}\left|\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \varphi\big(w,y(w),y'(w)\big)\,dw\right) - \int_{-\infty}^{t_1} f\big(w,y(w),y'(w)\big)\,dw\right)\right. \\
&\qquad \left. - \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \varphi\big(w,y(w),y'(w)\big)\,dw\right) - \int_{-\infty}^{t_2} f\big(w,y(w),y'(w)\big)\,dw\right)\right| \\
&\qquad + \frac{\overline{M}}{m_1^2}\left|a\left(t_1, \frac{(1+\tau(t_1))x(t_1)}{1+\tau(t_1)}, \frac{1}{\rho(t_1)}\rho(t_1)x'(t_1)\right)\right. \\
&\qquad \left. - a\left(t_2, \frac{(1+\tau(t_2))x(t_2)}{1+\tau(t_2)}, \frac{1}{\rho(t_2)}\rho(t_2)x'(t_2)\right)\right|.
\end{aligned}
$$

From (b), there exists $\delta_1 > 0$ such that

$$
\left|a\left(t_1, \big(1+\tau(t_1)\big)u, \frac{v}{\rho(t_1)}\right) - a\left(t_2, \big(1+\tau(t_2)\big)u, \frac{v}{\rho(t_2)}\right)\right| < \frac{m_1^2\epsilon}{2\overline{M}}
$$

for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta_1$ and $u, v \in [-r, r]$.

We see that

$$
\left|\Phi\left(a_- \int_{-\infty}^{+\infty} \varphi\big(w,y(w),y'(w)\big)\,dw\right) - \int_{-\infty}^{t} f\big(w,y(w),y'(w)\big)\,dw\right| \le \overline{M}
$$

and

$$\begin{aligned}&\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t_1}f\big(w,y(w),y'(w)\big)\,dw\right.\\&\qquad\left.-\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t_1}f\big(w,y(w),y'(w)\big)\,dw\right)\right|\\&\quad\le\int_{t_1}^{t_2}\big|f\big(w,y(w),y'(w)\big)\big|\,dw\le\int_{t_1}^{t_2}\phi_r(w)\,dw.\end{aligned}$$

Since Φ^{-1} is uniformly continuous on $[-\overline{M},\overline{M}]$, then there exists $\delta_2>0$ such that

$$\begin{aligned}&\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t_1}f\big(w,y(w),y'(w)\big)\,dw\right)\right.\\&\qquad\left.-\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t_2}f\big(w,y(w),y'(w)\big)\,dw\right)\right|<m_1\epsilon\end{aligned}$$

for $t_1,t_2\in[a,b]$ with $|t_1-t_2|<\delta_2$. Then

$$\big|\rho(t_1)\big(T_1(x,y)\big)'(t_1)-\rho(t_2)\big(T_1(x,y)\big)'(t_2)\big|<\frac{3\epsilon}{2},\quad t_1,t_2\in[a,b],|t_1-t_2|<\min\{\delta_1,\delta_2\}.$$

So $\{t\to\frac{(T_1(x,y))(t)}{1+\tau(t)}:(x,y)\in\Omega\}$ and $\{t\to\rho(t)(T_1(x,y))'(t):(x,y)\in\Omega\}$ are equicontinuous on $[a,b]$.

Similarly we can show that $\{t\to\frac{(T_2(x,y))(t)}{1+\sigma(t)}:(x,y)\in\Omega\}$ and $\{t\to\varrho(t)(T_2(x,y))'(t):(x,y)\in\Omega\}$ are equicontinuous on $[a,b]$. Then $T(\Omega)$ is equicontinuous on $[a,b]$.

(iii) Prove that

$$\begin{aligned}&\left\{t\to\frac{(T_1(x,y))(t)}{1+\tau(t)}:(x,y)\in\Omega\right\},\qquad\left\{t\to\frac{(T_2(x,y))(t)}{1+\sigma(t)}:(x,y)\in\Omega\right\},\\&\big\{t\to\rho(t)\big(T_1(x,y)\big)'(t):(x,y)\in\Omega\big\},\qquad\big\{t\to\varrho(t)\big(T_2(x,y)\big)'(t):(x,y)\in\Omega\big\}\end{aligned}$$

are equiconvergent at $t=\pm\infty$.

We have for $t<\xi$ by using (2.1) and (2.2)

$$\begin{aligned}&\left|\frac{1}{1+\tau(t)}\big(T_1(x,y)\big)(t)-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|\\&\quad\le\frac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\big|\phi\big(s,y(s),y'(s)\big)\big|\,ds\\&\qquad+\left|\frac{1}{1+\tau(t)}\int_t^{\xi}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)+\int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds\right.\\&\qquad\left.-\frac{\tau(t)}{1+\tau(t)}\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|+\frac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\big|\varphi\big(w,y(w),y'(w)\big)\big|\,dw\\&\quad\le\frac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\phi_r(w)\,dw\\&\qquad+\frac{1}{1+\tau(t)}\int_t^{\xi}\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)+\int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\right.\,ds\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw))}{\rho(s)}\bigg|\,ds+\frac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\big|\varphi\big(w,y(w),y'(w)\big)\big|\,dw \\
&\le \frac{2}{1+\tau(t)}\int_{-\infty}^{+\infty}\phi_r(w)\,dw \\
&\quad+\frac{1}{1+\tau(t)}\int_t^{\xi}\bigg|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)+\int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)}{\rho(s)a(s,x(s),x'(s))}\,ds \\
&\quad-\frac{\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw))}{\rho(s)}\bigg|\,ds \\
&\le \frac{2}{1+\tau(t)}\int_{-\infty}^{+\infty}\phi_r(w)\,dw \\
&\quad+\frac{\int_t^{\xi}\frac{|\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)+\int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)-\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw))|}{\rho(s)a(s,x(s),x'(s))}\,ds}{1+\tau(t)} \\
&\quad+\frac{1}{1+\tau(t)}\int_t^{\xi}\bigg|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw))}{\rho(s)a(s,x(s),x'(s))} \\
&\quad-\frac{\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw}{\rho(s)}\bigg|\,ds \\
&\le \frac{2}{1+\tau(t)}\int_{-\infty}^{+\infty}\phi_r(w)\,dw+\frac{\int_t^{\xi}|\frac{a_-}{\rho(s)a(s,x(s),x'(s))}-\frac{1}{\rho(s)}|\,ds}{1+\tau(t)}\int_{-\infty}^{+\infty}\phi_r(w)\,dw \\
&\quad+\frac{1}{m_1}\frac{\int_t^{\xi}\frac{|\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)+\int_{-\infty}^{s}f(w,y(w),y'(w))\,dw)-\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw))|}{\rho(s)}\,ds}{1+\tau(t)}.
\end{aligned}
$$

One sees that

$$
\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)+\int_{-\infty}^{s}f\big(w,y(w),y'(w)\big)\,dw\right|\le\overline{M}.
$$

Since Φ^{-1} is uniformly continuous on $[-\overline{M},\overline{M}]$, there exists $\delta_0>0$ such that

$$
\big|\Phi^{-1}(u_1)-\Phi^{-1}(u_1)\big|<\epsilon,\quad u_1,u_2\in[-\overline{M},\overline{M}],|u_1-u_2|<\delta_0.
$$

It is easy to see that there exists $S_1<\xi$ such that

$$
\begin{aligned}
&\left|\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)+\int_{-\infty}^{t}f\big(w,y(w),y'(w)\big)\,dw\right. \\
&\qquad\left.-\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)\right| \\
&\quad\le\int_{-\infty}^{t}\big|f\big(w,y(w),y'(w)\big)\big|\,dw \\
&\quad\le\int_{-\infty}^{t}\phi_r(w)\,dw<\delta_0,\quad t<T_1.
\end{aligned}\tag{2.16}
$$

On the other hand, there exists $S_2<\xi$ such that

$$
\big|a\big(t,x(t),x'(t)\big)-a_-\big|=\left|a\left(t,\big(1+\tau(t)\big)\frac{x(t)}{1+\tau(t)},\frac{1}{\rho(t)}\rho(t)x'(t)\right)-a_-\right|<\epsilon,\quad t<T_2
$$

and

$$\begin{aligned}
&\frac{\int_t^{\xi}|\frac{a_-}{\rho(s)a(s,x(s),x'(s))}-\frac{1}{\rho(s)}|\,ds}{1+\tau(t)}\\
&\quad\le\frac{\int_t^{\xi}\frac{1}{\rho(s)}|\frac{a_-}{a(s,x(s),x'(s))}-1|\,ds}{1+\tau(t)}\\
&\quad\le\frac{\int_{T_2}^{\xi}\frac{1}{\rho(s)}|\frac{a_-}{a(s,x(s),x'(s))}-1|\,ds+\int_t^{T_2}\frac{1}{\rho(s)}|\frac{a_-}{a(s,x(s),x'(s))}-1|\,ds}{1+\tau(t)}\\
&\quad\le\frac{\int_{T_2}^{\xi}\frac{1}{\rho(s)}|\frac{a_-}{m_1}+1|\,ds+\frac{1}{m_1}\int_t^{T_2}\frac{1}{\rho(s)}|a(s,x(s),x'(s))-a_-|\,ds}{1+\tau(t)}\\
&\quad\le\frac{[\frac{a_-}{m_1}+1]\int_{T_2}^{\xi}\frac{1}{\rho(s)}\,ds}{1+\tau(t)}+\frac{\frac{\epsilon}{m_1}\int_t^{T_2}\frac{1}{\rho(s)}\,ds}{1+\tau(t)}<\frac{[\frac{a_-}{m_1}+1]\int_{T_2}^{\xi}\frac{1}{\rho(s)}\,ds}{1+\tau(t)}+\frac{\epsilon}{m_1}.
\end{aligned}$$

There exists $S_3<\xi$ such that

$$\frac{2}{1+\tau(t)}<\epsilon,\qquad \frac{\int_t^{\xi}|\frac{a_-}{\rho(s)a(s,x(s),x'(s))}-\frac{1}{\rho(s)}|\,ds}{1+\tau(t)}<\left[\frac{a_-}{m_1}+1\right]\epsilon+\frac{\epsilon}{m_1},\quad t<S_3. \tag{2.17}$$

Hence for $t<\min\{S_1,S_2,S_3\}$ we have

$$\begin{aligned}
&\left|\frac{1}{1+\tau(t)}\big(T_1(x,y)\big)(t)-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|\\
&\quad<\epsilon\int_{-\infty}^{+\infty}\phi_r(w)\,dw+\left[\left(\frac{a_-}{m_1}+1\right)\epsilon+\frac{\epsilon}{m_1}\right]\int_{-\infty}^{+\infty}\phi_r(w)\,dw+\frac{1}{m_1}\frac{\int_t^{\xi}\frac{\epsilon}{\rho(s)}\,ds}{1+\tau(t)}\\
&\quad<\epsilon\int_{-\infty}^{+\infty}\phi_r(w)\,dw+\left[\left(\frac{a_-}{m_1}+1\right)\epsilon+\frac{\epsilon}{m_1}\right]\int_{-\infty}^{+\infty}\phi_r(w)\,dw+\frac{\epsilon}{m_1}.
\end{aligned}\tag{2.18}$$

It follows that $\{t\to\frac{(T_1(x,y))(t)}{1+\tau(t)}:(x,y)\in\Omega\}$ is equiconvergent at $t=-\infty$.

Furthermore, we have

$$\begin{aligned}
&\left|\rho(t)\big(T_1(x,y)\big)'(t)-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|\\
&\quad=\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{t}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right.\\
&\qquad\left.-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|\\
&\quad\le\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t}f\big(w,y(w),y'(w)\big)\,dw\right)\right.\\
&\qquad\left.-\Phi^{-1}\left(\phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)\right)\right|\Big/a\big(t,x(t),x'(t)\big)\\
&\qquad+\left|\frac{a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw}{a(t,x(t),x'(t))}-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right|\\
&\quad\le\frac{1}{m_1}\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\big(w,y(w),y'(w)\big)\,dw\right)-\int_{-\infty}^{t}f\big(w,y(w),y'(w)\big)\,dw\right)\right.
\end{aligned}$$

$$- \Phi^{-1}\left(\phi\left(a_- \int_{-\infty}^{+\infty} \varphi\big(w, y(w), y'(w)\big)\, dw\right)\right)\Bigg| + \int_{-\infty}^{+\infty} \phi_r(w)\, dw \left|\frac{a_-}{a(t,x(t),x'(t))} - 1\right|.$$

Similarly we can show that there exists $S < \xi$ such that

$$\left|\rho(t)\big(T_1(x,y)\big)'(t) - \int_{-\infty}^{+\infty} \varphi\big(w, y(w), y'(w)\big)\, dw\right| < \frac{\epsilon}{m_1} + \frac{\epsilon}{m_1}\int_{-\infty}^{+\infty} \phi_r(w)\, dw. \tag{2.19}$$

We see that $\{t \to \rho(t)(T_1(x,y))'(t) : (x,y) \in \Omega\}$ is equiconvergent at $t = -\infty$.

Now we show that $\{t \to \frac{(T_1(x,y))(t)}{1+\tau(t)} : (x,y) \in \Omega\}$ and $\{t \to \rho(t)(T_1(x,y))'(t) : (x,y) \in \Omega\}$ are equiconvergent at $t = +\infty$. In fact, for $t > \xi$, we have

$$\begin{aligned}
&\Bigg|\frac{(T_1(x,y))(t)}{1+\tau(t)} \\
&\quad - \int_\xi^t \frac{ds}{\rho(s)} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{a_+}\Bigg| \\
&\leq \frac{\int_{-\infty}^{+\infty} |\phi(s,y(s),y'(s))|\, ds}{1+\tau(t)} \\
&\quad + \Bigg|\frac{1}{1+\tau(t)}\int_\xi^t \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{s} f(w,y(w),y'(w))\, dw)}{\rho(s)a(s,x(s),x'(s))}\, ds \\
&\quad - \int_\xi^t \frac{ds}{\rho(s)} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{a_+}\Bigg| \\
&\leq \frac{\int_{-\infty}^{+\infty} \phi_r(w)\, dw}{1+\tau(t)} + \frac{1}{1+\tau(t)} \\
&\quad \times \int_\xi^t \Bigg|\frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{s} f(w,y(w),y'(w))\, dw)}{\rho(s)a(s,x(s),x'(s))} \\
&\quad - \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{\rho(s)a(s,x(s),x'(s))}\Bigg|\, ds \\
&\quad + \frac{1}{1+\tau(t)}\Bigg|\int_\xi^t \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{\rho(s)a(s,x(s),x'(s))}\, ds \\
&\quad - \int_\xi^t \frac{ds}{\rho(s)} \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{a_+}\Bigg| \\
&\leq \frac{\int_{-\infty}^{+\infty} \phi_r(w)\, dw}{1+\tau(t)} + \frac{1}{m_1}\frac{1}{1+\tau(t)} \\
&\quad \times \int_\xi^t \Bigg|\frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{s} f(w,y(w),y'(w))\, dw)}{\rho(s)} \\
&\quad - \frac{\Phi^{-1}(\Phi(a_- \int_{-\infty}^{+\infty} \varphi(w,y(w),y'(w))\, dw) - \int_{-\infty}^{+\infty} f(w,y(w),y'(w))\, dw)}{\rho(s)}\Bigg|\, ds \\
&\quad + \frac{\overline{M}}{1+\tau(t)}\int_\xi^t \frac{1}{\rho(s)}\left|\frac{1}{a(s,x(s),x'(s))} - \frac{1}{a_+}\right| ds
\end{aligned}$$

and

$$\begin{aligned}
&\left|\rho(t)\bigl(T_1(x,y)\bigr)'(t)-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a_+}\right|\\
&\quad\le\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{t}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right.\\
&\qquad\left.-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a_+}\right|\\
&\quad\le\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{t}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right.\\
&\qquad\left.-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right|\\
&\qquad+\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right.\\
&\qquad\left.-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a_+}\right|\\
&\quad\le\frac{1}{m_1}\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\bigl(w,y(w),y'(w)\bigr)\,dw\right)-\int_{-\infty}^{t}f\bigl(w,y(w),y'(w)\bigr)\,dw\right)\right.\\
&\qquad\left.-\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varphi\bigl(w,y(w),y'(w)\bigr)\,dw\right)-\int_{-\infty}^{+\infty}f\bigl(w,y(w),y'(w)\bigr)\,dw\right)\right|\\
&\qquad+\overline{M}\left|\frac{1}{a(t,x(t),x'(t))}-\frac{1}{a_+}\right|.
\end{aligned}$$

Similarly we can prove that

$$\begin{aligned}
&\left|\frac{(T_1(x,y))(t)}{1+\tau(t)}\right.\\
&\quad\left.-\int_\xi^t\frac{ds}{\rho(s)}\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a_+}\right|\to 0,\\
&\left|\rho(t)\bigl(T_1(x,y)\bigr)'(t)\right.\\
&\quad\left.-\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{+\infty}f(w,y(w),y'(w))\,dw)}{a_+}\right|\to 0
\end{aligned}$$

uniformly as $t\to+\infty$. Then $\{t\to\frac{(T_1(x,y))(t)}{1+\tau(t)}:(x,y)\in\Omega\}$ and $\{t\to\rho(t)(T_1(x,y))'(t):(x,y)\in\Omega\}$ are equiconvergent as $t\to+\infty$.

Similarly we can prove that $\{t\to\frac{(T_2(x,y))(t)}{1+\sigma(t)}:(x,y)\in\Omega\}$ and $\{t\to\varrho(t)(T_2(x,y))'(t):(x,y)\in\Omega\}$ are equiconvergent as $t\to\pm\infty$.

From the above discussion, we know from Claim 2.2 that $T(\Omega)$ is relatively compact. Then T is completely continuous. The proof is complete. □

3 Main theorem

In this section, the main results on the existence of solutions of BVP (1.1)-(1.2) are established. For $\varpi\in L^1(R)$, denote $\|\varpi\|_1=\int_{-\infty}^{+\infty}|\varpi(s)|\,ds$. We need the following assumption:

(B) there exist constants $a_{ij} \geq 0$ ($i = 1,2,3,4$, $j = 1,2,3$), $A_{ij} \geq 0$ ($i = 1,2$, $j = 1,2,3$), $\mu_i \geq 0$ ($i = 1,2$), nonnegative functions $\vartheta_i \in L^1(R)$ ($i = 1,2$) and $\varpi_i \in L^1(R)$ ($i = 1,2,3,4$) such that

$$\left|\phi\left(t, \big(1+\sigma(t)\big)u, \frac{v}{\varrho(t)}\right)\right| \leq \varpi_1(t)\big[a_{11}|u|^{\mu_1} + a_{12}|v|^{\mu_1} + a_{13}\big], \quad t \in R, u, v \in R,$$

$$\left|\varphi\left(t, \big(1+\sigma(t)\big)u, \frac{v}{\varrho(t)}\right)\right| \leq \varpi_2(t)\big[a_{21}|u|^{\mu_1} + a_{22}|v|^{\mu_1} + a_{23}\big], \quad t \in R, u, v \in R,$$

$$\left|f\left(t, \big(1+\sigma(t)\big)u, \frac{v}{\varrho(t)}\right)\right| \leq \vartheta_1(t)\big[A_{11}\Phi\big(|u|^{\mu_1}\big) + A_{12}\Phi\big(|v|^{\mu_1}\big) + A_{13}\big],$$
$$t \in R, u, v \in R,$$

$$\left|\chi\left(t, \big(1+\tau(t)\big)u, \frac{v}{\rho(t)}\right)\right| \leq \varpi_3(t)\big[a_{31}|u|^{\mu_2} + a_{32}|v|^{\mu_2} + a_{33}\big], \quad t \in R, u, v \in R,$$

$$\left|\psi\left(t, \big(1+\tau(t)\big)u, \frac{v}{\rho(t)}\right)\right| \leq \varpi_4(t)\big[a_{41}|u|^{\mu_2} + a_{42}|v|^{\mu_2} + a_{43}\big], \quad t \in R, u, v \in R,$$

$$\left|g\left(t, \big(1+\tau(t)\big)u, \frac{v}{\rho(t)}\right)\right| \leq \vartheta_2(t)\big[A_{21}\Psi\big(|u|^{\mu_2}\big) + A_{22}\Psi\big(|v|^{\mu_2}\big) + A_{23}\big],$$
$$t \in R, u, v \in R.$$

Denote

$$A_0 = \big(n_0^{-1}a_{-}a_{23} + a_{21} + a_{22}\big)\|\varpi_2\|_1,$$

$$\overline{A}_{10} = a_{13}\|\varpi_1\|_1 + \frac{\Phi^{-1}(\Phi(n_0A_0) + A_{13}\|\vartheta_1\|_1 + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)}\Phi(n_0))}{m_1},$$

$$\overline{B}_{10} = (a_{11} + a_{12})\|\varpi_1\|_1 + \frac{\nu_1(\frac{\Phi(n_0A_0)+A_{13}\|\vartheta_1\|_1}{\Phi(n_0)} + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)})}{m_1},$$

$$B_0 = \big(n_0^{-1}b_{-}a_{43} + a_{41} + a_{42}\big)\|\varpi_4\|_1,$$

$$\overline{A}_{20} = a_{33}\|\varpi_3\|_1 + \frac{\Psi^{-1}(\Psi(n_0B_0) + A_{23}\|\vartheta_2\|_1 + \frac{1+(A_{21}+A_{22})\|\vartheta_2\|_1\omega_2(1/B_0)}{\omega_2(1/B_0)}\Psi(n_0))}{m_2},$$

$$\overline{B}_{20} = (a_{31} + a_{32})\|\varpi_3\|_1 + \frac{\nu_2(\frac{\Psi(n_0B_0)+A_{23}\|\vartheta_2\|_1}{\Psi(n_0)} + \frac{1+(A_{21}+A_{22})\|\vartheta_2\|_1\omega_2(1/B_0)}{\omega_2(1/B_0)})}{m_2}.$$

Theorem 3.1 *Suppose that* (a)-(f) *hold. Then BVP* (1.1)-(1.2) *has at least one solution if one of the following items holds*:

(i) $\mu_1\mu_2 < 1$;

(ii) $\mu_1\mu_2 = 1$ *with* $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$ *or* $\overline{B}_{10}\overline{B}_{20}^{\mu_1} < 1$;

(iii) $\mu_1\mu_2 > 1$ *with* n_0 *being a positive integer and*

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \geq \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \geq \overline{B}_{10}.$$

Proof Let T be defined as in Section 2. By Lemma 2.1, T is completely continuous and $(x,y)\in E$ is a solution of BVP (1.1)-(1.2) if and only if $(x,y)\in E$ is a fixed point of T. We should define an open bounded subset Ω of E centered at zero such that $T(\overline{\omega})\subseteq\overline{\Omega}$. Then the fixed points of T are obtained by using the Schauder fixed point theorem.

For $(x,y)\in E$, by definition of T_1, we have

$$\begin{aligned}&\bigl[\Phi\bigl(\rho(t)a\bigl(t,x(t),x'(t)\bigr)\bigl(T_1(x,y)\bigr)'(t)\bigr)\bigr]'+f\bigl(t,y(t),y'(t)\bigr)=0,\quad t\in R,\\&\bigl(T_1(x,y)\bigr)(\xi)=\int_{-\infty}^{+\infty}\phi\bigl(s,y(s),y'(s)\bigr)\,ds,\\&\lim_{t\to-\infty}\rho(t)\bigl(T_1(x,y)\bigr)'(t)=\int_{-\infty}^{+\infty}\varphi\bigl(s,y(s),y'(s)\bigr)\,ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{|(T_1(x,y))(t)|}{1+\tau(t)}&\le\frac{|(T_1(x,y))(\xi)|}{1+\tau(t)}+\frac{|(T_1(x,y))(t)-(T_1(x,y))(\xi)|}{1+\tau(t)}\\&\le\int_{-\infty}^{+\infty}\bigl|\phi\bigl(s,y(s),y'(s)\bigr)\bigr|\,ds+\frac{|\int_\xi^t(T_1(x,y))'(w)\,dw|}{1+\tau(t)}\\&\le\int_{-\infty}^{+\infty}\bigl|\phi\bigl(s,y(s),y'(s)\bigr)\bigr|\,ds+\frac{|\int_\xi^t\frac{1}{\rho(s)}\rho(s)|(T_1(x,y))'(s)|\,ds|}{1+\tau(t)}\\&\le\int_{-\infty}^{+\infty}\left|\phi\left(s,\bigl(1+\sigma(s)\bigr)\frac{y(s)}{1+\sigma(s)},\frac{1}{\varrho(s)}\varrho(s)y'(s)\right)\right|\,ds\\&\quad+\sup_{t\in R}\rho(t)\bigl|\bigl(T_1(x,y)\bigr)'(t)\bigr|\\&\le\int_{-\infty}^{+\infty}\varpi_1(s)\left[a_{11}\left(\frac{|y(s)|}{1+\sigma(s)}\right)^{\mu_1}+a_{12}\bigl[\varrho(s)\bigl|y'(s)\bigr|\bigr]^{\mu_1}+a_{13}\right]ds\\&\quad+\sup_{t\in R}\rho(t)\bigl|\bigl(T_1(x,y)\bigr)'(t)\bigr|\\&\le\int_{-\infty}^{+\infty}\varpi_1(s)\,ds\bigl[(a_{11}+a_{12})\|y\|^{\mu_1}+a_{13}\bigr]+\sup_{t\in R}\rho(t)\bigl|\bigl(T_1(x,y)\bigr)'(t)\bigr|\\&=a_{13}\|\varpi_1\|_1+(a_{11}+a_{12})\|\varpi_1\|_1\|y\|^{\mu_1}+\sup_{t\in R}\rho(t)\bigl|\bigl(T_1(x,y)\bigr)'(t)\bigr|.\end{aligned}$$

On the other hand, we have from (2.1) and (2.2) that

$$\begin{aligned}&\rho(t)\bigl|\bigl(T_1(x,y)\bigr)'(t)\bigr|\\&\quad\le\left|\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}\varphi(w,y(w),y'(w))\,dw)-\int_{-\infty}^{t}f(w,y(w),y'(w))\,dw)}{a(t,x(t),x'(t))}\right|\\&\quad\le\frac{\Phi^{-1}(\Phi(a_-\int_{-\infty}^{+\infty}|\varphi(w,y(w),y'(w))|\,dw)+\int_{-\infty}^{+\infty}|f(w,y(w),y'(w))|\,dw)}{m_1}\\&\quad\le\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\varpi_2(s)\,ds\bigl[(a_{21}+a_{22})\|y\|^{\mu_1}+a_{23}\bigr]\right)\right.\\&\qquad\left.+\int_{-\infty}^{+\infty}\vartheta_1(s)\,ds\bigl[(A_{11}+A_{12})\Phi\bigl(\|y\|^{\mu_1}\bigr)+A_{13}\bigr]\right)\Big/m_1\\&\quad=\Phi^{-1}\bigl(\Phi\bigl(a_-a_{23}\|\varpi_2\|_1+(a_{21}+a_{22})\|\varpi_2\|_1\|y\|^{\mu_1}\bigr)\\&\qquad+A_{13}\|\vartheta_1\|_1+(A_{11}+A_{12})\|\vartheta_1\|_1\Phi\bigl(\|y\|^{\mu_1}\bigr)\bigr)/m_1.\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\Phi\big(a_{-}a_{23}\|\varpi_2\|_1 + (a_{21}+a_{22})\|\varpi_2\|_1\|y\|^{\mu_1}\big) \\
&\quad \le \begin{cases} \Phi(a_{-}a_{23}\|\varpi_2\|_1 + (a_{21}+a_{22})\|\varpi_2\|_1 n_0), & \|y\|^{\mu_1} \le n_0, \\ \Phi((n_0^{-1}a_{-}a_{23}\|\varpi_2\|_1 + (a_{21}+a_{22})\|\varpi_2\|_1)\|y\|^{\mu_1}), & \|y\|^{\mu_1} > n_0 \end{cases} \\
&\quad \le \Phi(n_0A_0) + \frac{\Phi(\|y\|^{\mu_1})}{\omega(1/A_0)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\rho(t)\big|\big(T_1(x,y)\big)'(t)\big| &\le \frac{\Phi^{-1}(\Phi(n_0A_0) + \frac{\Phi(\|y\|^{\mu_1})}{\omega_1(1/A_0)} + A_{13}\|\vartheta_1\|_1 + (A_{11}+A_{12})\|\vartheta_1\|_1\Phi(\|y\|^{\mu_1}))}{m_1} \\
&= \frac{\Phi^{-1}(\Phi(n_0A_0) + A_{13}\|\vartheta_1\|_1 + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A)}{\omega_1(1/A_0)}\Phi(\|y\|^{\mu_1}))}{m_1} \\
&\le \frac{\Phi^{-1}(\Phi(n_0A_0) + A_{13}\|\vartheta_1\|_1 + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)}\Phi(n_0))}{m_1} \\
&\quad + \frac{\Phi^{-1}((\frac{\Phi(n_0A_0)+A_{13}\|\vartheta_1\|_1}{\Phi(n_0)} + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)})\Phi(\|y\|^{\mu_1}))}{m_1} \\
&\le \frac{\Phi^{-1}(\Phi(n_0A_0) + A_{13}\|\vartheta_1\|_1 + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)}\Phi(n_0))}{m_1} \\
&\quad + \frac{\nu_1(\frac{\Phi(n_0A_0)+A_{13}\|\vartheta_1\|_1}{\Phi(n_0)} + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)})}{m_1}\|y\|^{\mu_1}.
\end{aligned}$$

So

$$\begin{aligned}
\frac{|(T_1(x,y))(t)|}{1+\tau(t)} &\le a_{13}\|\varpi_1\|_1 + (a_{11}+a_{12})\|\varpi_1\|_1\|y\|^{\mu_1} + \sup_{t\in R}\rho(t)\big|\big(T_1(x,y)\big)'(t)\big| \\
&\le a_{13}\|\varpi_1\|_1 + (a_{11}+a_{12})\|\varpi_1\|_1\|y\|^{\mu_1} \\
&\quad + \frac{\Phi^{-1}(\Phi(n_0A_0) + A_{13}\|\vartheta_1\|_1 + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)}\Phi(n_0))}{m_1} \\
&\quad + \frac{\nu_1(\frac{\Phi(n_0A_0)+A_{13}\|\vartheta_1\|_1}{\Phi(n_0)} + \frac{1+(A_{11}+A_{12})\|\vartheta_1\|_1\omega_1(1/A_0)}{\omega_1(1/A_0)})}{m_1}\|y\|^{\mu_1} = \overline{A}_{10} + \overline{B}_{10}\|y\|^{\mu_1}.
\end{aligned}$$

It follows that

$$\big\|T_1(x,y)\big\| \le \overline{A}_{10} + \overline{B}_{10}\|y\|^{\mu_1}. \tag{3.1}$$

Similarly, we can show that

$$\big\|T_2(x,y)\big\| \le \overline{A}_{20} + \overline{B}_{20}\|x\|^{\mu_2}. \tag{3.2}$$

For $r_1 > 0$, $r_2 > 0$, denote

$$\Omega = \big\{(x,y)\in E : \|x\| < r_1, \|y\| < r_2\big\}.$$

We will choose suitable positive constants r_1, r_2 such that $T(\overline{\Omega}) \subseteq \overline{\Omega}$. In fact, for $(x,y) \in \overline{\Omega}$, we have from (3.1) and (3.2)

$$\left\| T_1(x,y) \right\| \leq \overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1}$$

and

$$\left\| T_2(x,y) \right\| \leq \overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2}.$$

So one needs to seek $r_1, r_2 > 0$ such that

$$\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1} \leq r_1, \qquad \overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2} \leq r_2. \tag{3.3}$$

To satisfy (3.3), we firstly prove that there $r_1, r_2 > 0$ such that either

$$\frac{(\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1})^{\mu_2}}{r_2 - \overline{A}_{20}} \leq \frac{1}{\overline{B}_{20}} \quad \text{or} \quad \frac{(\overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2})^{\mu_1}}{r_1 - \overline{A}_{10}} \leq \frac{1}{\overline{B}_{10}}. \tag{3.4}$$

(i) $\mu_1\mu_2 < 1$. It is easy to see that we can choose $r_2 > 0$ sufficiently large such that

$$\frac{(\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1})^{\mu_2}}{r_2 - \overline{A}_{20}} \leq \frac{1}{\overline{B}_{20}}.$$

Then we choose $r_1 > 0$ such that

$$\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1} \leq r_1 \leq \left(\frac{r_2 - \overline{A}_{20}}{\overline{B}_{20}} \right)^{\frac{1}{\mu_2}}.$$

Hence there exist $r_1, r_2 > 0$ such that (3.3) holds. Let $\Omega_0 = \{(x,y) \in E : \|x\| < r_1, \|y\| < r_2\}$. So $T\overline{\Omega}_0 \subset \overline{\Omega}_0$. Thus the Schauder fixed point theorem [44] implies that the operator T has at least one fixed point in $\overline{\Omega}_0$. So BVP (1.1)-(1.2) has at least one solution.

(ii) $\mu_1\mu_2 = 1$. If $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$, we choose a positive integer n such that $\overline{B}_1^{\mu_2}\overline{B}_2 < 1$. It is easy to see that we can choose $r_2 > 0$ sufficiently large such that

$$\frac{(\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1})^{\mu_2}}{r_2 - \overline{A}_{20}} \leq \frac{1}{\overline{B}_{20}}.$$

Then we choose $r_1 > 0$ such that

$$\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1} \leq r_1 \leq \left(\frac{r_2 - \overline{A}_{20}}{\overline{B}_{20}} \right)^{\frac{1}{\mu_2}}.$$

Hence there exist $r_1, r_2 > 0$ such that (3.3) holds. Let $\Omega_0 = \{(x,y) \in E : \|x\| < r_1, \|y\| < r_2\}$.

If $\overline{B}_{10}\overline{B}_{20}^{\mu_1} < 1$, we choose a positive integer n such that $\overline{B}_2^{\mu_1}\overline{B}_1 < 1$. It is easy to see that we can choose $r_1 > 0$ sufficiently large such that

$$\frac{(\overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2})^{\mu_1}}{r_1 - \overline{A}_{10}} \leq \frac{1}{\overline{B}_{10}}.$$

Then we choose $r_2 > 0$ such that

$$\overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2} \le r_2 \le \left(\frac{r_1 - \overline{A}_{10}}{\overline{B}_{10}}\right)^{\frac{1}{\mu_1}}.$$

Hence there exist $r_1, r_2 > 0$ such that (3.3) holds. Let $\Omega_0 = \{(x,y) \in E : \|x\| < r_1, \|y\| < r_2\}$.

So $T\overline{\Omega}_0 \subset \overline{\Omega}_0$. Thus the Schauder fixed point theorem [44] implies that the operator T has at least one fixed point in $\overline{\Omega}_0$. So BVP (1.1)-(1.2) has at least one solution.

(iii) $\mu_1\mu_2 > 1$. If

$$\frac{1}{\mu_1\mu_2 - 1} \frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \ge \overline{B}_{20},$$

we choose $r_2 = \frac{\mu_1\mu_2\overline{A}_{20}}{\mu_1\mu_2-1}$, and it is easy to see that

$$\frac{r_2 - \overline{A}_{20}}{(\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1})^{\mu_2}} \ge \overline{B}_{20}.$$

So there exists $r_1 > 0$ such that

$$\overline{A}_{10} + \overline{B}_{10} r_2^{\mu_1} \le r_1 \le \left(\frac{r_2 - \overline{A}_{20}}{\overline{B}_{20}}\right)^{\frac{1}{\mu_2}}.$$

Hence there exist $r_1, r_2 > 0$ such that (3.3) holds. Let $\Omega_0 = \{(x,y) \in E : \|x\| < r_1, \|y\| < r_2\}$.

If

$$\frac{1}{\mu_1\mu_2 - 1} \frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \ge \overline{B}_{10},$$

we choose $r_1 = \frac{\mu_2\mu_1\overline{A}_{10}}{\mu_1\mu_2-1}$, and it is easily seen that

$$\frac{r_1 - \overline{A}_{10}}{(\overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2})^{\mu_1}} \ge \overline{B}_{10}.$$

So there exists $r_2 > 0$ such that

$$\overline{A}_{20} + \overline{B}_{20} r_1^{\mu_2} \le r_2 \le \left(\frac{r_1 - \overline{A}_{10}}{\overline{B}_{10}}\right)^{\frac{1}{\mu_1}}.$$

Hence there exist $r_1, r_2 > 0$ such that (3.3) holds. Let $\Omega_0 = \{(x,y) \in E : \|x\| < r_1, \|y\| < r_2\}$.

So $T\overline{\Omega}_0 \subset \overline{\Omega}_0$. Thus the Schauder fixed point theorem [44] implies that the operator T has at least one fixed point in $\overline{\Omega}_0$. So BVP (1.1)-(1.2) has at least one solution. □

4 Three examples

To show the application of Theorem 3.1, we give three examples.

Example 4.1 Consider the following boundary value problem of a second-order differential system:

$$\begin{aligned}
&x''(t) + \frac{e^{-t^2}}{\sqrt{\pi}}\left[A_{11}\left(\frac{y(t)}{1+|t|}\right)^{\mu_1} + A_{12}\left(|t|y'(t)\right)^{\mu_1} + A_{13}\right] = 0, \quad t \in R,\\
&y''(t) + \frac{1}{\pi(1+t^2)}\left[A_{21}\left(\frac{x(t)}{1+|t|}\right)^{\mu_2} + A_{22}\left(|t|x'(t)\right)^{\mu_2} + A_{32}\right] = 0, \quad t \in R,\\
&x(0) = a_1, \qquad \lim_{t\to-\infty} x'(t) = b_1,\\
&y(0) = a_2, \qquad \lim_{t\to-\infty} y'(t) = b_2,
\end{aligned} \tag{4.1}$$

where $a_1, b_1, a_2, b_2, A_{i,j}$ $(i,j = 1,2) \in R$, $\mu_1, \mu_2 \geq 0$ are constants. Then BVP (4.1) has at least one solution if one of the following items holds:

(i) $\mu_1\mu_2 < 1$;

(ii) $\mu_1\mu_2 = 1$ for sufficiently small $|A_{11}|$, $|A_{12}|$, $|A_{21}|$, and $|A_{22}|$;

(iii) $\mu_1\mu_2 > 1$ with n_0 being a positive integer and

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \geq \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \geq \overline{B}_{10},$$

where

$$\begin{aligned}
\overline{A}_{10} &= a_{13} + 2A_0 + A_{13} + A_{11} + A_{12} = a_1 + 2b_1 + A_{13} + A_{11} + A_{12},\\
\overline{B}_{10} &= a_{11} + a_{12} + 2A_0 + A_{13} + A_{11} + A_{12} = 2b_1 + A_{13} + A_{11} + A_{12},\\
\overline{A}_{20} &= a_{33} + 2B_0 + A_{23} + A_{21} + A_{22} = a_2 + 2b_2 + A_{23} + A_{21} + A_{22},\\
\overline{B}_{20} &= a_{31} + a_{32} + 2B_0 + A_{23} + A_{21} + A_{22} = 2b_2 + A_{23} + A_{21} + A_{22}.
\end{aligned}$$

Proof Corresponding to BVP (1.1)-(1.2), we choose $\xi = \eta = 0$ and

$$\begin{aligned}
&\rho(t) = \varrho(t) = 1, \qquad \Phi(s) = \Psi(s) = s, \qquad a(t,u,v) = b(t,u,v) = 1,\\
&\phi(t,u,v) = \frac{a_1 e^{-t^2}}{\sqrt{\pi}}, \qquad \varphi(t,u,v) = \frac{b_1}{\sqrt{\pi}}e^{-t^2},\\
&\chi(t,u,v) = \frac{a_2}{\sqrt{\pi}}e^{-t^2}, \qquad \psi(t,u,v) = \frac{b_2}{\sqrt{\pi}}e^{-t^2}
\end{aligned}$$

and

$$\begin{aligned}
f(t,u,v) &= \frac{e^{-t^2}}{\sqrt{\pi}}\left[A_{11}\left(\frac{u}{1+|t|}\right)^{\mu_1} + A_{12}\left(|t|v\right)^{\mu_1} + A_{13}\right],\\
g(t,u,v) &= \frac{1}{\pi(1+t^2)}\left[A_{21}\left(\frac{u}{1+|t|}\right)^{\mu_2} + A_{22}\left(|t|v\right)^{\mu_2} + A_{32}\right].
\end{aligned}$$

One sees that $\tau(t) = \sigma(t) = |t|$, $m_1 = M_1 = m_2 = M_2 = a_- = a_+ = b_- = b_+ = 1$, $\Phi^{-1}(x) = \Psi^{-1}(x) = x$, $\omega_1(x) = \nu_1(x) = \omega_2(x) = \nu_2(x) = x$, and we choose

$$\begin{aligned} &a_{11} = a_{12} = 0, \qquad a_{13} = a_1, \qquad a_{21} = a_{22} = 0, \qquad a_{23} = b_1, \\ &a_{31} = a_{32} = 0, \qquad a_{33} = a_2, \qquad a_{41} = a_{42} = 0, \qquad a_{43} = b_2 \end{aligned}$$

and

$$\varpi_1(t) = \varpi_2(t) = \varpi_3(t) = \varpi_4(t) = \frac{e^{-t^2}}{\sqrt{\pi}}, \qquad \vartheta_1(t) = \frac{e^{-t^2}}{\sqrt{\pi}}, \qquad \vartheta_2(t) = \frac{1}{\pi(1+t^2)};$$

we have

$$\begin{aligned} &\left|\phi\left(t, \bigl(1+\sigma(t)\bigr)u, \frac{v}{\varrho(t)}\right)\right| \le \varpi_1(t)\bigl[a_{11}|u|^{\mu_1} + a_{12}|v|^{\mu_1} + a_{13}\bigr], \quad t \in R, u, v \in R, \\ &\left|\varphi\left(t, \bigl(1+\sigma(t)\bigr)u, \frac{v}{\varrho(t)}\right)\right| \le \varpi_2(t)\bigl[a_{21}|u|^{\mu_1} + a_{22}|v|^{\mu_1} + a_{23}\bigr], \quad t \in R, u, v \in R, \\ &\left|f\left(t, \bigl(1+\sigma(t)\bigr)u, \frac{v}{\varrho(t)}\right)\right| \le \vartheta_1(t)\bigl[|A_{11}||u|^{\mu_1} + |A_{12}||v|^{\mu_1} + |A_{13}|\bigr], \quad t \in R, u, v \in R, \\ &\left|\chi\left(t, \bigl(1+\tau(t)\bigr)u, \frac{v}{\rho(t)}\right)\right| \le \varpi_3(t)\bigl[a_{31}|u|^{\mu_2} + a_{32}|v|^{\mu_2} + a_{33}\bigr], \quad t \in R, u, v \in R, \\ &\left|\psi\left(t, \bigl(1+\tau(t)\bigr)u, \frac{v}{\rho(t)}\right)\right| \le \varpi_4(t)\bigl[a_{41}|u|^{\mu_2} + a_{42}|v|^{\mu_2} + a_{43}\bigr], \quad t \in R, u, v \in R, \\ &\left|g\left(t, \bigl(1+\tau(t)\bigr)u, \frac{v}{\rho(t)}\right)\right| \le \vartheta_2(t)\bigl[|A_{21}\|u|^{\mu_2} + |A_{22}||v|^{\mu_2} + |A_{23}|\bigr], \quad t \in R, u, v \in R. \end{aligned}$$

So (a)-(f) mentioned in Section 1, and assumption (B) in Theorem 3.1 are satisfied. By direct computation, choosing $n_0 = 1$, we have

$$\begin{aligned} &\|\varpi_1\|_1 = \|\varpi_2\|_1 = \|\varpi_3\|_1 = \|\varpi_4\|_1 = \|\vartheta_1\|_1 = \|\vartheta_2\|_1 = 1, \\ &A_0 = a_{23} + a_{21} + a_{22} = b_1, \\ &\overline{A}_{10} = a_{13} + 2A_0 + A_{13} + A_{11} + A_{12} = a_1 + 2b_1 + A_{13} + A_{11} + A_{12}, \\ &\overline{B}_{10} = a_{11} + a_{12} + 2A_0 + A_{13} + A_{11} + A_{12} = 2b_1 + A_{13} + A_{11} + A_{12}, \\ &B_0 = a_{43} + a_{41} + a_{42} = b_2, \\ &\overline{A}_{20} = a_{33} + 2B_0 + A_{23} + A_{21} + A_{22} = a_2 + 2b_2 + A_{23} + A_{21} + A_{22}, \\ &\overline{B}_{20} = a_{31} + a_{32} + 2B_0 + A_{23} + A_{21} + A_{22} = 2b_2 + A_{23} + A_{21} + A_{22}. \end{aligned}$$

It follows from Theorem 3.1 that

(i) $\mu_1\mu_2 < 1$; BVP (4.1) has at least one solution;

(ii) $\mu_1\mu_2 = 1$ with $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$ or $\overline{B}_{10}\overline{B}_{20}^{\mu_1} < 1$; BVP (4.1) has at least one solution;

(iii) $\mu_1\mu_2 > 1$ with n_0 being a positive integer and

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \ge \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \geq \overline{B}_{10};$$

BVP (4.1) has at least one solution.

The proof is completed. □

Let

$$\Phi(x) = \Psi(x) = x^3, \quad x \in R, \qquad \rho(t) = \varrho(t) = \begin{cases} \sqrt{|t|}, & |t| \leq 1, \\ |t|, & |t| > 1, \end{cases}$$

$$p(t) = \begin{cases} \frac{1}{\sqrt{|t|}}, & t \in [-1,0) \cup (0,1], \\ \frac{1}{t^2}, & |t| > 1, \end{cases} \qquad q(t) = \begin{cases} \frac{1}{\sqrt[3]{t}}, & t \in [-1,0) \cup (0,1], \\ \frac{1}{t^3}, & |t| > 1. \end{cases}$$

One can see by direct computation that

$$\tau(t) = \sigma(t) = \left|\int_0^t \frac{du}{\rho(u)}\right| = \left|\int_0^t \frac{du}{\varrho(u)}\right| = \begin{cases} 2\sqrt{|t|}, & |t| \leq 1, \\ 2 + \ln|t|, & |t| > 1. \end{cases}$$

Example 4.2 Consider the following boundary value problem of a second-order differential system:

$$\begin{aligned}
&\left[\left(\rho(t)x'(t)\right)^3\right]' + p(t)\left[A_{11}\left(\frac{y(t)}{1+\sigma(t)}\right)^{3\mu_1} + A_{12}\left(\varrho(t)y'(t)\right)^{3\mu_1} + A_{13}\right] = 0, \quad t \in R, \\
&\left[\left(\varrho(t)y'(t)\right)^3\right]' + q(t)\left[A_{21}\left(\frac{x(t)}{1+\tau(t)}\right)^{3\mu_2} + A_{22}\left(\rho(t)x'(t)\right)^{3\mu_2} + A_{23}\right] = 0, \quad t \in R, \\
&x(0) = \int_{-\infty}^{+\infty} p(s)\left[a_{11}\left(\frac{y(s)}{1+\sigma(s)}\right)^{\mu_1} + a_{12}\left(\varrho(s)y'(s)\right)^{\mu_1} + a_{13}\right]ds, \\
&\lim_{t\to-\infty} \rho(t)x'(t) = \int_{-\infty}^{+\infty} p(s)\left[a_{21}\left(\frac{y(s)}{1+\sigma(s)}\right)^{\mu_1} + a_{22}\left(\varrho(s)y'(s)\right)^{\mu_1} + a_{23}\right]ds, \\
&y(0) = \int_{-\infty}^{+\infty} q(s)\left[a_{31}\left(\frac{x(s)}{1+\tau(s)}\right)^{\mu_2} + a_{32}\left(\rho(s)x'(s)\right)^{\mu_2} + a_{33}\right]ds, \\
&\lim_{t\to-\infty} \varrho(t)y'(t) = \int_{-\infty}^{+\infty} q(s)\left[a_{41}\left(\frac{x(s)}{1+\tau(s)}\right)^{\mu_2} + a_{42}\left(\rho(s)x'(s)\right)^{\mu_2} + a_{43}\right]ds,
\end{aligned} \tag{4.2}$$

where a_{ij} $(i = 1,2,3,4, j = 1,2,3)$, A_{ij} $(i = 1,2, j = 1,2,3)$ are nonnegative constants. Then BVP (4.2) has at least one solution if one of the following items holds:

(i) $\mu_1\mu_2 < 1$;

(ii) $\mu_1\mu_2 = 1$ for sufficiently small a_{ij} $(i = 1,2,3,4, j = 1,2,3)$, A_{ij} $(i = 1,2, j = 1,2)$;

(iii) $\mu_1\mu_2 > 1$ with n_0 being a positive integer and

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \geq \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \geq \overline{B}_{10},$$

where

$$\overline{A}_{10} = 6a_{13} + \sqrt[3]{432(a_{21}+a_{22}+a_{23})^3 + 6A_{13} + 6(A_{11}+A_{12})},$$
$$\overline{B}_{10} = 6(a_{11}+a_{12}) + +\sqrt[3]{432(a_{21}+a_{22}+a_{23})^3 + 6A_{13} + 6(A_{11}+A_{12})},$$
$$\overline{A}_{20} = 4a_{33} + \sqrt[3]{64(a_{41}+a_{42}+a_{43})^3 + 4A_{23} + 4(A_{21}+A_{22})},$$
$$\overline{B}_{20} = 4(a_{31}+a_{32}) + +\sqrt[3]{64(a_{41}+a_{42}+a_{43})^3 + 4A_{23} + 4(A_{21}+A_{22})}.$$

Proof Corresponding to BVP (1.1)-(1.2), we find that:

(a) $\rho, \varrho \in C^0(R,(0,\infty))$ are continuous on R and satisfy

$$\int_{-\infty}^{0} \frac{1}{\rho(s)}\,ds = +\infty, \qquad \int_{0}^{+\infty} \frac{1}{\rho(s)}\,ds = +\infty,$$
$$\int_{-\infty}^{0} \frac{1}{\varrho(s)}\,ds = +\infty, \qquad \int_{0}^{+\infty} \frac{1}{\varrho(s)}\,ds = +\infty.$$

(b) $a(t,x,y) = b(t,x,y) \equiv 1$ satisfies

$$\lim_{t\to\pm\infty} a\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right)$$
$$= 1 = a_{\pm} \quad \text{uniformly for } u, v \text{ in each bounded interval},$$
$$\lim_{t\to\pm\infty} b\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right)$$
$$= 1 = b_{\pm} \quad \text{uniformly for } u, v \text{ in each bounded interval},$$
$$a\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right) \equiv 1, \qquad m_1 = M_1 = 1,$$
$$b\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right) \equiv 1, \quad t \in R, u, v \in R, m_2,$$
$$(u,v) \to a\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right) \quad \text{is uniformly continuous for } t \in R,$$
$$(u,v) \to b\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right) \quad \text{is uniformly continuous for } t \in R.$$

(c) Φ, Ψ are quasi-Laplacian operators, the inverse operators of Φ, Ψ are $\Phi^{-1}(x) = \Psi^{-1}(x) = x^{\frac{1}{3}}$, respectively, the supporting functions of Φ and Φ^{-1} are denoted by $\omega_1(x) = x^3$ and $\nu_1(x) = x^{\frac{1}{3}}$, respectively, the supporting functions of Ψ and Ψ^{-1} by $\omega_2(x) = x^3$ and $\nu_2(x) = x^{\frac{1}{3}}$.

(d) f and g satisfy

$$\left|f\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right)\right| \le \big|p(t)\big|\big[A_{11}\Phi\big(|u|^{\mu_1}\big) + A_{12}\Phi\big(|v|^{\mu_1}\big) + A_{13}\big],$$
$$\left|g\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right)\right| \le \big|q(t)\big|\big[A_{21}\Psi\big(|u|^{\mu_2}\big) + A_{22}\Psi\big(|v|^{\mu_2}\big) + A_{23}\big],$$

one finds that f is a σ-Carathéodory function, and g is a τ-Carathéodory function.

(e) ϕ, φ, χ, ψ satisfy

$$\begin{aligned}
&\left|\phi\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right)\right| \le \big|p(t)\big|\big[a_{11}|u|^{\mu_1}+a_{12}|v|^{\mu_1}+a_{13}\big],\\
&\left|\varphi\left(t,\big(1+\sigma(t)\big)u,\frac{v}{\varrho(t)}\right)\right| \le \big|p(t)\big|\big[a_{21}|u|^{\mu_1}+a_{22}|v|^{\mu_1}+a_{23}\big],\\
&\left|\chi\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right)\right| \le \big|q(t)\big|\big[a_{31}|u|^{\mu_2}+a_{32}|v|^{\mu_2}+a_{33}\big],\\
&\left|\psi\left(t,\big(1+\tau(t)\big)u,\frac{v}{\rho(t)}\right)\right| \le \big|q(t)\big|\big[a_{41}|u|^{\mu_2}+a_{42}|v|^{\mu_2}+a_{43}\big],
\end{aligned}$$

and ϕ, φ are σ-Carathéodory functions, χ, ψ are τ-Carathéodory functions.

(f) $\xi = \eta = 0$.

So (a)-(f) mentioned in Section 1 and assumption (B) in Theorem 3.1 are satisfied. By direct computation we know that $\|\vartheta_1\|_1 = 6$, $\|\vartheta_2\|_1 = 4$, $\|\varpi_1\|_1 = \|\varpi_2\|_1 = 6$, and $\|\varpi_3\|_1 = \|\varpi_4\|_1 = 4$. By direct computation, we have

$$\begin{aligned}
&A_0 = 6(a_{23}+a_{21}+a_{22}),\\
&\overline{A}_{10} = 6a_{13} + \sqrt[3]{2A_0^3+6A_{13}+6(A_{11}+A_{12})},\\
&\overline{B}_{10} = 6(a_{11}+a_{12}) + +\sqrt[3]{2A_0^3+6A_{13}+6(A_{11}+A_{12})},\\
&B_0 = 4(a_{43}+a_{41}+a_{42}),\\
&\overline{A}_{20} = 4a_{33} + \sqrt[3]{2B_0^3+4A_{23}+4(A_{21}+A_{22})},\\
&\overline{B}_{20} = 4(a_{31}+a_{32}) + +\sqrt[3]{2B_0^3+4A_{23}+4(A_{21}+A_{22})}.
\end{aligned}$$

It follows from Theorem 3.1 that

(i) $\mu_1\mu_2 < 1$ implies that BVP (4.2) has at least one solution;

(ii) $\mu_1\mu_2 = 1$ with $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$ or $\overline{B}_{10}\overline{B}_{20}^{\mu_1} < 1$ implies that BVP (4.2) has at least one solution. One finds that $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$ or $\overline{B}_{10}\overline{B}_{20}^{\mu_1} < 1$ holds if a_{ij}, A_{ij} are sufficiently small. Then BVP (4.2) has at least one solution if a_{ij}, A_{ij} are sufficiently small;

(iii) $\mu_1\mu_2 > 1$ with $n_0 = 1$ and

$$\frac{1}{\mu_1\mu_2-1}\frac{\overline{A}_{20}}{[\overline{A}_{10}+\overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \ge \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2-1}\frac{\overline{A}_{10}}{[\overline{A}_{20}+\overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \ge \overline{B}_{10}$$

implies that BVP (4.2) has at least one solution.

The proof is completed. □

Example 4.3 Consider the following boundary value problem:

$$\begin{aligned}
&\left(\sqrt{|t|}\left(1+\frac{1}{1+t^2}\left(\arctan\left(t^2+x^2(t)+x'(t)^4\right)\right)^2\right)x'(t)\right)' \\
&\quad+\frac{e^{-t^2}}{\sqrt{\pi}}\left[A_{11}\left(\frac{y(t)}{1+2\sqrt{|t|}}\right)^{\mu_1}+A_{12}\left(\sqrt{|t|}y'(t)\right)^{\mu_1}+A_{13}\right]=0, \\
&\left(\sqrt{|t|}\left(1+\frac{1}{1+t^2}\left(\arctan\left(t^2+x^2(t)+x'(t)^4\right)\right)^2\right)y'(t)\right)' \\
&\quad+\frac{1}{\pi(1+t^2)}\left[A_{21}\left(\frac{x(t)}{1+2\sqrt{|t|}}\right)^{\mu_2}+A_{22}\left(\sqrt{|t|}x'(t)\right)^{\mu_2}+A_{23}\right]=0, \\
&x(0)=0, \qquad \lim_{t\to-\infty}\sqrt{|t|}x'(t)=0, \\
&y(0)=0, \qquad \lim_{t\to-\infty}\sqrt{|t|}y'(t)=0.
\end{aligned} \tag{4.3}$$

Corresponding to BVP (1.1)-(1.2), we have

$$\begin{aligned}
&\Phi(x)=\Psi(x)=x, \qquad \rho(t)=\varrho(t)=\sqrt{|t|}, \\
&a(t,u,v)=b(t,u,v)=1+\frac{1}{1+t^2}\left(\arctan\left(t^2+x^2(t)+x'(t)^4\right)\right)^2, \\
&\phi(t)=\varphi(t)=\chi(t)=\psi(t)=0, \qquad \xi=\eta=0, \\
&\vartheta_1(t)=\frac{e^{-t^2}}{\sqrt{\pi}}, \qquad \vartheta_2(t)=\frac{1}{\pi(1+t^2)}.
\end{aligned}$$

One sees that

$$\begin{aligned}
&\Phi^{-1}(x)=\Psi^{-1}(x)=\omega_1(x)=\omega_2(x)=\nu_1(x)=\nu_2(x)=x, \\
&a_\pm=b_\pm=1, \qquad m_1=m_2=1, \qquad M_1=M_2=1+\frac{\pi^2}{4}, \\
&\tau(t)=\sigma(t)=\left|\int_0^t\frac{1}{\sqrt{|s|}}\,ds\right|=2\sqrt{|t|}, \\
&\int_{-\infty}^0\frac{1}{\rho(s)}\,ds=\int_0^{+\infty}\frac{1}{\rho(s)}\,ds=\int_{-\infty}^0\frac{1}{\varrho(s)}\,ds=\int_0^{+\infty}\frac{1}{\varrho(s)}\,ds=+\infty.
\end{aligned}$$

On the other hand, suppose that f, g are continuous functions and there exist constants $A_{ij}\geq 0$ $(i=1,2,j=1,2,3)$, $\mu_i\geq 0$ $(i=1,2)$, and nonnegative functions $\vartheta_i\in L^1(R)$ $(i=1,2)$ such that

$$\begin{aligned}
&\phi\left(t,\left(1+\sigma(t)\right)u,\frac{v}{\varrho(t)}\right)=0, \qquad \varphi\left(t,\left(1+\sigma(t)\right)u,\frac{v}{\varrho(t)}\right)=0, \\
&\chi\left(t,\left(1+\sigma(t)\right)u,\frac{v}{\varrho(t)}\right)=0, \qquad \psi\left(t,\left(1+\sigma(t)\right)u,\frac{v}{\varrho(t)}\right)=0, \\
&\left|f\left(t,\left(1+\sigma(t)\right)u,\frac{v}{\varrho(t)}\right)\right|\leq\vartheta_1(t)\left[A_{11}|u|^{\mu_1}+A_{12}|v|^{\mu_1}+A_{13}\right], \quad t\in R,u,v\in R, \\
&\left|g\left(t,\left(1+\tau(t)\right)u,\frac{v}{\rho(t)}\right)\right|\leq\vartheta_2(t)\left[A_{21}|u|^{\mu_2}+A_{22}|v|^{\mu_2}+A_{23}\right], \quad t\in R,u,v\in R.
\end{aligned}$$

It is easy to see that (a)-(f) hold with $a_{ij} = 0$ ($i = 1,2,3,4$, $j = 1,2,3$), $\varpi_i(t) = 0$ ($i = 1,2,3,4$).

For ease of expression, choose $n_0 = 1$, by direct computation, we have

$$\begin{aligned}
&\|\varpi_1\|_1 = \|\varpi_2\|_1 = \|\varpi_3\|_1 = \|\varpi_4\|_1 = 0, \qquad \|\vartheta_1\|_1 = \|\vartheta_2\|_1 = 1,\\
&A_0 = (a_{23} + a_{21} + a_{22})\|\varpi_2\|_1 = 0,\\
&\overline{A}_{10} = a_{13}\|\varpi_1\|_1 + 2A_0 + A_{13}\|\vartheta_1\|_1 + (A_{11} + A_{12})\|\vartheta_1\|_1 = A_{13} + A_{11} + A_{12},\\
&\overline{B}_{10} = (a_{11} + a_{12})\|\varpi_1\|_1 + 2A_0 + A_{13}\|\vartheta_1\|_1 + (A_{11} + A_{12})\|\vartheta_1\|_1 = A_{13} + A_{11} + A_{12},\\
&B_0 = (a_{43} + a_{41} + a_{42})\|\varpi_4\|_1 = 0,\\
&\overline{A}_{20} = a_{33}\|\varpi_3\|_1 + 2B_0 + A_{23}\|\vartheta_2\|_1 + (A_{21} + A_{22})\|\vartheta_2\|_1 = A_{23} + A_{21} + A_{22},\\
&\overline{B}_{20} = (a_{31} + a_{32})\|\varpi_3\|_1 + 2B_0 + A_{23}\|\vartheta_2\|_1 + (A_{21} + A_{22})\|\vartheta_2\|_1 = A_{23} + A_{21} + A_{22}.
\end{aligned}$$

Then Theorem 3.1 implies that BVP (4.3) has at least one solution if one of the following items holds:

(i) $\mu_1\mu_2 < 1$;
(ii) $\mu_1\mu_2 = 1$ with $\overline{B}_{10}^{\mu_2}\overline{B}_{20} < 1$ or $\overline{B}_{20}^{\mu_1}\overline{B}_{10} < 1$;
(iii) $\mu_1\mu_2 > 1$ with

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{20}}{[\overline{A}_{10} + \overline{B}_{10}\overline{A}_{20}^{\mu_1}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_1}]^{\mu_2}} \geq \overline{B}_{20}$$

or

$$\frac{1}{\mu_1\mu_2 - 1}\frac{\overline{A}_{10}}{[\overline{A}_{20} + \overline{B}_{20}\overline{A}_{10}^{\mu_2}(\frac{\mu_1\mu_2}{\mu_1\mu_2-1})^{\mu_2}]^{\mu_1}} \geq \overline{B}_1.$$

Remark 4.1 In the above examples, f, g, ϕ, φ, ρ, ϱ, χ, and ψ are singular at 0. It is easily seen that the results in [1, 8, 24–27, 29–31] cannot be applied to solve BVP (4.1), BVP (4.2), and BVP (4.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Cahlon, B, Kulkarni, DM, Shi, P: Stepwise stability for the heat equation with a nonlocal constraint. SIAM J. Numer. Anal. **32**, 571-593 (1995)
2. Cannon, JR: The solution of the heat equation subject to the specification of energy. Q. Appl. Math. **21**, 155-160 (1963)
3. Choi, YS, Chan, KY: A parabolic equation with nonlocal boundary conditions arising from electrochemistry. Nonlinear Anal. **18**, 317-331 (1992)
4. Ewing, RE, Lin, T: A class of parameter estimation techniques for fluid flow in porous media. Adv. Water Resour. **14**, 89-97 (1991)

5. Shi, P, Shillor, M: Design of contact patterns in one dimensional thermoelasticity. In: Theoretical Aspects of Industrial Design. Society for Industrial and Applied Mathematics, Philadelphia (1992)
6. Samarski, AA: Some problems in the modern theory of differential equations. Differ. Uravn. **16**, 1221-1228 (1980)
7. Agarwal, RP: Boundary Value Problems for Higher Order Differential Equations. World Scientific, Singapore (1986)
8. Cupini, G, Macelli, C, Papalini, F: Heteroclinic solutions of boundary value problems on the real line involving general nonlinear differential operators. Differ. Integral Equ. **24**, 619-644 (2011)
9. Ge, W: Boundary Value Problems for Ordinary Differential Equations. Science Press, Beijing (2007)
10. Il'in, VA, Moiseev, EI: Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator. Differ. Equ. **23**, 979-987 (1987)
11. Rachunkova, I, Stanek, S, Tvrdy, M: Singularities and Laplacians in boundary value problems for nonlinear ordinary differential equations. In: Canada, A, Drabek, P, Fonde, A (eds.) Handbook of Differential Equations: Ordinary Differential Equations, vol. 3, pp. 606-723. Elsevier, Amsterdam (2006)
12. Kyoung, LE, Yong-Hoon, L: A global multiplicity result for two-point boundary value problems of p-Laplacian systems. Sci. China Math. **53**, 967-984 (2010)
13. Cheng, X, Zhang, Z: Positive solutions for a multi-parameter system of second-order ordinary differential equations. Sci. China Math. **54**, 959-972 (2011)
14. Gallardo, JM: Second order differential operators with integral boundary conditions and generation of analytic semigroups. Rocky Mt. J. Math. **30**, 1265-1291 (2000)
15. Karakostas, GL, Tsamatos, PC: Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary value problems. Electron. J. Differ. Equ. **2002**, 30 (2002)
16. Shi, P: Weak solution to evolution problem with a nonlocal constraint. SIAM J. Math. Anal. **24**, 46-58 (1993)
17. Yurchuk, NI: Mixed problem with an integral condition for certain parabolic equations. Differ. Equ. **22**, 1457-1463 (1986)
18. Denche, M, Marhoune, AL: High-order mixed-type differential equations with weighted integral boundary conditions. Electron. J. Differ. Equ. **2000**, 60 (2000)
19. Pulkina, LS: A non-local problem with integral conditions for hyperbolic equations. Electron. J. Differ. Equ. **1999**, 45 (1999)
20. Karakostas, GL, Tsamatos, PC: Existence results for some n-dimensional nonlocal boundary value problems. J. Math. Anal. Appl. **259**, 429-438 (2001)
21. Avramescu, C, Vladimirescu, C: Existence of homoclinic solutions to a nonlinear second order ODE. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **15**, 481-491 (2008)
22. Avramescu, C, Vladimirescu, C: Existence of solutions to second order ordinary differential equations having finite limits at $\pm\infty$. Electron. J. Differ. Equ. **2004**, 18 (2004)
23. Avramescu, C, Vladimirescu, C: Limits of solutions of a perturbed linear differential equation. Electron. J. Qual. Theory Differ. Equ. **2002**, 3 (2002)
24. Bianconi, B, Papalini, F: Non-autonomous boundary value problems on the real line. Discrete Contin. Dyn. Syst. **15**, 759-776 (2006)
25. Cabada, A, Cid, JA: Heteroclinic solutions for non-autonomous boundary value problems with singular Φ-Laplacian operators. Dynamical Systems, Differential Equations and Applications. 7th AIMS Conference, suppl. Discrete Contin. Dyn. Syst. **2009**, 118-122 (2009)
26. Calamai, A: Heteroclinic solutions of boundary value problems on the real line involving singular Φ-Laplacian operators. J. Math. Anal. Appl. **378**, 667-679 (2011)
27. Cupini, G, Macelli, C, Papalini, F: On the solvability of a boundary value problem on the real line. Bound. Value Probl. **2011**, 26 (2011)
28. Chandler-Wilde, SN, Zhang, B, Ross, CR: On the solvability of second kind integral equations on the real line. J. Math. Anal. Appl. **245**(1), 28-51 (2000)
29. Liu, Y: Solvability of boundary value problems for singular quasi-Laplacian differential equations on the whole line. Math. Model. Anal. **17**, 423-446 (2012)
30. Philos, CG, Purnaras, IK: A boundary value problem on the whole line to second order nonlinear differential equations. Georgian Math. J. **17**, 241-252 (2010)
31. Marcelli, C, Papalini, F: Heteroclinic connections for fully non-linear non-autonomous second-order differential equations. J. Differ. Equ. **241**, 160-183 (2007)
32. Karakostas, GL: Positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function. Electron. J. Differ. Equ. **2004**, 68 (2004)
33. Bebernes, J, Eberly, D: Mathematical Problems from Combustion Theory. Appl. Math. Sci., vol. 83. Springer, New York (1989)
34. Joseph, DD, Sparrow, EM: Nonlinear diffusion induced by nonlinear sources. Q. Appl. Math. **28**, 327-342 (1970)
35. Keller, H, Cohen, D: Some positone problems suggested by nonlinear heat generation. J. Math. Mech. **16**, 1361-1376 (1967)
36. Wang, J, Wang, ZW, Lu, LX, Zhu, Y, Wang, YG: Three-dimensional shock spectrum of critical component for nonlinear packaging system. Shock Vib. **18**(3), 437-445 (2011)
37. Yang, A, Miao, C, Ge, W: Solvability for second-order nonlocal boundary value problems with a p-Laplacian at resonance on a half line. Electron. J. Qual. Theory Differ. Equ. **2009**, 19 (2009)
38. Ge, W, Ren, J: An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian. Nonlinear Anal. **58**, 477-488 (2004)
39. Liu, Y, Wong, PJY: Unbounded solutions of BVP for second order ODE with p-Laplacian on the half line. Appl. Math. **58**(2), 179-204 (2013)
40. Yan, B, O'Regan, D, Agarwal, RP: Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. J. Comput. Appl. Math. **197**, 365-386 (2006)
41. Yan, B, O'Regan, D, Agarwal, RP: Multiple positive solutions for singular problems with mixed boundary data and derivative dependence. Math. Comput. Model. **45**, 404-430 (2007)
42. Yan, B, O'Regan, D, Agarwal, RP: Positive solutions for second order singular boundary value problems with derivative dependence on infinite intervals. Acta Appl. Math. **103**, 19-57 (2008)

43. Deren, FY, Hamal, NA: Second-order boundary value problems with integral boundary conditions on the real line. Electron. J. Differ. Equ. **2014**, 19 (2014)
44. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)