Li et al. *Boundary Value Problems* 2014, **2014**:197
http://www.boundaryvalueproblems.com/content/2014/1/197

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

General decay rate estimates for a semilinear parabolic equation with memory term and mixed boundary condition

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Abstract
This work is concerned with a mixed boundary value problem for a semilinear parabolic equation with a memory term. Under suitable conditions, we prove that the energy functional decays to zero as the time tends to infinity by the method of perturbation energy, in which the usual exponential and polynomial decay results are only special cases.

1 Introduction

Our main interest lies in the following semilinear heat equation with a memory term:

$$u_t - \Delta u + \int_0^t g(t-s)\,\mathrm{div}\big[a(x)\nabla u(s)\big]\,ds = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

subject to mixed boundary and initial conditions

$$-\frac{\partial u}{\partial \nu} + \int_0^t g(t-s)\big[a(x)\nabla u(s)\cdot \nu\big]\,ds = f(u), \quad (x,t) \in \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma_1 \times [0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbf{R}^n$ ($n \geq 1$) is a bounded domain with sufficiently smooth boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and Γ_0 and Γ_1 have positive measures, ν is the unit outward normal vector on $\partial\Omega$, and g, a, f, and u_0 are memory kernel, coefficient, nonlinear, and initial functions, respectively, satisfying appropriate conditions; see (H1)-(H4) in Section 2.

Many natural phenomena in engineering and physical science have been formulated with the nonlocal equation (1.1) as a mathematical model. For example, in the study of heat conduction in materials with memory, the classical Fourier law of the heat flux is replaced by the following form:

$$\vec{q} = -\lambda\left(\nabla u - \int_{-\infty}^t g(t-s)\nabla\big[a(x)u(x,s)\big]\,ds\right),$$

where u is the temperature, $\vec{q}$ the heat flux, λ the diffusion coefficient and the integral term represents the memory effect in the material. The memory kernel g is defined on $[0,\infty)$ and represents the negative derivative of the relation function of heat flux. The heat balance equation implies that $u(x,t)$ will satisfy (1.1), provided that the temperature is assumed to be known for $t \leq 0$. The study on this type of equations has drawn considerable attention; see [1–5]. From the mathematical point of view, one would expect the integral term in the equation above is dominated by the leading term $-\lambda\nabla u$. Hence, the theory of parabolic equations can be applied to this type of equations.

To motivate our work, let us recall some results on the global existence, blow-up solutions, and asymptotic properties of the initial boundary value problems for semilinear parabolic equations and systems with or without memory term. In the absence of the memory term ($g = 0$), there are many results on the global existence and finite time blow-up of the solutions for the semilinear parabolic equation; see the monographs [6, 7] and the survey papers [8–10]. Roughly summary, the global and nonglobal existences and the behavior of solutions depend on nonlinearity, dimension, initial data, and nonlinear boundary flux. Concerning systems, we refer to [11].

When a memory term exists ($g \neq 0$), Olmstead *et al.* [12] considered the non-Newtonian fluid equation

$$u_t - \int_{-\infty}^{t} g(t,s)u_{xx}(x,s)\,ds = au - u^3, \quad (x,t) \in (0,\pi) \times (0,\infty),$$

subject to homogeneous Dirichlet boundary condition, and discussed that the bifurcation behavior. In [13], Bellout studied the following equation:

$$u_t - \Delta u - \int_0^t (u+\lambda)^p\,ds = f(x), \quad (x,t) \in \Omega \times (0,\infty),$$

with homogeneous Dirichlet boundary condition, where $f(x) \geq 0$ is a smooth function and $\lambda > 0$. The author established the existence and the uniqueness of the local classical solution, and obtained some criteria for solutions to blow up in a finite time. Moreover, he obtained some results on the blow-up points under some suitable assumptions. In [14], Yamada investigated the stability properties of the global solutions of the following nonlocal Volterra diffusion equation:

$$u_t - \Delta u + \int_0^t g(t-s)u(x,s)\,ds = (a - bu)u, \quad (x,t) \in \Omega \times (0,\infty).$$

Moreover, there have also been published many other results for single equations with memory. We refer the readers to [15, 16] and the references therein. Concerning systems, similar examples exist in the works of Pao [17] and Yamada [18], as well as others.

Recently, Messaoudi [19] studied the semilinear heat equation with a source term of power form and homogeneous Dirichlet boundary condition

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty),$$

where the relaxation function $g : \mathbf{R}^+ \to \mathbf{R}^+$ is a bounded C^1-function and $p > 2$, and proved the existence of a blow-up solution with positive initial energy by the convexity method.

Later, Fang and Sun [20] improved the results of [19], when $|u|^{p-2}u$ is replaced with a fully nonlinear source term $f(u)$. In [21], Berrimi and Messaoudi considered the quasilinear parabolic system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s) = 0, \quad (x,t) \in \Omega \times (0,\infty),$$

subject to homogeneous Dirichlet boundary condition, and proved that if $A(t) \in C(\mathbf{R}^+)$ a bounded square matrix such that

$$\bigl(A(t)v, v\bigr) \geq c_0|v|^2, \quad \forall t \in \mathbf{R}^+, v \in \mathbf{R}^n,$$

then the solution with small initial energy decays exponentially for $m = 2$ and polynomially for $m > 2$. Thereafter, Messaoudi and Tellab [22] established a general decay result from which the usual exponential and polynomial decay results are only special cases.

On the other hand, for the uniform decay of solutions of the viscoelastic equation with a nonlinear source or variable diffusion coefficient $a(x)$, we reader to [23, 24].

Accessing the relevant papers, one can find that research on the asymptotic behavior of the solution for the nonlocal semilinear parabolic equation (1.1) with mixed boundary conditions has not been started yet. Very recently, for mixed boundary problem (1.1)-(1.4) with generalized Lewis functions, Fang and Qiu [25] proved the existence and uniqueness of global solution and the energy functional decays exponentially or polynomially to zero as the time tends to infinity by the technique of Lyapunov functional. Motivated by this observation, we intend to study the generalized property of energy decay for the initial mixed boundary value problem (1.1)-(1.4) using the technique of perturbation energy.

The rest of our paper is organized as follows: In Section 2, we present some assumptions, lemmas, and an energy functional, and give the energy decay results in Section 3.

2 Preliminaries

Throughout this paper, We use the standard Lebesgue space $L^p(\Omega)$, $L^p(\Gamma_0)$ and the Sobolev space $H^1(\Omega)$, $H^1(\Gamma_0)$ with their usual scalar products and norms. To simplify the notations, we denote $\|u\|_{L^p(\Omega)}$ and $\|u\|_{L^p(\Gamma_0)}$ by $\|u\|_p$ and $\|u\|_{p,\Gamma_0}$, respectively.

We give the following general hypotheses on the memory kernel g, coefficient a, nonlinearity function f, and initial function u_0:

(H1) $g : \mathbf{R}^+ \to \mathbf{R}^+$ *is a nonincreasing differentiable function such that* $g(0) > 0$, *and there exists a differentiable function* ξ *satisfying*

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0,$$

where $\xi(t) > 0$, $\xi'(t) \leq 0$, $\forall t > 0$ *and* $\int_0^\infty \xi(t)\,dt = +\infty$.

(H2) $a : \Omega \to \mathbf{R}^+$ *is a nonnegative bounded function such that* $a(x) \geq a_0 > 0$ *and*

$$1 - \|a\|_{L^\infty} \int_0^\infty g(s)\,ds = l > 0.$$

(H3) *The function* $f : \mathbf{R} \to \mathbf{R}$ *is Lipschitz continuous and satisfies*

$$f(s)s \geq 2F(s) \geq 0, \quad s \in \mathbf{R},$$

where $F(u) := \int_0^u f(s)\,ds$.

(H4) *(Compatibility condition) The initial function u_0 satisfies*

$$u_0 \in V \cap H^2(\Omega), \quad -\frac{\partial u_0}{\partial \nu} = f(u_0),$$

where the set $V = \{u | u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\}$.

Remark 1 There are many functions satisfying (H1) and (H2). Examples of such functions are

$$\begin{aligned} &g_1(t) = c(1+t)^{\nu}, \quad \text{for } c > 0, \nu < -1; \\ &g_2(t) = ce^{-d(t+1)^{\nu}}, \quad \text{for } c > 0, d > 0 \text{ and } 0 < \nu \le 1; \\ &g_3(t) = c\frac{[\ln(1+t)]^{\nu}}{1+t}, \quad \text{for } c > 0, \nu < -1, \end{aligned}$$

with $a(x) = \frac{1}{x^2+1}$.

Remark 2 The condition $1 - \|a\|_{L^\infty} \int_0^\infty g(s)\,ds = l > 0$ is necessary to guarantee the parabolicity of problem (1.1)-(1.4).

In order to define an energy functional $E(t)$ of problem (1.1)-(1.4), we give the following computation:

Multiplying (1.1) by u_t, integrating the result over Ω, and using Green's formula, we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_\Omega k(x,t)|\nabla u|^2\,dx + \int_\Omega (g \circ \nabla u)\,dx + 2\int_{\Gamma_0} F(u)\,d\Gamma\right) \\ &\quad + \int_\Omega u_t^2\,dx + \frac{1}{2}\int_\Omega g(t)a(x)|\nabla u|^2\,dx - \frac{1}{2}\int_\Omega \left(g' \circ \nabla u\right)dx = 0, \end{aligned} \tag{2.1}$$

where we apply the fact that

$$\begin{aligned} &\int_\Omega \int_0^t g(t-s)a(x)\nabla u(s)\nabla u_t(t)\,ds\,dx \\ &\quad = -\frac{1}{2}\int_\Omega g(t)a(x)\left|\nabla u(t)\right|^2 dx + \frac{1}{2}\int_\Omega \left(g' \circ \nabla u\right)dx \\ &\qquad - \frac{1}{2}\frac{d}{dt}\int_\Omega \left[(g \circ \nabla u) - \int_0^t g(s)a(x)\left|\nabla u(t)\right|^2 ds\right] dx. \end{aligned}$$

Therefore, we can define an energy functional $E(t)$ of problem (1.1)-(1.4) as

$$E(t) = \frac{1}{2}\int_\Omega k(x,t)|\nabla u|^2\,dx + \frac{1}{2}\int_\Omega (g \circ \nabla u)\,dx + \int_{\Gamma_0} F(u)\,d\Gamma, \tag{2.2}$$

where

$$\begin{aligned} &1 \ge k(x,t) = 1 - a(x)\int_0^t g(s)\,ds > 1 - \left\|a(x)\right\|_{L^\infty}\int_0^\infty g(s)\,ds > 0, \\ &(g \circ \nabla u) = \int_0^t g(t-s)\left|\sqrt{a(x)}\left(\nabla u(t) - \nabla u(s)\right)\right|^2 ds. \end{aligned}$$

We recall the trace Sobolev embedding $V \hookrightarrow L^2(\Gamma_0)$, and the embedding inequality $\|u\|_{2,\Gamma_0} \le B_* \|\nabla u\|_2$, where B_* is the optimal constant.

One can have the following nonincreasing property on $E(t)$.

Lemma 1 *The energy functional $E(t)$ is nonnegative and satisfies*

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_\Omega \left(g' \circ \nabla u\right) dx - \frac{1}{2}\int_\Omega g(t)\left|\sqrt{a(x)}\nabla u(t)\right|^2 dx - \int_\Omega |u_t|^2 \, dx \le 0. \tag{2.3}$$

Secondly, we give the definition of a weak solution of (1.1)-(1.4).

Definition 1 A weak solution of (1.1)-(1.4) is a function $u \in C([0,T];V) \cap C^1([0,T];L^2(\Omega))$, $T > 0$, which satisfies

$$\begin{aligned} &\int_0^t \int_\Omega u_t \phi(x,s)\,dx\,ds + \int_0^t \int_\Omega \nabla u \nabla \phi(x,s)\,dx\,ds \\ &\quad - \int_0^t \int_\Omega \int_0^s a(x) g(t-s) \nabla u(\tau) \nabla \phi(x,s)\,d\tau\,dx\,ds - \int_0^t \int_{\Gamma_0} f(u)\phi(x,s)\,d\Gamma\,ds = 0, \end{aligned}$$

for all $t \in [0,T)$ and all $\phi \in C([0,T];V)$.

Remark 3 By using a similar argument in [25], one can show the existence and uniqueness of the global solution to problem (1.1)-(1.4) with assumptions (H1)-(H4) by the technique of Galerkin, the contraction mapping principle, and a continuation argument.

3 General energy decay rate

In this section, we establish the estimates of general uniform energy decay rates and introduce a perturbed energy functional

$$G(t) = E(t) + \varepsilon_1 \varphi(t) + \varepsilon_2 \psi(t),$$

to show the uniform decay of the solution, where ε_1 and ε_2 are positive constants, and

$$\varphi(t) = \frac{1}{2}\int_\Omega u^2\,dx, \tag{3.1}$$

$$\psi(t) = \int_\Omega u \int_0^t g(t-s) a(x) u(s)\,ds\,dx. \tag{3.2}$$

We can choose small ε_1 and ε_2, if needed, so that

$$\frac{1}{2}E(t) \le G(t) \le \frac{3}{2}E(t). \tag{3.3}$$

Indeed, through a simple calculation, we deduce that

$$\begin{aligned} \left|\varphi(t)\right| &= \frac{1}{2}\int_\Omega u^2\,dx \le \frac{1}{2}C_*^2 \|\nabla u\|_2^2 \le c_1 E(t), \\ \left|\psi(t)\right| &\le \left|\int_\Omega u \int_0^t g(t-s) a(x)\big(u(s) - u(t)\big)\,ds\,dx\right| + \left|\int_\Omega u \int_0^t g(t-s) a(x) u(t)\,ds\,dx\right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\delta\|u\|_2^2 + \frac{C_*^2(1-l)}{2\delta}\left|\int_\Omega (g\circ\nabla u)\,dx\right| + \left|\int_\Omega\int_0^t g(t-s)a(x)\left|u(t)\right|^2\,ds\,dx\right| \\
&\le \left(\frac{1}{2}\delta + 1 - l\right)C_*^2\|\nabla u\|_2^2 + \frac{C_*^2(1-l)}{2\delta}\int_\Omega (g\circ\nabla u)\,dx \le c_2E(t),
\end{aligned}
$$

where C_* is an embedding constant satisfying the Poincaré inequality, $\|u\|_2 \le C_*\|\nabla u\|_2$. Hence, we have

$$
(1 - \varepsilon_1c_1 - \varepsilon_2c_2)E(t) \le G(t) \le (1 + \varepsilon_1c_1 + \varepsilon_2c_2)E(t).
$$

Thus, selecting $\varepsilon_1 = \varepsilon_2 = \frac{1}{2(c_1+c_2)}$, we get (3.3).

We now give precise estimates of the derivatives $\varphi'(t)$ and $\psi'(t)$ which will be used in the proof of our main results.

Lemma 2 *Suppose that assumptions* (H1)-(H3) *hold and* $u_0 \in V$. *If* u *is a solution of problem* (1.1)-(1.4), *then* $\varphi(t)$ *satisfies*

$$
\varphi'(t) \le \frac{1-l}{2l}\int_\Omega (g\circ\nabla u)\,dx - \frac{l}{2}\|\nabla u\|_2^2 - \int_{\Gamma_0} f(u)u\,d\Gamma. \tag{3.4}
$$

Proof of Lemma 2 From (3.1) and (1.1), we have

$$
\begin{aligned}
\varphi'(t) &= \int_\Omega uu_t\,dx \\
&= -\|\nabla u\|_2^2 + \int_\Omega a(x)\nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_{\Gamma_0} f(u)u\,d\Gamma.
\end{aligned} \tag{3.5}
$$

The second term in the right-hand side of (3.5) is

$$
\begin{aligned}
&\int_\Omega a(x)\nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&\quad\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t a(x)g(t-s)\left|\nabla u(s)\right|\,ds\right)^2dx \\
&\quad\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t a(x)g(t-s)\left[\left|\nabla u(s)-\nabla u(t)\right| + \left|\nabla u(t)\right|\right]ds\right)^2dx \\
&\quad\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)\int_\Omega\left(\int_0^t a(x)g(t-s)\left|\nabla u(t)\right|\,ds\right)^2dx \\
&\qquad + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t a(x)g(t-s)\left|\nabla u(s)-\nabla u(t)\right|\,ds\right)^2dx \\
&\quad\le \frac{1}{2}\left[1 + (1+\eta)(1-l)^2\right]\|\nabla u\|_2^2 + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)\int_\Omega (g\circ\nabla u)\,dx.
\end{aligned}
$$

Then we can deduce

$$
\varphi'(t) \le \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)\int_\Omega (g\circ\nabla u)\,dx + \frac{1}{2}\left[(1+\eta)(1-l)^2 - 1\right]\|\nabla u\|_2^2 - \int_{\Gamma_0} f(u)u\,d\Gamma.
$$

By taking $\eta = \frac{l}{1-l}$ in the inequality above, we obtain (3.4), which completes the proof. □

Lemma 3 *Suppose that assumptions* (H1)-(H3) *hold and* $u_0 \in V$. *If* u *is a solution of problem* (1.1)-(1.4), *then* $\psi(t)$ *satisfies*

$$\begin{aligned}
\psi'(t) \le{}& \left[1 - \frac{l}{2} + (1-l)^2 + \frac{1}{2}C_*^2 + C_*^2 g(0)\|a\|_{L^\infty} + \frac{1}{2}\left(C_*^2 + B_*^2\right)(1+\delta)(1-l)^2\right]\|\nabla u\|_2^2 \\
&+ \left[\frac{1-l}{2l} + \frac{1-l}{2}B_*^2 g(0)\left(1 + \frac{1}{\delta}\right)\right]\int_\Omega (g \circ \nabla u)\,dx \\
&- \frac{1}{2}C_*^2\left(1 + \frac{1}{\delta}\right)\|a\|_{L^\infty}\int_\Omega \left(g' \circ \nabla u\right)dx + \frac{1}{2}\int_{\Gamma_0}\left|f(u)\right|^2\,d\Gamma.
\end{aligned} \tag{3.6}$$

Proof of Lemma 3 From (3.2) and (1.1), we have

$$\begin{aligned}
\psi'(t) ={}& \int_\Omega u_t \int_0^t g(t-s)a(x)u(s)\,ds\,dx + \int_\Omega u \int_0^t g'(t-s)a(x)u(s)\,ds\,dx \\
&+ \int_\Omega u^2 g(0)a(x)\,dx \\
={}& -\int_\Omega \nabla u(t)\int_0^t g(t-s)a(x)\nabla u(s)\,ds\,dx + \int_\Omega \left(\int_0^t g(t-s)a(x)\nabla u(s)\,ds\right)^2 dx \\
&+ \int_\Omega u^2 g(0)a(x)\,dx - \int_{\Gamma_0} f(u)\int_0^t g(t-s)a(x)u(s)\,ds\,d\Gamma \\
&+ \int_\Omega u \int_0^t g'(t-s)a(x)u(s)\,ds\,dx.
\end{aligned} \tag{3.7}$$

Now, we estimate the five terms in the right-hand side of (3.7):

$$\begin{aligned}
&-\int_\Omega \nabla u(t)\int_0^t g(t-s)a(x)\nabla u(s)\,ds\,dx \\
&\quad\le \frac{1}{2}\left[1 + (1+\eta)(1-l)^2\right]\|\nabla u\|_2^2 + \frac{1}{2}\left(1 + \frac{1}{\eta}\right)(1-l)\int_\Omega (g \circ \nabla u)\,dx,
\end{aligned} \tag{3.8}$$

$$\int_\Omega \left(\int_0^t g(t-s)a(x)\nabla u(s)\,ds\right)^2 dx \le (1-l)^2\|\nabla u\|_2^2, \tag{3.9}$$

$$\begin{aligned}
&\int_\Omega u \int_0^t g'(t-s)a(x)u(s)\,ds\,dx \\
&\quad\le \frac{1}{2}C_*^2\|\nabla u\|_2^2 + \frac{1}{2}C_*^2\int_\Omega\left(\int_0^t g'(t-s)a(x)\left|\nabla u(s)\right|ds\right)^2 dx \\
&\quad\le \frac{1}{2}C_*^2\|\nabla u\|_2^2 + \frac{1}{2}C_*^2\int_\Omega\left(\int_0^t g'(t-s)a(x)\left(\left|\nabla u(s) - \nabla u(t)\right| + \left|\nabla u(t)\right|\right)ds\right)^2 dx \\
&\quad\le \frac{1}{2}C_*^2\|\nabla u\|_2^2 + \frac{1}{2}C_*^2(1+\delta)\int_\Omega\left(\int_0^t g'(t-s)a(x)\left|\nabla u(t)\right|ds\right)^2 dx \\
&\qquad+ \frac{1}{2}C_*^2\left(1 + \frac{1}{\delta}\right)\int_\Omega\left(\int_0^t g'(t-s)a(x)\left|\nabla u(s) - \nabla u(t)\right|ds\right)^2 dx \\
&\quad\le \left[\frac{1}{2}C_*^2 + \frac{1}{2}C_*^2(1+\delta)\|a\|_{L^\infty}^2 g^2(0)\right]\|\nabla u\|_2^2 \\
&\qquad- \frac{1}{2}C_*^2\left(1 + \frac{1}{\delta}\right)\|a\|_{L^\infty} g(0)\int_\Omega\left(g' \circ \nabla u\right)dx,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&-\int_{\Gamma_0} f(u)\int_0^t g(t-s)a(x)u(s)\,ds\,d\Gamma \\
&\quad\le \frac{1}{2}\int_{\Gamma_0}\bigl|f(u)\bigr|^2\,d\Gamma+\frac{1}{2}B_*^2(1+\delta)(1-l)^2\bigl\|\nabla u(t)\bigr\|_2^2 \\
&\qquad+\frac{1}{2}B_*^2\left(1+\frac{1}{\delta}\right)(1-l)\int_\Omega (g\circ\nabla u)\,dx,
\end{aligned}\tag{3.11}$$

$$\int_\Omega u^2 g(0)a(x)\,dx\le C_*^2 g(0)\|a\|_{L^\infty}\|\nabla u\|_2^2.\tag{3.12}$$

Combining inequalities (3.7)-(3.12), we deduce

$$\begin{aligned}
\psi'(t)\le{}&\biggl[\frac{1}{2}+\frac{1}{2}(1+\eta)(1-l)^2+(1-l)^2+\frac{1}{2}C_*^2+C_*^2 g(0)\|a\|_{L^\infty} \\
&+\frac{1}{2}\bigl(C_*^2+B_*^2\bigr)(1+\delta)(1-l)^2\biggr]\|\nabla u\|_2^2 \\
&+\biggl[\frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)+\frac{1}{2}B_*^2 g(0)\left(1+\frac{1}{\delta}\right)(1-l)\biggr]\int_\Omega (g\circ\nabla u)\,dx \\
&-\frac{1}{2}C_*^2\left(1+\frac{1}{\delta}\right)\|a\|_{L^\infty}\int_\Omega \bigl(g'\circ\nabla u\bigr)\,dx+\frac{1}{2}\int_{\Gamma_0}\bigl|f(u)\bigr|^2\,d\Gamma.
\end{aligned}$$

By taking $\eta=\frac{l}{1-l}$ in the inequality above, we get (3.6), which completes the proof. □

Using the above estimates (3.4) and (3.6), we can obtain the general uniform decay estimate of the energy functional.

Theorem 1 *Suppose that assumptions* (H1)-(H4) *hold and* $u_0\in V$. *Then, for each* $t_0>0$, *there exist two positive constants* c *and* λ *for which the solution of* (1.1)-(1.4) *satisfies*

$$E(t)\le ce^{-\lambda\int_{t_0}^t \xi(s)\,ds},\quad \forall t\ge t_0.$$

Proof of Theorem 1 Since $g(t)$ is positive, we have

$$\int_0^t g(s)\,ds\ge\int_0^{t_0} g(s)\,ds=g_0>0,\quad t\ge t_0.$$

By (2.1), (H1), and Lemmas 2 and 3, we have

$$\begin{aligned}
G'(t)={}&E'(t)+\varepsilon_1\varphi'(t)+\varepsilon_2\psi'(t) \\
\le{}&\frac{1}{2}\int_\Omega\bigl(g'\circ\nabla u\bigr)\,dx-\frac{1}{2}\int_\Omega g(t)\bigl|\sqrt{a(x)}\nabla u(t)\bigr|^2\,dx \\
&-\int_\Omega |u_t|^2\,dx+\frac{1-l}{2l}\varepsilon_1\int_\Omega (g\circ\nabla u)\,dx-\frac{l}{2}\varepsilon_1\|\nabla u\|_2^2 \\
&-\varepsilon_1\int_{\Gamma_0} f(u)u\,d\Gamma+\biggl[1-\frac{l}{2}+(1-l)^2+\frac{1}{2}C_*^2+C_*^2 g(0)\|a\|_{L^\infty} \\
&+\frac{1}{2}\bigl(C_*^2+B_*^2\bigr)(1+\delta)(1-l)^2\biggr]\varepsilon_2\|\nabla u\|_2^2
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1-l}{2l}+\frac{1-l}{2}B_*^2g(0)\left(1+\frac{1}{\delta}\right)\right]\varepsilon_2\int_\Omega (g\circ\nabla u)\,dx \\
&-\frac{1}{2}C_*^2\left(1+\frac{1}{\delta}\right)\|a\|_{L^\infty}\varepsilon_2\int_\Omega \left(g'\circ\nabla u\right)dx+\frac{1}{2}\varepsilon_2\int_{\Gamma_0}\left|f(u)\right|^2\,d\Gamma \\
\le{}& -\int_\Omega |u_t|^2\,dx-\varepsilon_1\int_{\Gamma_0} f(u)u\,d\Gamma \\
&+\left\{\frac{1-l}{2l}\varepsilon_1+\left[1-\frac{l}{2}+(1-l)^2+\frac{1}{2}C_*^2+C_*^2g(0)\|a\|_{L^\infty}\right.\right. \\
&\left.\left.+\frac{1}{2}\left(C_*^2+B_*^2\right)(1+\delta)(1-l)^2\right]\varepsilon_2-a_0g_0\right\}\|\nabla u\|_2^2 \\
&+\frac{1-l}{2l}\left(\varepsilon_1+\varepsilon_2+lB_*^2g(0)\left(1+\frac{1}{\delta}\right)\varepsilon_2\right)\int_\Omega (g\circ\nabla u)\,dx \\
&+\left[\frac{1}{2}-\frac{1}{2}C_*^2\left(1+\frac{1}{\delta}\right)\|a\|_{L^\infty}\varepsilon_2\right]\int_\Omega \left(g'\circ\nabla u\right)dx+\frac{1}{2}\varepsilon_2\int_{\Gamma_0}\left|f(u)\right|^2\,d\Gamma,
\end{aligned}
$$

for all $t \ge t_0$. By choosing ε_1 and ε_2 so that

$$
\begin{aligned}
&\frac{1-l}{2l}\varepsilon_1+\left[1-\frac{l}{2}+(1-l)^2+\frac{1}{2}C_*^2+C_*^2g(0)\|a\|_{L^\infty}\right. \\
&\quad\left.+\frac{1}{2}\left(C_*^2+B_*^2\right)(1+\delta)(1-l)^2\right]\varepsilon_2-a_0g_0<0, \\
&\frac{1}{2}-\frac{1}{2}C_*^2\left(1+\frac{1}{\delta}\right)\|a\|_{L^\infty}\varepsilon_2>0,
\end{aligned}
$$

we deduce

$$
G'(t)\le -cE(t)+C\int_\Omega (g\circ\nabla u)\,dx,\quad \forall t\ge t_0. \tag{3.13}
$$

Multiplying (3.13) by $\xi(t)$, one can see that

$$
\begin{aligned}
\xi(t)G'(t)&\le -c\xi(t)E(t)+C\xi(t)\int_\Omega (g\circ\nabla u)\,dx \\
&\le -c\xi(t)E(t)-C\int_\Omega \left(g'\circ\nabla u\right)dx \\
&\le -c\xi(t)E(t)-CE'(t),
\end{aligned}
$$

from (3.13) and (H2). Let $L(t)=\xi(t)G(t)+CE(t)$ and then $L(t)\sim E(t)$. Hence, we arrive at

$$
L'(t)\le -c\xi(t)E(t)\le -\lambda\xi(t)L(t),\quad \forall t\ge t_0, \tag{3.14}
$$

by (H1) and (H3), where λ is a positive constant. A simple integration leads to

$$
L(t)\le L(t_0)e^{-\lambda\int_{t_0}^t \xi(s)\,ds},\quad \forall t\ge t_0.
$$

Again, employing $L(t)$ is equivalent to $E(t)$ leads to,

$$
E(t)\le ce^{-\lambda\int_{t_0}^t \xi(s)\,ds},\quad \forall t\ge t_0, \tag{3.15}
$$

where c is a positive constant. This completes the proof. □

Remark 4 The exponential and polynomial decay estimates are only particular cases of Theorem 1. We illustrate the energy decay rate:

(i) If

$$\xi(t) = \alpha, \quad \alpha > 0,$$

then (3.15) gives the exponential decay estimate

$$E(t) \leq ce^{-\lambda\alpha t}.$$

Similarly, if

$$\xi(t) = \alpha(1+t)^{-1}, \quad \alpha > 0,$$

then we obtain the polynomial decay estimate

$$E(t) \leq c(1+t)^{-\lambda\alpha}.$$

(ii) If

$$g(t) = \alpha e^{-\alpha_1(\ln(1+t))^\nu}$$

with $\alpha, \alpha_1 > 0$ and $\nu > 1$, then H_1 holds for

$$\xi(t) = \frac{\alpha_1\nu(\ln(1+t))^{\nu-1}}{1+t}.$$

Thus, (3.15) gives the estimate

$$E(t) \leq ce^{-\lambda\alpha_1(\ln(1+t))^\nu}.$$

(iii) If

$$g(t) = \frac{\alpha}{(2+t)^\nu(\ln(2+t))^{\alpha_1}}$$

with $\alpha > 0$, $\nu > 1$ and $\alpha_1 \in R$ (or $\nu = 1$ and $\alpha_1 > 1$), then for

$$\xi(t) = \frac{\nu(\ln(2+t)) + \alpha_1}{(2+t)(\ln(2+t))},$$

we obtain from (3.15)

$$E(t) \leq \frac{c}{[(2+t)^\nu(\ln(2+t))^{\alpha_1}]^\lambda}.$$

Remark 5 It can be seen that the estimate (3.15) is also true for $t \in [0, t_0]$ by the continuity and boundedness of $E(t)$ and $\xi(t)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Acknowledgements
This work is supported by the Natural Science Foundation of Shandong Province of China (ZR2012AM018) and the Fundamental Research Funds for the Central Universities (No. 201362032). The authors would like to sincerely thank all the reviewers for their insightful and constructive comments.

Received: 24 February 2014 Accepted: 1 August 2014 Published online: 26 September 2014

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doi:10.1186/s13661-014-0197-0
Cite this article as: Li et al.: **General decay rate estimates for a semilinear parabolic equation with memory term and mixed boundary condition.** *Boundary Value Problems* 2014 **2014**:197.