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Shrinking projection methods for solving split equilibrium problems and fixed point problems for asymptotically nonexpansive mappings in Hilbert spaces

Uamporn Witthayarat[1], Afrah A N Abdou[2*] and Yeol Je Cho[2,3*]

*Correspondence:
aabdou@kau.edu.sa;
yjcho@gnu.ac.kr
[2]Department of Mathematics, King Abdulaziz University, Jeddah, 21589, Saudi Arabia
[3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, 660-701, Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we propose a new iterative sequence for solving common problems which consist of split equilibrium problems and fixed point problems for asymptotically nonexpansive mappings in the framework of Hilbert spaces and prove some strong convergence theorems of the generated sequence $\{x_n\}$ by the shrinking projection method. Our results improve and extend the previous results given in the literature.

MSC: 54E70; 47H25

Keywords: split equilibrium problem; asymptotically nonexpansive mapping; fixed point problem; Hilbert space

1 Introduction

Throughout this paper, let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all positive integers, respectively. Let H be a real Hilbert space and C be a nonempty closed convex subset of H.

A mapping $T : C \times C \to \mathbb{R}$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \le k_n\|x - y\|$$

for all $x, y \in C$. It is easy to see that, if $k_n \equiv 1$, then T is said to be *nonexpansive*. We denote the set of fixed point of T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. There are many iterative methods for solving a fixed point problem corresponding to an asymptotically nonexpansive mapping (see also [1–3]).

Recall that a Hilbert space H satisfies *Opial's condition* [4], that is, for any subsequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the following inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in H$ with $y \neq x$. Furthermore, a Hilbert space H has a Kadec-Klee property, *i.e.*, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. In fact, from

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2,$$

we can conclude that a Hilbert space has a Kadec-Klee property.

In 1994, Blum and Oettli [5] introduced the *equilibrium problem* which is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

They denoted the solution set of problem (1.1) as $EP(F)$. Since the well-known problems were variational problems, complementary problems, fixed point problems, saddle point problems and other problems proposed from the equilibrium problem, it has become the most attractive topic for many mathematicians [6–8]. They have widely spread its applications to other applied disciplines including physics, chemistry, economics and engineering (see, for example, [9–12]).

In 1997, Combettes and Hirstoaga [13] proposed an iterative method for solving problem (1.1) by the assumption that $EP(F) \neq \emptyset$. Moreover, there are many new iteratively generated sequences for solving this problem together with fixed point problems (see [14–17]).

Later, the so-called *split equilibrium problem* was introduced (shortly, *SEP*). Let H_1, H_2 be two real Hilbert spaces. Let C, Q be closed convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Further, let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions. The *SEP* is to find the element $x^* \in C$ such that

$$F_1(x^*, y) \geq 0 \quad \text{for all } y \in C \tag{1.2}$$

and such that

$$Ax^* \in Q \text{ solves } F_2(Ax^*, v) \geq 0 \quad \text{for all } v \in Q. \tag{1.3}$$

The solution sets of problems (1.2) and (1.3) are symbolized by $EP(F_1)$ and $EP(F_2)$, respectively. Therefore, we denote $\Omega = \{v \in C : v \in EP(F_1) \text{ such that } Av \in EP(F_2)\}$ as the solution set of *SEP*.

Clearly, the *SEP* contains two equilibrium problems, that is, we find out the solution of one equilibrium problem, *i.e.*, its image under a given bounded linear operator, must be the solution of another equilibrium problem. In order to find a common solution of equilibrium problems, it has been mostly considered in the same spaces. However, we normally found that, in the real-life problems, it may be considered in different spaces. That is how the *SEP* works very well for this case (see, for example, [18]). Moreover, the split variational inequality problem (shortly, *SVIP*) is its special case, which is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0 \quad \text{for all } x \in C, \tag{1.4}$$

and corresponding to

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^*\rangle \geq 0 \quad \text{for all } y \in Q, \tag{1.5}$$

where $f : H_1 \to H_1$ and $g : H_1 \to H_2$ are nonlinear mappings and $A : H_1 \to H_2$ is a bounded linear operator (see [19]).

In 2012, He [18] proposed the new algorithm for solving a split equilibrium problem and investigated the convergence behavior in several ways including both weak and strong convergence. Moreover, they gave some examples and mentioned that there exist many SEPs, and the new methods for solving it further need to be explored in the future. Later, in 2013, Kazmi and Rizvi [20] considered the iterative method to compute the common approximate solution of a split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping in the framework of real Hilbert spaces. They generated the sequence iteratively as follows:

$$\begin{cases} u_n = J_{r_n}^{F_1}(I + \gamma A^*(J_{r_n}^{F_2} - I)A)x_n, \\ y_n = P_C(u_n - \lambda_n D u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n \end{cases} \tag{1.6}$$

for each $n \geq 0$, where $A : H_1 \to H_2$ is a bounded linear operator, $D : C \to H_1$ is a τ-inverse strongly monotone mapping, $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions. They found that, under the sufficient conditions of r_n, λ_n, γ, β_n and γ_n, the generated sequence $\{x_n\}$ converges strongly to a common solution of all mentioned problems.

Recently, in 2014, Bnouhachem [21] introduced a new iterative method for solving split equilibrium problem and hierarchical fixed point problems by defining the sequence $\{x_n\}$ as follows:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I + \gamma A^*(T_{r_n}^{F_2} - I)A)x_n, \\ y_n = \beta_n S x_n + (1 - \beta_n)u_n, \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))] \end{cases} \tag{1.7}$$

for each $n \geq 0$, where S, T are nonexpansive mappings, $F : C \to C$ is a k-Lipschitz mapping and η-strongly monotone, $U : C \to C$ is a τ-Lipschitz mapping. Also, they proved some strong convergence theorems for the proposed iteration under some appropriate conditions.

In this paper, motivated and inspired by the results [18, 20, 21] and the recent works in this field, we introduce the shrinking projection method for solving split equilibrium problems and fixed point problems for asymptotically nonexpansive mappings in the framework of Hilbert spaces and prove some strong convergence theorems for the proposed new iterative method. In fact, our results improve and extend the results given by some authors.

2 Preliminaries

In this section, we recall some concepts including the assumption which will be needed for the proof of our main result.

Let H be a Hilbert space and C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_c x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* from H onto C. It is well known that P_C is a firmly nonexpansive mapping from H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0$$

for all $y \in C$. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in H$. Moreover, we can investigate that, for each $\lambda \in (0, 2\alpha]$, $I - \lambda A$ is a nonexpansive mapping of C into H (see [22]).

Lemma 2.1 *In a Hilbert space H, the following identity holds*:

$$\big\|\lambda x + (1 - \lambda)y\big\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

Lemma 2.2 [23] *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the following properties*:

(1) $x_n \rightharpoonup z$;

(2) $Tx_n - x_n \to 0$.

Then $z \in F(T)$.

Assumption 2.3 [24] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.4 [24] *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *For any $x \in H$ and $r > 0$, define a mapping $T_r^F : H \to C$ by*

$$T_r^F(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}.$$

Then T_r^F is well defined and the following hold:

(1) *T_r^F is single-valued*;

(2) T_r^F *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \le \langle T_r^F x - T_r^F y, x - y\rangle;$$

(3) $F(T_r^F) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

3 Main results

In this section, we prove some strong convergence theorems of an iterative algorithm for solving a split equilibrium together with a fixed point problem revolving an asymptotically nonexpansive mapping in the framework of Hilbert spaces.

Theorem 3.1 *Let* H_1, H_2 *be two real Hilbert spaces and* C, Q *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *be two bifunctions satisfying conditions* (A1)-(A4) *and* F_2 *be upper semi-continuous in the first argument. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping and* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $F(T) \cap \Omega \neq \emptyset$, *where* $\Omega = \{v \in C : v \in EP(F_1)$ *such that* $Av \in EP(F_2)\}$, *and let* $x_0 \in C$. *For* $C_1 = C$ *and* $x_1 = P_{C_1}x_0$, *define a sequence* $\{x_n\}$ *iteratively as follows*:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{3.1}$$

for each $n \ge 1$, *where* $\theta_n = (1 - \alpha_n)(k_n^2 - 1)\sup\{\|x_n - z\|^2 : z \in \Omega\}$, $0 \le \alpha_n \le a < 1$ *for all* $n \in \mathbb{N}$, $0 < b \le r_n < \infty$, $\gamma \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Then the sequence* $\{x_n\}$ *generated by* (3.1) *strongly converges to a point* $z_0 \in F(T) \cap \Omega$.

Proof First of all, we investigate that, for each $n \in \mathbb{N}$, $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{2L}$-inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $(I - T_{r_n}^{F_2})$ is $\frac{1}{2}$-inverse strongly monotone, it follows that

$$\begin{aligned} &\|A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay\|^2 \\ &\quad = \langle A^*(I - T_{r_n}^{F_2})(Ax - Ay), A^*(I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\ &\quad = \langle (I - T_{r_n}^{F_2})(Ax - Ay), AA^*(I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\ &\quad \le L\langle (I - T_{r_n}^{F_2})(Ax - Ay), (I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\ &\quad = L\|(I - T_{r_n}^{F_2})(Ax - Ay)\|^2 \\ &\quad \le 2L\langle x - y, A^*(I - T_{r_n}^{F_2})(Ax - Ay)\rangle \end{aligned}$$

for all $x, y \in H$, from which it can be concluded that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{2L}$-inverse strongly monotone mapping. Moreover, we claim that since $\gamma \in (0, \frac{1}{L})$, $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive.

Next, we show that $F(T) \cap \Omega \subset C_{n+1}$ for all $n \in \mathbb{N}$. Let $p \in F(T) \cap \Omega$, *i.e.*, $T_{r_n}^{F_1} p = p$ and $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$. By mathematical induction, we have $p \in C = C_1$ and hence $F(T) \cap \Omega \subset C_1$. Let $F(T) \cap \Omega \subset C_k$ for some $k \in \mathbb{N}$. It follows that

$$\begin{aligned} \|u_k - p\| &= \left\| T_{r_k}^{F_1}\left(I - \gamma A^*\left(I - T_{r_k}^{F_2}\right)A\right)x_k - T_{r_k}^{F_1}\left(I - \gamma A^*\left(I - T_{r_k}^{F_2}\right)A\right)p \right\| \\ &\le \left\| \left(I - \gamma A^*\left(I - T_{r_k}^{F_2}\right)A\right)x_k - \left(I - \gamma A^*\left(I - T_{r_k}^{F_2}\right)A\right)p \right\| \\ &\le \|x_k - p\| \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\|y_k - p\|^2 \\ &\quad = \left\| \alpha_k x_k + (1 - \alpha_k)T^n u_k - p \right\|^2 \\ &\quad \le \alpha_k \|x_k - p\|^2 + (1 - \alpha_k)\left\| T^n u_k - T^n - p \right\|^2 - \alpha_k(1 - \alpha_k)\left\| x_k - p - \left(T^n u_k - T^n p\right) \right\|^2 \\ &\quad \le \alpha_k \|x_k - p\|^2 + (1 - \alpha_k)\|u_k - p\|^2 - \alpha_k(1 - \alpha_k)\left\| x_k - T^n u_k \right\|^2 \\ &\quad \le \alpha_k \|x_k - p\|^2 + (1 - \alpha_k)k_k^2 \|x_k - p\|^2 \\ &\quad = \|x_k - p\|^2 + (1 - \alpha_k)\left(k_k^2 - 1\right)\|x_k - p\|^2 \\ &\quad \le \|x_k - p\|^2 + (1 - \alpha_k)\left(k_k^2 - 1\right)M_k^2 \\ &\quad = \|x_k - p\|^2 + \theta_k, \end{aligned} \tag{3.3}$$

where $M_k = \sup\{\|x_k - z\| : z \in \Omega\}$ and $\theta_k = (1 - \alpha_k)(k_k^2 - 1)M_k^2$. It can be concluded that $p \in C_{k+1}$ and $F(T) \cap \Omega \subset C_{k+1}$ and, further, $F(T) \cap \Omega \subset C_n$ for all $n \in \mathbb{N}$.

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. First, it is obvious that $C_1 = C$ is closed and convex. By induction, we suppose that C_k is closed and convex for some $k \in \mathbb{N}$. Let $z_m \in C_{k+1} \subset C_k$ with $z_m \to z$. Since C_k is closed, it follows that $x \in C_k$ and $\|y_k - z_m\|^2 \le \|z_m - x_k\|^2 + \theta_k$. Then we have

$$\begin{aligned} \|y_k - z\|^2 &= \|y_k - z_m + z_m - z\|^2 \\ &= \|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle \\ &\le \|z_m - x_k\|^2 + \theta_k + \|z_m - z\|^2 + 2\|y_k - z_m\|\|z_m - z\|. \end{aligned}$$

Letting $m \to \infty$, we have

$$\|y_k - z\|^2 \le \|z - x_k\|^2 + \theta_k,$$

which means that $z \in C_{k+1}$. Let $x, y \in C_{k+1} \subset C_k$ and $z = \alpha x + (1 - \alpha)y$ for any $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$, $\|y_k - x\|^2 \le \|x - x_k\|^2 + \theta_k$ and $\|y_k - y\|^2 \le \|x - x_k\|^2 + \theta_k$ and so

$$\begin{aligned} \|y_k - z\|^2 &= \left\| y_k - \left(\alpha x + (1 - \alpha)y\right) \right\|^2 \\ &= \left\| \alpha(y_k - x) + (1 - \alpha)(y_k - y) \right\|^2 \\ &= \alpha \|y_k - x\|^2 + (1 - \alpha)\|y_k - y\|^2 - \alpha(1 - \alpha)\left\| y_k - x - (y_k - y) \right\|^2 \end{aligned}$$

$$\begin{aligned}
&\le \alpha\bigl(\|x-x_k\|^2+\theta_k\bigr)+(1-\alpha)\bigl(\|y-x_k\|^2+\theta_k\bigr)-\alpha(1-\alpha)\|y-x\|^2\\
&\le \alpha\|x-x_k\|^2+\theta_k+(1-\alpha)\|y-x_k\|^2-\alpha(1-\alpha)\bigl\|(x_k-x)-(x_k-y)\bigr\|^2\\
&= \alpha\|x-x_k\|^2+(1-\alpha)\|y-x_k\|^2-\alpha(1-\alpha)\bigl\|(x_k-x)-(x_k-y)\bigr\|^2+\theta_k\\
&= \bigl\|\alpha(x_k-x)+(1-\alpha)(x_k-y)\bigr\|^2+\theta_k\\
&= \|x_k-z\|^2+\theta_k.
\end{aligned}$$

Therefore, $z \in C_{k+1}$ and hence C_{k+1} is closed and convex. It is immediately concluded that C_n is closed and convex for all $n \in \mathbb{N}$, which implies that $\{x_n\}$ is well defined.

Next, from $x_n = P_{C_n}x_0$, we have

$$\langle x_0-x_n, x_n-y\rangle \ge 0$$

for all $y \in C_n$. Since $p \in F(T)\cap\Omega$, we have

$$\langle x_0-x_n, x_n-p\rangle \ge 0$$

for all $p \in F(T)\cap\Omega$, that is, we have

$$0 \le \langle x_0-x_n, x_n-p\rangle \le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-p\|.$$

This implies that

$$\|x_n-x_0\| \le \|x_0-p\| \tag{3.4}$$

for all $n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0-x_n, x_n-x_{n+1}\rangle \ge 0 \tag{3.5}$$

for all $n \in \mathbb{N}$, and so we have

$$\begin{aligned}
0 &\le \langle x_0-x_n, x_n-x_{n+1}\rangle\\
&= \langle x_0-x_n, x_n-x_0+x_0-x_{n+1}\rangle\\
&\le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-x_{n+1}\|.
\end{aligned}$$

Hence we have

$$\|x_n-x_0\|^2 \le \|x_0-x_n\|\|x_0-x_{n+1}\|, \tag{3.6}$$

that is, $\|x_n-x_0\| \le \|x_0-x_{n+1}\|$ for all $n \in \mathbb{N}$. From (3.4), it follows that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-x_0\|$ exists.

Next, we show that $\|x_n-x_{n+1}\| \to 0$. From (3.5), we have

$$\begin{aligned}
&\|x_n-x_{n+1}\|^2\\
&\quad= \bigl\|(x_n-x_0)+(x_0-x_{n+1})\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= \|x_0 - x_{n+1}\|^2 - \|x_n - x_0\|^2.
\end{aligned}
$$

Since the limit of $\{\|x_n - x_0\|\}$ exists, we have

$$
\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.7}
$$

Thus, by (3.7) and (3.14), we have

$$
\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \tag{3.8}
$$

as $n \to \infty$. Furthermore, since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \big\| T_{r_n}^{F_1}\big(x_n - \gamma A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big) - T_{r_n}^{F_1}\big(p - \gamma A^*\big(I - T_{r_n}^{F_2}\big)Ap\big)\big\|^2 \\
&\le \big\|\big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)A\big)x_n - \big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)A\big)p\big\|^2 \\
&\quad - \big\|\big(I - T_{r_n}^{F_1}\big)\big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big) - \big(I - T_{r_n}^{F_1}\big)\big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)Ap\big)\big\|^2 \\
&= \big\|x_n - p - \gamma\big(A^*\big(I - T_{r_n}^{F_2}\big)Ax_n - A^*\big(I - T_{r_n}^{F_2}\big)Ap\big)\big\|^2 - \big\|z_n - T_{r_n}^{F_1}z_n\big\|^2 \\
&= \|x_n - p\|^2 - 2\gamma\big\langle x_n - p, A^*\big(I - T_{r_n}^{F_2}\big)Ax_n - A^*\big(I - T_{r_n}^{F_2}\big)Ap\big\rangle \\
&\quad + \gamma^2\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n - A^*\big(I - T_{r_n}^{F_2}\big)Ap\big\|^2 - \big\|z_n - T_{r_n}^{F_1}z_n\big\|^2 \\
&\le \|x_n - p\|^2 + \gamma\left(\gamma - \frac{1}{L}\right)\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\|^2 - \big\|z_n - T_{r_n}^{F_1}z_n\big\|^2,
\end{aligned}
$$

where $z_n = (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n$. Moreover,

$$
\begin{aligned}
\|y_n - p\|^2 &= \big\|\alpha_n x_n + (1 - \alpha_n)T^n u_n - p\big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2\bigg[\|x_n - p\|^2 \\
&\quad + \gamma\left(\gamma - \frac{1}{L}\right)\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\|^2 - \big\|z_n - T_{r_n}^{F_1}z_n\big\|^2\bigg] \\
&= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2\|x_n - p\|^2 - (1 - \alpha_n)k_n^2\big\|z_n - T_{r_n}^{F_1}z_n\big\|^2 \\
&\quad + (1 - \alpha_n)k_n^2\gamma\left(\gamma - \frac{1}{L}\right)\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\|^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&(1 - \alpha_n)k_n^2\bigg[\gamma\left(\frac{1}{L} - \gamma\right)\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\|^2 + \big\|z_n - T_{r_n}^{F_1}z_n\big\|^2\bigg] \\
&\quad \le \big(\alpha_n + (1 - \alpha_n)k_n^2\big)\|x_n - p\|^2 - \|y_n - p\|^2.
\end{aligned} \tag{3.9}
$$

Letting $\rho_n = k_n - 1$. Then it is clear that $\rho_n \to 0$ as $n \to \infty$ and, by (3.9), we exactly have

$$\begin{aligned}(1-\alpha_n)k_n^2\biggl[\gamma\left(\frac{1}{L}-\gamma\right)\bigl\|A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n)\bigr\|^2+\bigl\|z_n-T_{r_n}^{F_1}z_n\bigr\|^2\biggr]&\\ \le \alpha_n\|x_n-p\|^2+(1-\alpha_n)(\rho_n+1)^2\|x_n-p\|^2-\|y_n-p\|^2&\\ \le \|x_n-p\|^2-\|y_n-p\|^2+(1-\alpha_n)\bigl(\rho_n^2+2\rho_n\bigr)\|x_n-p\|^2&\\ \le \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|+(1-\alpha_n)\bigl(\rho_n^2+2\rho_n\bigr)\|x_n-p\|^2.&\end{aligned}$$

By (3.8) and $\rho_n \to 0$ as $n \to \infty$, we have

$$\bigl\|A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\| \to 0, \qquad \bigl\|z_n-T_{r_n}^{F_1}z_n\bigr\| \to 0 \tag{3.10}$$

as $n \to \infty$. Furthermore, since A is linear bounded and so is A^*, we can conclude that

$$\lim_{n\to\infty}\bigl\|\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\| = 0. \tag{3.11}$$

Next, we show that $\|u_n - x_n\| \to 0$. We investigate the following:

$$\begin{aligned}\|u_n-x_n\| &= \bigl\|T_{r_n}^{F_1}z_n-x_n\bigr\|\\ &\le \bigl\|T_{r_n}^{F_1}z_n-z_n\bigr\|+\|z_n-x_n\|\\ &= \bigl\|T_{r_n}^{F_1}z_n-z_n\bigr\|+\bigl\|\bigl(I-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)A\bigr)x_n-x_n\bigr\|\\ &= \bigl\|T_{r_n}^{F_1}z_n-z_n\bigr\|+\gamma\bigl\|A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\|.\end{aligned} \tag{3.12}$$

Consequently, by (3.12), we can conclude that

$$\|u_n-x_n\| \to 0. \tag{3.13}$$

Next, we show that $\|T^nx_n - x_n\| \to 0$. We first consider

$$\|y_n-x_n\| = \bigl\|\alpha_nx_n+(1-\alpha_n)T^nu_n\bigr\| = (1-\alpha_n)\bigl\|T^nu_n-x_n\bigr\|,$$

and since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n-x_{n+1}\|^2 \le \|x_n-x_{n+1}\|^2+\theta_n,$$

which means that

$$\|y_n-x_{n+1}\| \le \|x_n-x_{n+1}\|+\sqrt{\theta_n}. \tag{3.14}$$

Hence,

$$\begin{aligned}\bigl\|T^nu_n-x_n\bigr\| &= \frac{1}{1-\alpha_n}\|y_n-x_n\|\\ &\le \frac{1}{1-a}\bigl(\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\\ &\le \frac{1}{1-a}\bigl(\|x_n-x_{n+1}\|+\sqrt{\theta_n}\bigr)+\frac{1}{1-a}\|x_{n+1}-x_n\|,\end{aligned}$$

and so $\|T^n u_n - x_n\| \to 0$. Consider

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n u_n\| + \|T^n u_n - x_n\| \\ &\le k_n\|x_n - u_n\| + \|T^n u_n - x_n\|.\end{aligned}$$

Therefore, we have $\|T^n x_n - x_n\| \to 0$ as $n \to \infty$. Putting $k_\infty = \sup\{k_n : n \ge 1\} < \infty$, we deduce that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le k_\infty\|x_n - T^n x_n\| + (1 + k_\infty)\|x_n - x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|.\end{aligned}$$

Hence we have $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. Without loss of generality, since $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup x^*$. It is easy to see that $x^* \in C_n$ for all $n \ge 1$. On the other hand, we have

$$\|x_n - x_0\| \le \|x^* - x_0\|.$$

It follows that

$$\|x^* - x_0\| \le \liminf_{n\to\infty}\|x_n - x_0\| \le \limsup_{n\to\infty}\|x_n - x_0\| \le \|x^* - x_0\|$$

and so

$$\lim_{n\to\infty}\|x_n - x_0\| = \|x^* - x_0\|.$$

Hence $\|x_n\| \to \|x^*\|$. Since every Hilbert space has the Kadec-Klee property, we immediately have $x_n \to x^*$.

Finally, we prove that $x^* \in F(T) \cap \Omega$. Since $x_n \to x^*$ and $x_n - Tx_n \to 0$ as $n \to \infty$, consider

$$\begin{aligned}\|x^* - Tx^*\| &\le \|x^* - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tx^*\| \\ &\le (1 + k_1)\|x^* - x_n\| + \|x_n - Tx_n\|.\end{aligned}$$

We can see that $\|x^* - Tx^*\| = 0$ and, further, $x^* \in F(T)$. Therefore, we have $x^* \in F(T)$.

Next, we show that $x^* \in \Omega$. By (3.1), $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2}))$, that is,

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \ge 0$$

for all $y \in C$. From (A2), it follows that

$$-\frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F_1(y, u_n)$$

for all $y \in C$. Since $\|A^*(T_{r_n}^{F_2} - I)Ax_n\| \to 0$, $\|u_n - x_n\| \to 0$ and $\|x_n - x^*\| \to 0$ as $n \to \infty$, we have

$$F_1(y, x^*) \le 0$$

for all $y \in C$. Let $y_t = ty + (1-t)x^*$ for any $0 < t \leq 1$ and $y \in C$. It means that $y_t \in C$ and hence

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, x^*) \leq tF_1(y_t, y),$$

and then $F_1(y_t, y) \geq 0$. Letting $t \to 0$, we immediately have $F_1(x^*, y) \geq 0$, *i.e.*, $x^* \in EP(F_1)$.

Next, we show that $Ax^* \in EP(F_2)$. Since A is a bounded linear operator and (3.11), we have

$$\|T_{r_n}^{F_2}Ax_n - Ax^*\| \leq \|T_{r_n}^{Ax_n}\| - \|Ax_n - Ax^*\| \to 0$$

as $n \to \infty$, which yields that $T_{r_n}^{F_2}Ax_n \to Ax^*$. By the definition of $T_{r_n}^{F_2}$, we have

$$F_2(T_{r_n}^{F_2}Ax_n, y) + \frac{1}{r_n}\langle y - T_{r_n}^{F_2}Ax_n, T_{r_n}^{F_2}Ax_n - Ax_n\rangle \geq 0 \tag{3.15}$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument, taking lim sup in (3.15), it follows that

$$F_2(Ax^*, y) \geq 0$$

for all $x, y \in C$, from which it can be concluded that $Ax^* \in EP(F_2)$. Consequently, $x^* \in \Omega$. This completes the proof. □

In Theorem 3.1, if the mapping T is a nonexpansive mapping, then we immediately have the following.

Corollary 3.2 *Let H_1, H_2 be two real Hilbert spaces and C, Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying conditions* (A1)-(A4) *and F_2 be upper semi-continuous in the first argument. Let $T : C \to C$ be a nonexpansive mapping and $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{v \in C : v \in EP(F_1)$ such that $Av \in EP(F_2)\}$, and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ iteratively as follows*:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{3.16}$$

for each $n \in \mathbb{N}$, where $M_n = \sup\{\|x_n - z\| : x \in \Omega\}$ and $\theta_n = (1-\alpha_n)(k_n^2 - 1)M_n^2$, $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$, $0 < b \leq r_n < \infty$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the sequence $\{x_n\}$ generated by (3.16) *strongly converges to a point $z_0 \in F(T) \cap \Omega$.*

If $H_1 = H_2$, $C = Q$ and $A = I$ in Theorem 3.1, then we have the following.

Corollary 3.3 *Let H be a real Hilbert space and C be a nonempty closed convex subset of a Hilbert space H. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4) *and F_2 be upper semi-continuous in the first argument. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Suppose that $F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{v \in C : v \in EP(F_1) \cap EP(F_2)\}$, and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ iteratively as follows*:

$$\begin{cases} u_n = T_{r_n}^{F_1} T_{r_n}^{F_2} x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{3.17}$$

for each $n \in \mathbb{N}$, where $M_n = \sup\{\|x_n - z\| : z \in \Omega\}$ and $\theta_n = (1-\alpha_n)(k_n^2 - 1)(M_n)^2$, $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq r_n < \infty$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (3.17) *strongly converges to a point $z_0 \in F(T) \cap \Omega$.*

4 Applications

4.1 Applications to split variational inequality problems

Firstly, we point out the so-called variational inequality problem (shortly, *VIP*), which is to find a point $x^* \in C$ which satisfies the following inequality:

$$\langle Ax^*, z - x^* \rangle \geq 0$$

for all $z \in C$. Its solution set is symbolized by $VI(A, C)$.

In 2012, Censor *et al.* [19] proposed the *split variational inequality problem* (shortly, *SVIP*) which is formulated as follows:

$$\text{Find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in C$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \quad \text{for all } y \in Q,$$

where $A : C \to C$ is a bounded linear operator. The solution set of split variational inequality problem is denoted by the *SVIP*.

Setting $F_1(x, y) = \langle f(x), y - x \rangle$ and $F_2(x, y) = \langle g(x), y - x \rangle$, it is clear that F_1, F_2 satisfy conditions (A1)-(A4), where f and g are η_1- and η_2-inverse strongly monotone mappings, respectively. Then, by Theorem 3.1, we get the following.

Theorem 4.1 *Let H_1, H_2 be two real Hilbert spaces and C, Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let f and g be η_1- and η_2-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying conditions* (A1)-(A4), *which are defined by f and g, and F_2 be upper semi-continuous in the first argument. Let $T : C \to C$ be an asymptotically nonexpansive mapping and $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{v \in C : v \in EP(F_1)$ such that $Av \in EP(F_2)\}$, and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$,* *define a sequence* $\{x_n\}$ *iteratively as follows*:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{4.1}$$

for each $n \in \mathbb{N}$, *where* $M_n = \sup\{\|x_n - z\| : z \in \Omega\}$ *and* $\theta_n = (1-\alpha_n)(k_n^2 - 1)M_n^2$, $0 \leq \alpha_n \leq a < 1$ *for all* $n \in \mathbb{N}$, $0 < b \leq r_n < \infty$, $\gamma \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Then the sequence* $\{x_n\}$ *generated by* (4.1) *strongly converges to a point* $z_0 \in F(T) \cap \Omega$.

Proof The desired result can be proved directly through Theorem 3.1. □

4.2 Applications to split optimization problems

In this section, we mention applications to the *split optimization problem*, which is to find $x^* \in C$ such that

$$f(x^*) \geq f(x) \text{ for all } x \in C \text{ satisfying } Ax^* = y^* \in Q \text{ solves } g(y^*) \geq g(y) \tag{4.2}$$

for all $y \in Q$. We symbolize Γ for the solution set of the split optimization problem.

Let $f : C \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ be two functions satisfying the following assumption:

(1) for each $x, y \in C$, $f(tx + (1-t)y) \leq f(y)$, and for each $u, v \in Q$, $g(tu + (1-t)v) \leq g(v)$;

(2) $f(x)$ is concave and upper semi-continuous for all $x \in C$ and $g(u)$ is concave and upper semi-continuous for all $u \in Q$.

Let $F_1(x, y) = f(x) - f(y)$ for all $x, y \in C$ and $F_2(u, v) = g(u) - g(v)$ for all $u, v \in Q$. If f and g satisfy conditions (1) and (2), then it is clear that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying conditions (A1)-(A4). Therefore, by Theorem 3.1, we have the following.

Theorem 4.2 *Let* H_1, H_2 *be two real Hilbert spaces and* C, Q *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $f : C \to \mathbb{R}$ *and* $g : Q \to \mathbb{R}$ *be two functions satisfying conditions* (1) *and* (2). *Let* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *be two bifunctions satisfying conditions* (A1)-(A4) *and* F_2 *be upper semi-continuous in the first argument. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $F(T) \cap \Gamma \neq \emptyset$ *and let* $x_0 \in C$. *For* $C_1 = C$ *and* $x_1 = P_{C_1} x_0$, *define a sequence* $\{x_n\}$ *iteratively as follows*:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{4.3}$$

for each $n \in \mathbb{N}$, *where* $0 \leq \alpha_n \leq a < 1$, $0 < b \leq r_n < \infty$, *and* $\gamma \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Then the sequence* $\{x_n\}$ *generated by* (4.3) *strongly converges to a point* $z_0 \in F(T) \cap \Gamma$.

Proof The desired result can be proved directly through Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, School of Science, University of Phayao, Phayao, 56000, Thailand. [2]Department of Mathematics, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, 660-701, Korea.

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