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A class of singularly perturbed delayed boundary value problem in the critical case

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Abstract

Based on the boundary layer function method, a class of generalized high-dimensional delayed systems in the critical case is studied. In the framework of this paper, we not only construct the asymptotic expansion of the solution for the original equation but also give the proof of the uniformly valid asymptotic expansion. Meanwhile, we give an example to demonstrate our result.

MSC: 34B15; 34E10; 37C10

Keywords: critical case; singular perturbation; boundary layer; approximate solution

1 Introduction

The dielectric constants of the basic semiconductor devices are usually small, and therefore boundary value problems for semiconductor equations with higher derivative terms can be regarded as singular perturbation problems with small parameters. The singular perturbation approach to semiconductor problems was started by Vasil'eva, Kardosysove and Stelmach. It is a kind of specially effective uncoupled method for solving semiconductor device simulation problems and is a general analysis method for finding an approximate solution of the systems in the natural sciences and engineering. During the past decade the method of averaging [1], the boundary layer function method, the principle of matched asymptotic expansion [2] and multiple scales [3], to name but a few, have been developed and refined. Using these methods, many approximate solution problems, including organic compounds reaction, neural networks [4], polymer rheology [5], and so on were solved. However, studies of the singularly perturbed delayed boundary value problem in the critical case rarely appeared. Until now, singularly perturbed initial value problems in the critical case have been solved. But, for boundary value problems, only some special cases have been considered [6]. The reason for it lies is two-fold. Firstly, the smooth approximate solution on the intersection of the two adjacent intervals is not easy to get. Secondly, the complex boundary layer and the internal layer of the zero order equations are very difficult.

In recent years, the great importance of semiconductor device simulation caused intensive work on the subject [7–14]. However, in practical applications, we need to consider the effect of delays. As is well known, there are a close relation between the current state and the previous state for the electric field, the electron density, and the hole density in a semiconductor device. Since the electric field is produced by the potential difference of the device and the time that the movement of electrons makes the electric field reach a balanced equilibrium is very short, the delays are easier to miss. In fact, we know that even a small delay is also likely to have an important impact on the stability of the system. Therefore it is necessary for us to consider the influence of delays and to study nonlinear systems which are more realistic. Our work is based on such a background.

In this paper, using the boundary layer function method, we consider the asymptotic behavior of solution for the problem, and the formal asymptotic solution is also constructed. The asymptotical behavior of solutions for the singular perturbation problem is usually refer to as 'under certain conditions the solution $x(t,\mu)$ of the original problem converges to the solution $\bar{x}(t,\mu)$ of the degradation problem as $\mu \to 0$'. The asymptotic behavior of a solution and the research of using a degenerate solution to approximate the exact solution provide a theoretical basis for constructing the numerical simulation of the semiconductor devices.

We consider the following quasi-linear delayed boundary value problems:

$$\mu \frac{dz}{dt} = A\bigl(u(t), u(t-\sigma), t\bigr)y + \mu B\bigl(u(t), u(t-\sigma), t\bigr), \tag{1.1}$$

$$\mu \frac{dy}{dt} = f\bigl(z(t), z(t-\sigma), t\bigr), \quad 0 \le t \le T, \tag{1.2}$$

$$\mu \frac{du}{dt} = C\bigl(u(t), u(t-\sigma), t\bigr)y + \mu D\bigl(u(t), u(t-\sigma), t\bigr), \tag{1.3}$$

with the initial and boundary value conditions

$$z(t,\mu) = \rho(t), \qquad u(t,\mu) = \boldsymbol{\theta}(t), \quad -\sigma \le t \le 0, \tag{1.4}$$

$$z(T,\mu) = z^T, \tag{1.5}$$

where $\mu > 0$ is a small parameter, $\sigma > 0$ is also a small shifting parameter, y and z are scalar functions, u is a k-dimensional vector function and $\sigma \le T \le 2\sigma$, functions A, B, C, and D are analytic in some domain $G(u,[u],t)$, f is analytic in $I(z,[z],t)$. It is easy to see that in this case the system is quasi-linear because it is linear with respect to y. The choice of such a system is motivated in part by its occurrence in the study of applied problems from semiconductor theory. The symbol [ ] stands for the deviation operation, namely, $[u] \equiv u(t-\sigma)$.

(H_1') *Suppose that* $Af_z > 0$.

Next, we will use the method of fractional steps to discuss the systems (1.1)-(1.5). Setting $\mu = 0$ and obtaining the so-called left degenerate problems

$$A\bigl(\bar{u}(t), \theta(t-\sigma), t\bigr)\bar{y}(t) = 0, \tag{1.6}$$

$$f\bigl(\bar{z}(t), \rho(t-\sigma), t\bigr) = 0, \tag{1.7}$$

$$C\bigl(\bar{u}(t), \theta(t-\sigma), t\bigr)\bar{y}(t) = 0, \tag{1.8}$$

on the interval $[0,\sigma]$ and the right degenerate problems

$$A\bigl(\bar{\bar{u}}(t), \bar{u}(t), t\bigr)\bar{y}(t) = 0, \tag{1.9}$$

$$f(\bar{\bar{z}}(t), \bar{z}(t), t) = 0, \tag{1.10}$$

$$C(\bar{\bar{u}}(t), \bar{u}(t), t)\bar{y}(t) = 0, \tag{1.11}$$

on the interval $[\sigma, T]$, respectively.

The left degenerate problems are algebraic equations. We suppose that systems (1.6)-(1.8) have a family of solutions

$$\bar{z}(t) = \bar{z}^{(-)}(t), \qquad \bar{y}(t) = \bar{y}^{(-)}(t) = 0, \qquad \bar{u}(t) = \bar{u}^{(-)}(t). \tag{1.12}$$

Similarly, substituting (1.12) into the right degenerate problems gives

$$\bar{\bar{z}}(t) = \bar{z}^{(+)}(t), \qquad \bar{\bar{y}}(t) = \bar{y}^{(+)}(t) = 0, \qquad \bar{\bar{u}}(t) = \bar{u}^{(+)}(t). \tag{1.13}$$

Here $\bar{u}^{(-)}(t)$ and $\bar{u}^{(+)}(t)$ are two arbitrary k-dimensional vector functions. The matrix F_x (evaluated at $z = \bar{z}^{(-)}(t)$, $y = \mu = 0$, $u = \bar{u}^{(-)}(t)$ and $z = \bar{z}^{(+)}(t)$, $y = \mu = 0$, $u = \bar{u}^{(+)}(t)$), for $F = (Ay\, f\, Cy)^T$ and $x = (z\, y\, u)^T$, is equal to the block matrix

$$\begin{pmatrix} 0 & A & 0 \\ f_z & 0 & 0 \\ 0 & C & 0 \end{pmatrix}.$$

It is easy to see that F_x has $\lambda \equiv 0$ as an eigenvalue of multiplicity k as well as two eigenvalues of opposite signs in the domain $G(\bar{u}^{(\mp)}(t), [\bar{u}^{(\mp)}(t)], t)$, namely $\lambda_{1,2} = \pm\sqrt{Af_z}$. Thus we have indeed a *critical conditionally stable case.*

From (1.12) and (1.13), we obtain the degenerate solution on the interval $[0, T]$, that is,

$$\bar{x} = \begin{cases} \bar{x}^{(-)}(t), & 0 \le t \le \sigma, \\ \bar{x}^{(+)}(t), & \sigma \le t \le T. \end{cases} \tag{1.14}$$

In general, the right limit solution is not equal to the left limit solution at $t = \sigma$, *i.e.*,

$$\bar{x}(\sigma-) = \lim_{t\to\sigma^+} \bar{x}^{(-)}(t) \neq \lim_{t\to\sigma^-} \bar{x}^{(+)}(t) = \bar{x}(\sigma+).$$

It turns out that the internal layer may occur at $t = \sigma$. Besides, boundary layers may occur at $t = 0$ and $t = T$, because the solution of reduced problems may not satisfy the initial boundary value conditions. Since the reduced problem has a family of solutions $\bar{x} = \bar{x}(t)$, our questions consequently arise as previously. Under what conditions will the solution of the systems (1.1)-(1.5) converge to one of the solutions of this family as $\mu \to 0$, and, in particular, to which one? In this paper, we will discuss this problem and construct the uniformly valid asymptotic solution $x(t, \mu)$ on the interval $[0, T]$.

2 Construction of the formal asymptotics

We seek a solution of systems (1.1)-(1.5) in the form

$$x(t, \mu) = \begin{cases} \sum_{i=0}^{\infty} \mu^i \{\bar{x}_i(t) + L_i x(\tau_0) + Q_i^{(-)} x(\tau_c)\}, & 0 \le t \le \sigma, \\ \sum_{i=0}^{\infty} \mu^i \{\bar{\bar{x}}_i(t) + Q_i^{(+)} x(\tau_c) + R_i x(\tau_1)\}, & \sigma \le t \le T, \end{cases} \tag{2.1}$$

where $\tau_0 = \frac{t}{\mu}$, $\tau_c = \frac{t-\sigma}{\mu}$, $\tau_1 = \frac{t-T}{\mu}$. By the initial and boundary value conditions, we obtain

$$\begin{aligned}
&\bar{u}_0(0) + L_0u(0) = \theta(0), \qquad \bar{u}_k(0) = -L_ku(0),\\
&\bar{z}_0(0) + L_0z(0) = \rho(0), \qquad \bar{z}_k(0) = -L_kz(0),\\
&\bar{\bar{z}}_0(T) + R_0z(0) = z^T, \qquad \bar{\bar{z}}_k(T) = -R_kz(0).
\end{aligned}$$

In order to make $x(t,\mu)$ continuous at $t = \sigma$, the following equation must be satisfied:

$$x^{(-)}(\sigma,\mu) = x^{(+)}(\sigma,\mu). \tag{2.2}$$

From the continuity condition (2.2), we get

$$\begin{aligned}
&\bar{z}_0(\sigma) + Q_0^{(-)}z(0) = \bar{\bar{z}}_0(\sigma) + Q_0^{(+)}z(0) = p_0,\\
&\bar{z}_k(\sigma) + Q_k^{(-)}z(0) = \bar{\bar{z}}_k(\sigma) + Q_k^{(+)}z(0) = p_k,
\end{aligned}$$

where p_k ($k \geq 0$) are unknown parameters.

2.1 The construction of asymptotic solutions on the interval $[0,\sigma]$

First, we consider the regular part of the solution. Substituting (1.12) into the original problem and setting $\mu = 0$, we obtain the so-called degenerate problems of $\bar{x}_0(t)$:

$$\begin{aligned}
&A\big(\bar{u}_0(t),\theta(t-\sigma),t\big)\bar{y}_0(t) = 0,\\
&f\big(\bar{z}_0(t),\rho(t-\sigma),t\big) = 0,\\
&C\big(\bar{u}_0(t),\theta(t-\sigma),t\big)\bar{y}_0(t) = 0.
\end{aligned} \tag{2.3}$$

The root of (2.3) is

$$\bar{z}_0(t) = \bar{\alpha}(t), \qquad \bar{y}_0(t) = 0, \qquad \bar{u}_0(t) = \bar{\boldsymbol{\gamma}}(t), \tag{2.4}$$

where $\bar{\boldsymbol{\gamma}}(t)$ is an arbitrary k-dimensional vector function.

The equations of $L_0x(\tau_0)$ are

$$\begin{aligned}
&\frac{d}{d\tau_0}L_0z = \tilde{A}\big(\bar{\boldsymbol{\gamma}}(0) + L_0u(\tau_0),\theta(-\sigma),0\big)L_0y(\tau_0),\\
&\frac{d}{d\tau_0}L_0y = \tilde{f}\big(\bar{\alpha}(0) + L_0z(\tau_0),\rho(-\sigma),0\big) - \bar{f}\big(\bar{\alpha}(0),\rho(-\sigma),0\big),\\
&\frac{d}{d\tau_0}L_0u = \tilde{C}\big(\bar{\boldsymbol{\gamma}}(0) + L_0u(\tau_0),\theta(-\sigma),0\big)L_0y(\tau_0),
\end{aligned} \tag{2.5}$$

with the initial and boundary conditions

$$L_0z(0) = \rho(0) - \bar{\alpha}(0), \qquad L_0u(0) = \theta(0) - \bar{\boldsymbol{\gamma}}(0), \qquad L_0x(+\infty) = 0, \tag{2.6}$$

where $\bar{\boldsymbol{\gamma}}(0)$ is unknown, and the initial value of $L_0y(\tau_0)$ is as yet arbitrary. We will use this arbitrariness to ensure that $L_0x(\tau_0)$ satisfies condition $L_0x(+\infty) = 0$.

From (2.5), we have

$$\frac{dL_0 u}{dL_0 z} = \frac{\tilde{C}(\bar{\boldsymbol{\gamma}}(0) + L_0 u(\tau_0), \theta(0), 0)}{\tilde{A}(\bar{\boldsymbol{\gamma}}(0) + L_0 u(\tau_0), \theta(0), 0)}. \tag{2.7}$$

Integrating the above formula after separating variable and using implicit function theorem, we find that (2.7) has a solution. Let us denote by

$$L_0 u = \bar{U}_0\bigl(\bar{\boldsymbol{\gamma}}(0), L_0 z\bigr) \tag{2.8}$$

the solution of this system such that $L_0 u = 0$ for $L_0 z = 0$, that is, $\bar{U}_0(\bar{\boldsymbol{\gamma}}(0), 0) = 0$.

The condition (H_1') shows that there exists a unique solution in a certain neighborhood of the point $L_0 z = 0$. Then substituting (2.8) into (2.5), we obtain the system of equations

$$\begin{aligned} &\frac{d}{d\tau_0} L_0 z = \tilde{A}\bigl(\bar{\boldsymbol{\gamma}}(0) + \bar{U}_0\bigl(\bar{\gamma}(0), L_0 z\bigr), \theta(0), 0\bigr) L_0 y(\tau_0), \\ &\frac{d}{d\tau_0} L_0 y = \tilde{f}\bigl(\bar{\alpha}(0) + L_0 z(\tau_0), \rho(-\sigma), 0\bigr) - \bar{f}\bigl(\bar{\alpha}(0), \rho(-\sigma), 0\bigr). \end{aligned} \tag{2.9}$$

The equilibrium point $(0,0)$ of (2.9) is a saddle, since the roots of the corresponding characteristic equation are clearly equal to $\pm\sqrt{\tilde{A}\tilde{f}_{L_0 z}}$ and, by virtue of the condition (H_1'), are real and have opposite signs.

By integrating (2.9), we get the separate equation of the saddle point

$$L_1\colon L_0 y(\tau_0) = -\left(2\int_0^{L_0 z} \frac{\tilde{f}(\bar{\alpha}(0) + \xi, \rho(0), 0) - \bar{f}(\bar{\alpha}(0), \rho(0), 0)}{\tilde{A}(\bar{\boldsymbol{\gamma}}(0) + \bar{U}_0(\bar{\boldsymbol{\gamma}}(0), \xi), \rho(0), 0)}\, d\xi\right)^{\frac{1}{2}}. \tag{2.10}$$

Equations (2.8) and (2.10) give an analytic expression of the one-dimensional manifold $\bar{\Omega}_0$ having the property that if $L_0 x(0) \in \bar{\Omega}_0$, then $L_0 x(\tau_0) \in \bar{\Omega}_0$ for $\tau_0 > 0$, and the inequality

$$\bigl\| L_0 x(\tau_0)\bigr\| \le c e^{-\sigma_0 \tau_0} \quad (\tau_0 \ge 0) \tag{2.11}$$

is satisfied. In order to make the solution of system (2.5) satisfy an exponential decay estimation, we need the following assumption.

(H_2) *Suppose that* $L_0 z(0) \cap \bar{\Omega}_0 \neq \emptyset$.

Substituting (2.6) into (2.8), we have

$$\theta(0) - \bar{\boldsymbol{\gamma}}(0) = \bar{U}_0\bigl(\bar{\boldsymbol{\gamma}}(0), \rho(0) - \bar{\alpha}(0)\bigr), \tag{2.12}$$

which represents a system of k scalar equations in the k unknown components of the vector $\bar{\boldsymbol{\gamma}}(0)$. The following condition is concerned with the solvability of $\bar{\boldsymbol{\gamma}}(0)$.

(H_3) *Suppose that* (2.12) *has a root* $\bar{\boldsymbol{\gamma}}(0) = \bar{\boldsymbol{\gamma}}^0$.

By means of (H_3) and by taking (2.10) into account, we can determine the initial value $L_0 y(0)$, and consequently, $L_0 x(\tau_0)$ satisfies an exponential decay. For determining $L_0 x(\tau_0)$, it is necessary to substitute (2.10) into (2.9) and to solve the resulting scalar equation for

$L_0z(\tau_0)$ with the initial value condition $L_0z(0) = \rho(0) - \bar{\alpha}(0)$. By virtue of (2.8) and (2.10) we obtain $L_0u(\tau_0)$ and $L_0y(\tau_0)$. Thus $L_0x(\tau_0)$ are completely determined, while for the as yet unknown function $\bar{\boldsymbol{\gamma}}(t)$ we only know its initial value $\bar{\boldsymbol{\gamma}}^0$. The function $\bar{\boldsymbol{\gamma}}(t)$ need to be determined in the first approximation.

For $\bar{x}_1(t)$, we get

$$\begin{aligned}
&\bar{A}\big(\bar{\boldsymbol{\gamma}}(t),\theta(t-\sigma),t\big)\bar{y}_1(t) + \bar{B}\big(\bar{\boldsymbol{\gamma}}(t),\theta(t-\sigma),t\big) = \bar{\alpha}'(t),\\
&f_z\big(\bar{\alpha}(t),\rho(t-\sigma),t\big)\bar{z}_1(t) = 0,\\
&\bar{C}\big(\bar{\boldsymbol{\gamma}}(t),\theta(t-\sigma),t\big)\bar{y}_1(t) + \bar{D}\big(\bar{\boldsymbol{\gamma}}(t),\theta(t-\sigma),t\big) = \bar{\boldsymbol{\gamma}}'(t),
\end{aligned}$$

which has a solution

$$\bar{z}_1(t) = 0, \qquad w\bar{y}_1(t) = \frac{\bar{\alpha}'(t) - \bar{B}}{\bar{A}}, \qquad \frac{d\bar{\boldsymbol{\gamma}}}{dt} = \frac{\bar{C}(\bar{\alpha}'(t) - \bar{B})}{\bar{A}} + \bar{D}, \tag{2.13}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are all taken value at the point $(\bar{\boldsymbol{\gamma}}(t),\theta(t-\sigma),t)$. From the existence of a solution for the initial value problem, (2.13) together with the initial condition $\bar{\boldsymbol{\gamma}}(0) = \bar{\boldsymbol{\gamma}}^0$ has the solution $\bar{\boldsymbol{\gamma}} = \bar{\boldsymbol{\gamma}}(t)$ for $0 \le t \le \sigma$. Therefore $\bar{x}_0(t)$ can be completely determined. Equation (2.13) has only determined $\bar{z}_1(t)$ and $\bar{y}_1(t)$, while $\bar{u}_1(t)$ is yet unknown. We will take advantage of the first approximation to determine $\bar{u}_1(t)$.

The equations for $L_1x(\tau_0)$ have the form

$$\begin{aligned}
\frac{d}{d\tau_0}L_1z &= \tilde{A}\big(\bar{\boldsymbol{\gamma}}(0) + L_0u(\tau_0),\theta(-\sigma),0\big)L_1y(\tau_0)\\
&\quad + \tilde{A}_u\big(\bar{u}_1(0) + L_1u(\tau_0)\big)L_0y(\tau_0) + \varphi_1(\tau_0),\\
\frac{d}{d\tau_0}L_1y &= \tilde{f}_zL_1z(\tau_0) + \varphi_2(\tau_0),\\
\frac{d}{d\tau_0}L_1u &= \tilde{C}\big(\bar{\boldsymbol{\gamma}}(0) + L_0u(\tau_0),\theta(-\sigma),0\big)L_1y(\tau_0)\\
&\quad + \tilde{C}_u\big(\bar{u}_1(0) + L_1u(\tau_0)\big)L_0y(\tau_0) + \varphi_3(\tau_0),
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
\varphi_1(\tau_0) &= \tilde{A}\bar{y}_1(\mu\tau_0) + \big[\tilde{A}_u\boldsymbol{\gamma}'(0)\tau_0 + \tilde{A}_\theta\theta'(0)\tau_0 + \tilde{A}_t\tau_0\big]L_0y(\tau_0)\\
&\quad - \bar{A}\bar{y}_1(\mu\tau_0) + (\tilde{B} - \bar{B}),\\
\varphi_2(\tau_0) &= \tilde{f}_z\bar{\alpha}'(0)\tau_0 + \tilde{f}_\rho\rho'(0)\tau_0 + \tilde{f}_t\tau_0,\\
\varphi_3(\tau_0) &= \tilde{C}\bar{y}_1(\mu\tau_0) + \big[\tilde{C}_u\boldsymbol{\gamma}'(0)\tau_0 + \tilde{C}_\theta\theta'(0)\tau_0 + \tilde{C}_t\tau_0\big]L_0y(\tau_0)\\
&\quad - \bar{C}\bar{y}_1(\mu\tau_0) + (\tilde{D} - \bar{D}),
\end{aligned}$$

here $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ take values at the point $(\bar{\boldsymbol{\gamma}}(0),\theta(-\sigma),0)$, $\tilde{A}$, $\tilde{A}_u$, $\tilde{A}_\theta$, $\tilde{A}_t$, $\tilde{B}$, $\tilde{C}$, $\tilde{C}_u$, $\tilde{C}_\theta$, $\tilde{C}_t$, $\tilde{D}$ take values at the point $(\bar{\boldsymbol{\gamma}}(0) + L_0u(\tau_0),\theta(-\sigma),0)$, and $\tilde{f}_z, \tilde{f}_\rho, \tilde{f}_t$ take values at the point $(\bar{\alpha}(0) + L_0z(\tau_0),\rho(-\sigma),0)$.

The initial and boundary conditions of $L_1x(\tau_0)$ are

$$L_1z(0) = 0, \qquad L_1u(0) = -\bar{u}_1(0), \qquad L_1x(+\infty) = 0. \tag{2.15}$$

Like the case of $L_0x(\tau_0)$, we will take advantage of the arbitrariness of $L_1y(0)$ and choose the appropriate value to guarantee that $L_1y(0)$ satisfies condition (2.15). For this purpose, we introduce the diagonalization transform

$$L_1z = \delta_1, \qquad L_1y = \delta_2, \qquad L_1u = \delta_3 + \frac{\delta_1\tilde{C}}{\tilde{A}}.$$

It is easy to verify that we obtain the system

$$\begin{aligned}
\frac{d\delta_1}{d\tau_0} &= \frac{\tilde{A}_u\tilde{C}}{\tilde{A}}L_0y(\tau_0)\delta_1 + \tilde{A}\delta_2 + \tilde{A}_uL_0y(\tau_0)\bigl(\delta_3 + \bar{u}_1(0)\bigr) + \varphi_1(\tau_0), \\
\frac{d\delta_2}{d\tau_0} &= \tilde{f}_z\delta_1 + \varphi_2(\tau_0), \\
\frac{d\delta_3}{d\tau_0} &= \left(\tilde{C}_u - \frac{\tilde{C}\tilde{A}_u}{\tilde{A}}\right)L_0y(\tau_0)\bigl(\delta_3 + \bar{u}_1(0)\bigr) + \left(\varphi_3(\tau_0) - \frac{\tilde{C}}{\tilde{A}}\varphi_1(\tau_0)\right).
\end{aligned} \tag{2.16}$$

Obviously, (2.16) has already separated δ_3 from δ_1 and δ_2 completely. The initial and boundary conditions of δ_1, δ_2, and δ_3 are $\delta_1(0) = 0$, $\delta_3(0) = -\bar{u}_1(0)$, and $\delta_i(+\infty) = 0$ ($i = 1, 2, 3$). Together with initial condition, the solution of δ_3 in (2.16) is

$$\delta_3(\tau_0) = -\bar{u}_1(0) + \int_0^{\tau_0} \Phi(\tau_0)\Phi^{-1}(s)\varphi_4(s)\,ds,$$

where $\varphi_4(\tau_0) = \varphi_3(\tau_0) - \frac{\varphi_1(\tau_0)\tilde{C}}{\tilde{A}}$, $\Phi(\tau_0)$ is the fundamental matrix of the corresponding homogeneous equation with $\Phi(0) = E_k$.

The condition $\delta_3(+\infty) = 0$ uniquely determines $\bar{u}_1(0)$:

$$\bar{u}_1(0) = \Phi(\tau_0)\int_0^{+\infty} \Phi^{-1}(s)\varphi_4(s)\,ds.$$

Since $\Phi(\tau_0)$ exponentially converges to $\Phi(+\infty)$ as $\tau_0 \to +\infty$, the exponential estimate of $\delta_3(\tau_0)$ can be obtained. After determining $\delta_3(\tau_0)$, the first two questions in (2.16) can be rewritten as

$$\begin{aligned}
\frac{d\delta_1}{d\tau_0} &= \frac{\tilde{A}_u\tilde{C}}{\tilde{A}}L_0y(\tau_0)\delta_1 + \tilde{A}\delta_2 + \varphi_5(\tau_0), \\
\frac{d\delta_2}{d\tau_0} &= \tilde{f}_z\delta_1 + \varphi_2(\tau_0),
\end{aligned} \tag{2.17}$$

where $\varphi_5(\tau_0) = \tilde{A}_uL_0y(\tau_0)(\delta_3 + \bar{u}_1(0)) + \varphi_1(\tau_0)$, and $\varphi_5(\tau_0)$ is an exponentially decreasing function. The homogeneous equation of (2.17) is the variational equation of (2.9). Therefore, by Lemma 4.5 in [9], (2.17) has a unique solution which is satisfied with conditions $\delta_1(0) = 0$, $\delta_i(+\infty) = 0$ ($i = 1, 2, 3$), and which is exponentially decreasing. Thus $L_1x(\tau_0)$ is determined.

As for the unknown function $\bar{u}_1(t)$, the determination of $\bar{u}_1(t)$ is entirely similar to the case of $\bar{\boldsymbol{y}}(t)$. We have already solved its initial value $\bar{u}_1(0)$. The fundamental difference between the system of $\bar{u}_1(t)$ and the system of $\bar{\boldsymbol{y}}(t)$ is that the equation of $\bar{u}_1(t)$ is linear, while the equation of $\bar{\boldsymbol{y}}(t)$ is nonlinear. Hence we can construct an asymptotic solution up to an arbitrary order.

Next, we will construct the right boundary layer terms. The determination of $Q_i^{(-)}x(\tau_c)$ is similar to that of $L_i x(\tau_0)$. For $Q_0^{(-)}x(\tau_c)$, we have

$$\begin{aligned}
&\frac{d}{d\tau_c}Q_0^{(-)}z = \hat{A}\bigl(\bar{\boldsymbol{y}}(\sigma) + Q_0^{(-)}u(\tau_c), \theta(0), \sigma\bigr)Q_0^{(-)}y(\tau_c),\\
&\frac{d}{d\tau_c}Q_0^{(-)}y = \hat{f}\bigl(\bar{\alpha}(\sigma) + Q_0^{(-)}z(\tau_c), \rho(0), \sigma\bigr) - \check{f}\bigl(\bar{\alpha}(\sigma), \rho(0), \sigma\bigr),\\
&\frac{d}{d\tau_c}Q_0^{(-)}u = \hat{C}\bigl(\bar{\boldsymbol{y}}(\sigma) + Q_0^{(-)}u(\tau_c), \theta(0), \sigma\bigr)Q_0^{(-)}y(\tau_c),
\end{aligned} \tag{2.18}$$

and the initial and boundary value conditions

$$Q_0^{(-)}z(0) = p_0 - \bar{\alpha}(\sigma), \qquad Q_0^{(-)}x(-\infty) = 0, \tag{2.19}$$

where p_0 is an undetermined parameter. The difference between (2.18) and (2.5) is that $\bar{\boldsymbol{y}}(\sigma)$ in (2.18) is a known quantity, while $\bar{\boldsymbol{y}}(0)$ in (2.5) is unknown.

From (2.18), we obtain

$$\frac{dQ_0^{(-)}u}{dQ_0^{(-)}z} = \frac{\hat{C}(\bar{\boldsymbol{y}}(\sigma) + L_0u(\tau_c), \theta(\sigma), \sigma)}{\hat{A}(\bar{\boldsymbol{y}}(\sigma) + L_0u(\tau_c), \theta(\sigma), \sigma)}. \tag{2.20}$$

Integrating the above formula after separating the variable and using the implicit function theorem, one can claim that the system (2.20) has a solution. Let us denote by

$$Q_0^{(-)}u = \hat{U}_0\bigl(\bar{\boldsymbol{y}}(\sigma), Q_0^{(-)}z\bigr) \tag{2.21}$$

the solution of (2.20) satisfying the condition $Q_0^{(-)}u = 0$ for $Q_0^{(-)}z = 0$, namely $\hat{U}_0(\bar{\boldsymbol{y}}(\sigma), 0) = 0$.

Then substituting (2.21) into the first two equations of (2.18), we get

$$\begin{aligned}
&\frac{d}{d\tau_c}Q_0^{(-)}z = \hat{A}\bigl(\bar{\boldsymbol{y}}(\sigma) + \hat{U}_0\bigl(\bar{\boldsymbol{y}}(\sigma), Q_0^{(-)}z\bigr), \theta(0), \sigma\bigr)Q_0^{(-)}y(\tau_c),\\
&\frac{d}{d\tau_c}Q_0^{(-)}y = \hat{f}\bigl(\bar{\alpha}(\sigma) + Q_0^{(-)}z(\tau_c), \rho(0), \sigma\bigr) - \check{f}\bigl(\bar{\alpha}(\sigma), \rho(0), \sigma\bigr).
\end{aligned} \tag{2.22}$$

From this we obtain the separate equation of the saddle point

$$L_2\colon Q_0^{(-)}y(\tau_c) = \left(2\int_0^{Q_0^{(-)}z} \frac{\hat{f}(\bar{\alpha}(\sigma) + \xi, \rho(0), \sigma) - \check{f}(\bar{\alpha}(\sigma), \rho(0), \sigma)}{\hat{A}(\bar{\boldsymbol{y}}(\sigma) + \hat{U}_0(\bar{\boldsymbol{y}}(\sigma), \xi), \theta(0), \sigma)}\,d\xi\right)^{\frac{1}{2}}. \tag{2.23}$$

Equations (2.21) and (2.23) give an analytic expression of the one-dimensional manifold $\bar{\Omega}_1$, which is similar to that for the manifold $\bar{\Omega}_0$. Therefore we require that the following condition is established.

(H_4) *Suppose that* $Q_0^{(-)}z(0) \cap \bar{\Omega}_1 \neq \emptyset$.

Substituting $Q_0^{(-)}z(0)$ into (2.21) and (2.23), we get the initial values $Q_0^{(-)}u(0)$ and $Q_0^{(-)}y(0)$, and these initial values are related to the unknown parameter p_0.

For $Q_1^{(-)}z(\tau_c)$, we have

$$\begin{aligned} \frac{d}{d\tau_c}Q_1^{(-)}z &= \hat{A}Q_1^{(-)}y(\tau_C) + \hat{A}_u\big(\bar{u}_1(\sigma) + Q_1^{(-)}u(\tau_c)\big)Q_0^{(-)}y(\tau_c) + \psi_1(\tau_c), \\ \frac{d}{d\tau_c}Q_1^{(-)}y &= \hat{f}_z Q_1^{(-)}z(\tau_c) + \psi_1(\tau_c), \\ \frac{d}{d\tau_c}Q_1^{(-)}u &= \hat{C}Q_1^{(-)}y(\tau_c) + \hat{C}_u\big(\bar{u}_1(\sigma) + Q_1^{(-)}u(\tau_c)\big)Q_0^{(-)}y(\tau_c) + \psi_3(\tau_c), \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} \psi_1(\tau_c) &= \hat{A}\bar{y}_1(\sigma + \mu\tau_c) + \big[\hat{A}_u\bar{\boldsymbol{y}}'(\sigma)\tau_c + \hat{A}_\theta\theta'(\sigma)\tau_c + \hat{A}_t\tau_c\big]Q_0^{(-)}y(\tau_c) \\ &\quad - \check{A}\bar{y}_1(\sigma + \mu\tau_c) + (\hat{B} - \check{B}), \\ \psi_2(\tau_c) &= \hat{f}_z\bar{\alpha}'(\sigma)\tau_c + \hat{f}_\rho\rho'(\sigma)\tau_c + \hat{f}_t\tau_c, \\ \psi_3(\tau_c) &= \hat{C}\bar{y}_1(\sigma + \mu\tau_c) + \big[\hat{C}_u\bar{\boldsymbol{y}}'(\sigma)\tau_c + \hat{C}_\theta\theta'(\sigma)\tau_c + \hat{C}_t\tau_c\big]Q_0^{(-)}y(\tau_c) \\ &\quad - \check{C}\bar{y}_1(\sigma + \mu\tau_c) + (\hat{D} - \check{D}), \end{aligned}$$

here $\check{A}$, $\check{B}$, $\check{C}$, $\check{D}$ take values at the point $(\bar{\boldsymbol{y}}(\sigma), \theta(0), \sigma)$, $\hat{A}$, $\hat{A}_u$, $\hat{A}_\theta$, $\hat{A}_t$, $\hat{B}$, $\hat{C}$, $\hat{C}_u$, $\hat{C}_\theta$, $\hat{C}_t$, $\hat{D}$ take values at the point $(\bar{\boldsymbol{y}}(\sigma) + Q_0^{(-)}u(\tau_c), \theta(0), \sigma)$, and $\hat{f}_z, \hat{f}_\rho, \hat{f}_t$ take values at the point $(\bar{\alpha}(\sigma) + Q_0^{(-)}z(\tau_c), \rho(0), \sigma)$. The initial and boundary conditions of $Q_1^{(-)}z(\tau_c)$ are

$$Q_1^{(-)}z(0) = p_1 - \bar{z}_1(\sigma), \qquad Q_1^{(-)}x(-\infty) = 0. \tag{2.25}$$

Analogously to the determination of $L_1x(\tau_0)$, performing the diagonalization transform, $Q_1^{(-)}z = \delta_4$, $Q_1^{(-)}y = \delta_5$, $Q_1^{(-)}u = \delta_6 + \frac{\delta_4\hat{C}}{\hat{A}}$. We obtain from (2.24) the system

$$\begin{aligned} \frac{d\delta_4}{d\tau_c} &= \frac{\hat{A}_u\hat{C}}{\hat{A}}Q_0^{(-)}y(\tau_c)\delta_4 + \hat{A}\delta_5 + \hat{A}_uQ_0^{(-)}y(\tau_c)\big(\delta_6 + \bar{u}_1(\sigma)\big) + \psi_1(\tau_c), \\ \frac{d\delta_5}{d\tau_c} &= \hat{f}_z\delta_4 + \psi_2(\tau_c), \\ \frac{d\delta_6}{d\tau_c} &= \left(\hat{C}_u - \frac{\hat{C}\hat{A}_u}{\hat{A}}\right)Q_0^{(-)}y(\tau_c)\big(\delta_6 + \bar{u}_1(\sigma)\big) + \left(\varphi_3(\tau_c) - \frac{\hat{C}}{\hat{A}}\varphi_1(\tau_c)\right), \end{aligned} \tag{2.26}$$

with the initial and boundary value conditions

$$\delta_4(0) = p_1, \qquad \delta_6(0) = Q_1^{(-)}u(0) - \frac{p_1\hat{C}}{\hat{A}}, \qquad \delta_i(-\infty) = 0 \quad (i = 4, 5, 6).$$

Under the initial and boundary value conditions, the solution of the third equation of (2.26) is $\delta_6(\tau_c) = Q_1^{(-)}u(0) - \frac{p_1\hat{C}}{\hat{A}} + \int_0^{\tau_c}\hat{\Phi}(\tau_c)\hat{\Phi}^{-1}(s)\psi_4(s)\,ds$, where $\psi_4(\tau_c) = \psi_3(\tau_c) - \frac{\psi_1(\tau_c)\hat{C}}{\hat{A}}$, $\hat{\Phi}(\tau_c)$ is the fundamental matrix of the corresponding homogeneous equation with $\Phi(0) = E_k$.

The condition $\delta_6(+\infty) = 0$ uniquely determines $Q_1^{(-)}u(0)$:

$$Q_1^{(-)}u(0) = \frac{p_1\hat{C}}{\hat{A}} - \hat{\Phi}(+\infty)\int_0^{+\infty}\hat{\Phi}^{-1}(s)\psi_4(s)\,ds.$$

After computing $\delta_6(\tau_c)$, we transform the expression for the first two equations of (2.26)

$$\begin{aligned}\frac{d\delta_4}{d\tau_c} &= \frac{\hat{A}_u\hat{C}}{\hat{A}}Q_0^{(-)}y(\tau_c)\delta_4 + \hat{A}\delta_5 + \psi_5(\tau_c),\\ \frac{d\delta_5}{d\tau_c} &= \hat{f}_z\delta_4 + \psi_2(\tau_c),\end{aligned} \tag{2.27}$$

where $\varphi_5(\tau_c) = \hat{A}_u Q_0^{(-)}y(\tau_c)(\delta_6 + \bar{u}_1(\sigma)) + \psi_1(\tau_c)$. The homogeneous equation of (2.27) is the variational equation of (2.22). Therefore, by Lemma 4.5 in [9], (2.17) has a unique solution which is satisfied with the conditions $\delta_4(0) = p_1$ and $\delta_i(-\infty) = 0$ ($i = 4, 5, 6$). It should be noted that $Q_0^{(-)}x(\tau_c)$ is related to p_1, and we will use the continuous condition to determine p_k ($k \geq 0$).

2.2 The construction of asymptotic solutions on the interval [σ, T]

By analogy with the regular part on the interval $[0, \sigma]$, one can get the degenerate problem

$$\begin{aligned}&A(\bar{\bar{u}}_0(t), \bar{\boldsymbol{\gamma}}(t), t)\bar{\bar{y}}_0(t) = 0,\\ &f(\bar{\bar{z}}_0(t), \bar{\alpha}(t), t) = 0,\\ &C(\bar{\bar{u}}_0(t), \bar{\boldsymbol{\gamma}}(t), t)\bar{\bar{y}}_0(t) = 0.\end{aligned} \tag{2.28}$$

Clearly, the solution of (2.28) is

$$\bar{\bar{z}}_0(t) = \bar{\bar{\alpha}}(t), \qquad \bar{\bar{y}}_0(t) = 0, \qquad \bar{\bar{u}}_0(t) = \bar{\bar{\boldsymbol{\gamma}}}(t), \tag{2.29}$$

where $\bar{\bar{\boldsymbol{\gamma}}}(t)$ is an arbitrary k-dimensional vector function.

Now we consider the left boundary system

$$\begin{aligned}\frac{d}{d\tau_c}Q_0^{(+)}z &= \tilde{\bar{\bar{A}}}(\bar{\bar{\boldsymbol{\gamma}}}(\sigma) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{\gamma}}(\sigma) + L_0u, \sigma)Q_0^{(+)}y(\tau_c),\\ \frac{d}{d\tau_c}Q_0^{(+)}y &= \tilde{\bar{\bar{f}}}(\bar{\bar{\alpha}}(\sigma) + Q_0^{(+)}z(\tau_c), \bar{\alpha}(\sigma) + L_0z, \sigma) - \bar{\bar{f}}(\bar{\bar{\alpha}}(\sigma), \bar{\alpha}(\sigma), \sigma),\\ \frac{d}{d\tau_c}Q_0^{(+)}u &= \tilde{\bar{\bar{C}}}(\bar{\bar{\boldsymbol{\gamma}}}(\sigma) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{\gamma}}(\sigma) + L_0u, \sigma)Q_0^{(+)}y(\tau_c),\end{aligned} \tag{2.30}$$

and the initial and boundary conditions

$$Q_0^{(+)}z(0) = p_0 - \bar{\bar{\alpha}}(\sigma), \qquad Q_0^{(+)}x(-\infty) = 0. \tag{2.31}$$

Note that p_0 and $\bar{\bar{\boldsymbol{\gamma}}}(0)$ in (2.30)-(2.31) are both unknown and the initial value of $Q_0^{(+)}y(\tau_c)$ is arbitrary.

We first consider the stability manifold $\bar{\bar{\Omega}}_0$. From (2.30), we have

$$\frac{dQ_0^{(+)}u}{dQ_0^{(+)}z} = \frac{\tilde{\bar{C}}(\bar{\bar{\boldsymbol{y}}}(\sigma) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{y}}(\sigma) + L_0 u, \sigma)}{\tilde{\bar{A}}(\bar{\bar{\boldsymbol{y}}}(\sigma) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{y}}(\sigma) + L_0 u, \sigma)}. \tag{2.32}$$

Similarly, we denote the solution of (2.32) by

$$Q_0^{(+)}u = \bar{\bar{U}}_0\big(\bar{\bar{\boldsymbol{y}}}(\sigma), Q_0^{(+)}z\big), \tag{2.33}$$

satisfying the condition $Q_0^{(+)}u = 0$ for $Q_0^{(+)}z = 0$, namely $\bar{\bar{U}}_0(\bar{\bar{\boldsymbol{y}}}(\sigma), 0) = 0$.

By assumption (H_1'), one can claim that there exists a unique solution in a certain neighborhood of the point $Q_0^{(+)}z = 0$. Then substituting (2.33) into (2.30), we get

$$\begin{aligned}
\frac{dQ_0^{(+)}z}{d\tau_c} &= \tilde{\bar{A}}\big(\bar{\bar{\boldsymbol{y}}}(\sigma) + \bar{\bar{U}}_0\big(\bar{\bar{\boldsymbol{y}}}(\sigma), Q_0^{(+)}z\big), \bar{\boldsymbol{y}}(\sigma) + L_0 u, \sigma\big) Q_0^{(+)}y(\tau_c), \\
\frac{dQ_0^{(+)}y}{d\tau_c} &= \tilde{\bar{f}}\big(\bar{\bar{\alpha}}(\sigma) + Q_0^{(+)}z(\tau_c), \bar{\alpha}(\sigma) + L_0 z, \sigma\big) - \bar{\bar{f}}\big(\bar{\bar{\alpha}}(\sigma), \bar{\alpha}(\sigma), \sigma\big).
\end{aligned} \tag{2.34}$$

The equilibrium point $(0, 0)$ on the phase plane $(Q_0^{(+)}z, Q_0^{(+)}y)$ is also a saddle point. Integrating (2.34), we can get a separate equation for the saddle point,

$$L_3\colon Q_0^{(+)}y(\tau_c) = -\left(2\int_0^{Q_0^{(+)}z} \frac{\tilde{\bar{f}}(\bar{\bar{\alpha}}(\sigma) + \xi, \bar{\alpha}(\sigma) + L_0 z, \sigma) - \bar{\bar{f}}(\bar{\bar{\alpha}}(\sigma), \bar{\alpha}(\sigma), \sigma)}{\tilde{\bar{A}}(\bar{\bar{\boldsymbol{y}}}(\sigma) + \bar{\bar{U}}_0(\bar{\bar{\boldsymbol{y}}}(\sigma), \xi), \bar{\boldsymbol{y}}(\sigma) + L_0 u, \sigma)}\, d\xi\right)^{\frac{1}{2}}. \tag{2.35}$$

Equations (2.35) and (2.33) give an analytic representation of the one-dimensional manifold $\tilde{\Omega}_0$. In order to make sure (2.32) has a solution, we need the condition that follows:

(H_5) *Suppose that* $Q_0^{(+)}z(0) \cap \tilde{\Omega}_0 \neq 0$.

Putting (2.31) into (2.33), we have

$$Q_0^{(+)}u(0) = \bar{\bar{U}}_0\big(\bar{\bar{\boldsymbol{y}}}(\sigma), p_0 - \bar{\bar{\alpha}}(\sigma)\big), \tag{2.36}$$

which represents a system of k scalar equations in the k unknown components of the vector $\bar{\bar{\boldsymbol{y}}}(\sigma)$.

(H_6) *Suppose that* (2.36) *has a solution* $\bar{\bar{\boldsymbol{y}}}(\sigma) = \bar{\bar{\boldsymbol{y}}}^0$.

Here $\bar{\bar{\boldsymbol{y}}}^0$ is related to p_0, namely $\bar{\bar{\boldsymbol{y}}}^0 = \bar{\bar{\boldsymbol{y}}}^0(p_0, \bar{\bar{\alpha}}(\sigma))$. $Q_0^{(+)}x(\tau_c)$ can be determined in the same manner as $L_0 x(\tau_0)$. $Q_0^{(+)}x(\tau_c)$ is also related to p_0. For $\bar{\bar{\boldsymbol{y}}}(t)$, we only know its initial value $\bar{\boldsymbol{y}}^0$. The determination of $\bar{\bar{\boldsymbol{y}}}(t)$ should be in the next step.

For $\bar{\bar{x}}_1(t)$, we have

$$\begin{aligned}
&\bar{\bar{A}}\big(\bar{\bar{\boldsymbol{y}}}(t), \bar{\boldsymbol{y}}(t), t\big)\bar{\bar{y}}_1(t) + \bar{\bar{B}}\big(\bar{\bar{\boldsymbol{y}}}(t), \bar{\boldsymbol{y}}(t), t\big) = \bar{\bar{\alpha}}'(t), \\
&f_z\big(\bar{\bar{\alpha}}(t), \bar{\alpha}(t), t\big)\bar{\bar{z}}_1(t) = 0, \\
&\bar{\bar{C}}\big(\bar{\bar{\boldsymbol{y}}}(t), \bar{\boldsymbol{y}}(t), t\big)\bar{\bar{y}}_1(t) + \bar{\bar{D}}\big(\bar{\bar{\boldsymbol{y}}}(t), \bar{\boldsymbol{y}}(t), t\big) = \bar{\bar{\boldsymbol{y}}}'(t),
\end{aligned}$$

whose solution is

$$\bar{\bar{z}}_1(t) = 0, \qquad \bar{\bar{y}}_1(t) = \frac{\bar{\bar{\alpha}}'(t) - \bar{\bar{B}}}{\bar{\bar{A}}}, \qquad \frac{d\bar{\bar{\boldsymbol{\gamma}}}}{dt} = \frac{\bar{\bar{C}}(\bar{\bar{\alpha}}'(t) - \bar{\bar{B}})}{\bar{\bar{A}}} + \bar{\bar{D}}. \tag{2.37}$$

By the existence of solution for the initial value problem, (2.37) together with the initial condition $\bar{\bar{\boldsymbol{\gamma}}}(\sigma) = \bar{\bar{\boldsymbol{\gamma}}}^0$ has a solution $\bar{\bar{\boldsymbol{\gamma}}} = \bar{\bar{\boldsymbol{\gamma}}}(t)$ for $\sigma \le t \le T$. Therefore $\bar{\bar{x}}_0(t)$ can be completely determined. For $\bar{\bar{x}}_1(t)$, the determination of it is similar to that of $\bar{x}_1(t)$. Equation (2.37) has only determined $\bar{\bar{z}}_1(t)$ and $\bar{\bar{y}}_1(t)$, while $\bar{\bar{u}}_1(t)$ is yet unknown. We need use the first approximation of the left boundary term to determine $\bar{\bar{u}}_1(t)$.

For $Q_1^{(+)}(\tau_c)$, we obtain

$$\begin{aligned}
\frac{d}{d\tau_c}Q_1^{(+)}z &= \tilde{\bar{\bar{A}}}\big(\bar{\bar{\boldsymbol{\gamma}}}(\sigma) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{\gamma}}(\sigma) + L_0u(\tau_c), \sigma\big)Q_1^{(+)}y(\tau_c) \\
&\quad + \tilde{\bar{\bar{A}}}_u\big(\bar{\bar{u}}_1(\sigma) + Q_1^{(+)}u(\tau_c)\big)Q_0^{(+)}y(\tau_c) + \tilde{\varphi}_1(\tau_c), \\
\frac{d}{d\tau_c}Q_1^{(+)}y &= \tilde{\bar{\bar{f}}}_z Q_1^{(+)}z(\tau_c) + \tilde{\varphi}_2(\tau_c), \\
\frac{d}{d\tau_c}Q_1^{(+)}u &= \tilde{\bar{\bar{C}}}\big(\bar{\bar{\boldsymbol{\gamma}}}(0) + Q_0^{(+)}u(\tau_c), \bar{\boldsymbol{\gamma}}(\sigma), \sigma\big)Q_1^{(+)}y(\tau_c) \\
&\quad + \tilde{\bar{\bar{C}}}_u\big(\bar{\bar{u}}_1(0) + Q_1^{(+)}u(\tau_c)\big)Q_0^{(+)}y(\tau_c) + \tilde{\varphi}_3(\tau_c),
\end{aligned} \tag{2.38}$$

where the expression of $\tilde{\varphi}_i(\tau_c)$ $(i = 1,2,3)$ is similar to that of $\varphi_i(\tau_c)$ $(i = 1,2,3)$. The initial and boundary conditions of $Q_1^{(+)}x(\tau_c)$ are

$$Q_1^{(+)}z(0) = p_1 - \bar{\bar{z}}_1(\sigma), \qquad Q_1^{(+)}x(+\infty) = 0. \tag{2.39}$$

Similar to $L_1x(\tau_0)$, we introduce the diagonalization transform $Q_1^{(+)}z = \tilde{\delta}_1$, $Q_1^{(+)}y = \tilde{\delta}_2$, $Q_1^{(+)}u = \tilde{\delta}_3 + \frac{\tilde{\delta}_1\tilde{\bar{\bar{C}}}}{\tilde{\bar{\bar{A}}}}$. The system (2.38) takes the form

$$\begin{aligned}
\frac{d\tilde{\delta}_1}{d\tau_c} &= \frac{\tilde{\bar{\bar{A}}}_u\tilde{\bar{\bar{C}}}}{\tilde{\bar{\bar{A}}}}Q_0^{(+)}y(\tau_c)\tilde{\delta}_1 + \tilde{\bar{\bar{A}}}\tilde{\delta}_2 + \tilde{\bar{\bar{A}}}_u Q_0^{(+)}y(\tau_c)\big(\tilde{\delta}_3 + \bar{\bar{u}}_1(\sigma)\big) + \tilde{\varphi}_1(\tau_c), \\
\frac{d\tilde{\delta}_2}{d\tau_c} &= \tilde{\bar{\bar{f}}}_z\tilde{\delta}_1 + \tilde{\varphi}_2(\tau_c), \\
\frac{d\tilde{\delta}_3}{d\tau_c} &= \left(\tilde{\bar{\bar{C}}}_u - \frac{\tilde{\bar{\bar{C}}}\tilde{\bar{\bar{A}}}_u}{\tilde{\bar{\bar{A}}}}\right)Q_0^{(+)}y(\tau_c)\big(\tilde{\delta}_3 + \bar{\bar{u}}_1(\sigma)\big) + \left(\tilde{\varphi}_3(\tau_c) - \frac{\tilde{\bar{\bar{C}}}}{\tilde{\bar{\bar{A}}}}\tilde{\varphi}_1(\tau_c)\right),
\end{aligned} \tag{2.40}$$

with the initial and boundary value conditions

$$\tilde{\delta}_1(0) = p_1, \qquad \tilde{\delta}_3(0) = Q_1^{(+)}u(0) - \frac{p_1\tilde{\bar{\bar{C}}}}{\tilde{\bar{\bar{A}}}}, \qquad \tilde{\delta}_i(+\infty) = 0 \quad (i = 1,2,3).$$

Then the solution of the third equation of (2.40) is

$$\tilde{\delta}_3(\tau_c) = Q_1^{(+)}u(0) - \frac{p_1\tilde{\bar{\bar{C}}}}{\tilde{\bar{\bar{A}}}} + \int_0^{\tau_c}\tilde{\Phi}(\tau_c)\tilde{\Phi}^{-1}(s)\tilde{\varphi}_4(s)\,ds,$$

where $\tilde{\varphi}_4(\tau_c) = \tilde{\varphi}_3(\tau_c) - \frac{\tilde{\varphi}_1(\tau_c)\tilde{\bar{C}}}{\tilde{\bar{A}}}$, $\tilde{\Phi}(\tau_c)$ is the fundamental solution matrix of the corresponding homogeneous equations with $\tilde{\Phi}(0) = E_k$.

The condition $\tilde{\delta}_3(-\infty) = 0$ uniquely determines $Q_1^{(+)}u(0)$:

$$Q_1^{(+)}u(0) = \frac{p_1\tilde{\bar{C}}}{\tilde{\bar{A}}} - \tilde{\Phi}(-\infty)\int_0^{-\infty}\tilde{\Phi}^{-1}(s)\tilde{\varphi}_4(s)\,ds.$$

After determining $\tilde{\delta}_3(\tau_0)$, the first two equations of (2.40) can be rewritten as

$$\begin{aligned}\frac{d\tilde{\delta}_1}{d\tau_c} &= \frac{\tilde{\bar{A}}_u\tilde{\bar{C}}}{\tilde{\bar{A}}}Q_0^{(+)}y(\tau_c)\tilde{\delta}_1 + \tilde{\bar{A}}\tilde{\delta}_2 + \tilde{\varphi}_5(\tau_c),\\ \frac{d\tilde{\delta}_2}{d\tau_c} &= \tilde{\bar{f}}_z\tilde{\delta}_1 + \tilde{\varphi}_2(\tau_c).\end{aligned}\tag{2.41}$$

The homogeneous equations of (2.41) are the variational equations of (2.34). Thus (2.41) has a unique solution, which satisfies the conditions $\tilde{\delta}_1(0) = p_1$, $\tilde{\delta}_i(+\infty) = 0$, and which meets the exponential decay estimation. Hence $Q_1^{(+)}x(\tau_c)$ is completely determined. As for $\bar{\bar{u}}_1(t)$, we have obtained its initial value $\bar{\bar{u}}_1(\sigma)$. The determination of $\bar{\bar{u}}_1(t)$ is in the next step, which is completely similar to that of $\bar{\bar{\boldsymbol{y}}}(t)$. Then we will apply the continuity condition to solve p_k ($k \geq 0$).

By means of the continuous property of solution, we obtain, in view of (2.2),

$$\begin{aligned}&\mu^0\colon \bar{y}_0(\sigma) + Q_0^{(-)}y(0) = \bar{\bar{y}}_0(\sigma) + Q_0^{(+)}y(0),\\ &\mu^1\colon \bar{y}_1(\sigma) + Q_1^{(-)}y(0) = \bar{\bar{y}}_1(\sigma) + Q_1^{(+)}y(0),\\ &\ldots,\\ &\mu^k\colon \bar{y}_k(\sigma) + Q_k^{(-)}y(0) = \bar{\bar{y}}_k(\sigma) + Q_k^{(+)}y(0),\\ &\ldots.\end{aligned}\tag{2.42}$$

From the first relation of (2.42), we get an equation involving p_0,

$$\begin{aligned}M(p_0) :=& -\left(2\int_0^{p_0-\bar{\alpha}(\sigma)}\frac{\hat{f}(\bar{\alpha}(\sigma)+\xi,\rho(0),\sigma) - \check{f}(\bar{\alpha}(\sigma),\rho(0),\sigma)}{\hat{A}(\bar{\boldsymbol{y}}(\sigma)+\hat{U}_0(\bar{\boldsymbol{y}}(\sigma),\xi),\theta(0),\sigma)}\,d\xi\right)^{\frac{1}{2}}\\ &- \left(2\int_0^{p_0-\bar{\bar{\alpha}}(\sigma)}\frac{\tilde{\bar{f}}(\bar{\bar{\alpha}}(\sigma)+\xi,\bar{\alpha}(\sigma)+L_0z,\sigma) - \bar{\bar{f}}(\bar{\bar{\alpha}}(\sigma),\bar{\alpha}(\sigma),\sigma)}{\tilde{\bar{A}}(\bar{\bar{\boldsymbol{y}}}^0(p_0,\bar{\bar{\alpha}}(\sigma)) + \bar{\bar{U}}_0(\bar{\bar{\boldsymbol{y}}}^0(p_0,\bar{\bar{\alpha}}(\sigma)),\xi),\bar{\boldsymbol{y}}(\sigma)+L_0u,\sigma)}\,d\xi\right)^{\frac{1}{2}}\\ =& \;0.\end{aligned}\tag{2.43}$$

(H_7) *Suppose that the equation $M(p_0) = 0$ has a solution and $\frac{dM}{dp}|_{p=p_0} \neq 0$.*

Similarly, by the kth continuous condition, p_k ($k \geq 1$) can be solved in turn. Thus, $Q^{(+)}x(\tau_c)$ is determined. The solvable method of $Rx(\tau_1)$ is analogous to that of $Q^{(-)}x(\tau_c)$ and it is omitted here.

3 The existence of a solution and the estimate of the remainder term

We first introduce a curve L in the space of the variables (x,t). The curve L is composed of the following six pieces: $L_1 = \{(x,t) : \bar{x}_0(0) + L_0x(\tau_0), \tau_0 \geq 0, t = 0\}$, $L_2 = \{(x,t) : \bar{x}_0(t), 0 \leq t \leq$

$\sigma\}$, $L_3 = \{(x,t) : \bar{x}_0(\sigma) + Q_0^{(-)}x(\tau_c), \tau_c \leq 0, t = \sigma\}$, $L_4 = \{(x,t) : \bar{\bar{x}}_0(\sigma) + Q_0^{(+)}x(\tau_c), \tau_c \geq 0, t = \sigma\}$, $L_5 = \{(x,t) : \bar{\bar{x}}_0(t), \sigma \leq t \leq T\}$, and $L_6 = \{(x,t) : \bar{\bar{x}}_0(T) + R_0x(\tau_1), \tau_1 \leq 0, t = T\}$. We denote the projection of L onto the space of the variables (x,t) by $\tilde{L}$. Then we take the domain $G(u,t)$ in the condition H_1 to be an arbitrary δ-tube of $\tilde{L}$.

(H_1) *Suppose that the functions A, B, C, and D have continuous partial derivatives with respect to each argument up to order* $(n+2)$ *inclusive in some* δ*-tube of* $\tilde{L}$ *and* $Af_z > 0$.

Denote the kth partial sum of the series (2.1) by

$$X_k(t,\mu) = \begin{cases} \sum_{i=0}^{k} \mu^i\{\bar{x}_i(t) + L_ix(\tau_0) + Q_i^{(-)}x(\tau_c)\}, & 0 \leq t \leq \sigma, \\ \sum_{i=0}^{k} \mu^i\{\bar{\bar{x}}_i(t) + Q_i^{(+)}x(\tau_c) + R_0x(\tau_1)\}, & \sigma \leq t \leq T. \end{cases} \tag{3.1}$$

Theorem *Suppose that* (H_1)-(H_7) *hold. Then there exist positive constants* μ *and c such that for* $0 < \mu \leq \mu_0$ *there exists a unique solution* $x(t,\mu)$ *of the problem* (1.1)-(1.5) *lying in a* $c\delta$*-tube of L. Moreover, the following asymptotic expansion holds*:

$$\|x(t,\mu) - X_n(t,\mu)\| \leq c\mu^{n+1}, \quad 0 \leq t \leq T. \tag{3.2}$$

Proof Let $\zeta = z - Z_{n+1}$, $\eta = y - Y_{n+1}$, $w = u - U_{n+1}$, where $(z\ y\ u)$ is an exact solution of the problem (1.1)-(1.5), and $(Z_{n+1}\ Y_{n+1}\ U_{n+1})$ is partial sum of (3.1). Substituting $(Z_{n+1}\ Y_{n+1}\ U_{n+1})$ into (1.1)-(1.5) and separating the linear part of the zeroth approximation, we obtain for $(\zeta\ \eta\ w)$ the boundary value problem on the intervals $[0,\sigma]$ and $[\sigma,T]$, respectively, namely,

$$\mu\frac{d\zeta}{dt} = A(\bar{U}_0, \theta(t-\sigma), t)\eta + A_\mu(\bar{U}_0, \theta(t-\sigma), t)w + G_1(\eta, w, t, \mu), \tag{3.3}$$

$$\mu\frac{d\eta}{dt} = f_\mu(\bar{Z}_0, \rho(t-\sigma), t)\zeta + G_2(\zeta, t, \mu), \tag{3.4}$$

$$\mu\frac{dw}{dt} = C(\bar{U}_0, \theta(t-\sigma), t)\eta + C_\mu(\bar{U}_0, \theta(t-\sigma), t)w + G_3(\eta, w, t, \mu) \tag{3.5}$$

and

$$\mu\frac{d\zeta}{dt} = A(\bar{\bar{U}}_0, \bar{U}_0, t)\eta + A_\mu(\bar{\bar{U}}_0, \bar{U}_0, t)w + G_4(\eta, w, t, \mu), \tag{3.6}$$

$$\mu\frac{d\eta}{dt} = f_\mu(\bar{\bar{Z}}_0, \bar{Z}_0, t)\zeta + G_5(\zeta, t, \mu), \tag{3.7}$$

$$\mu\frac{dw}{dt} = C(\bar{\bar{U}}_0, \bar{U}_0, t)\eta + C_\mu(\bar{\bar{U}}_0, \bar{U}_0, t)w + G_6(\eta, w, t, \mu), \tag{3.8}$$

where we have the functions

$$\begin{aligned} G_1(\eta, w, t, \mu) = {} & A(\bar{U}_{n+1} + w, \theta(t-\sigma), t)(Y_{n+1} + \eta) + \mu B(\bar{U}_{n+1} + w, \theta(t-\sigma), t) \\ & - \mu\frac{dU_{n+1}}{dt} - A(\bar{U}_0, \theta(t-\sigma), t)\eta - A_\mu(\bar{U}_0, \theta(t-\sigma), t)w, \\ G_2(\zeta, t, \mu) = {} & f(\bar{Z}_{n+1} + \zeta, \rho(t-\sigma), t) - f_\mu(\bar{Z}_0, \rho(t-\sigma), t)\zeta - \mu\frac{dZ_{n+1}}{dt}, \end{aligned}$$

$G_i(\eta, w, t, \mu)$ $(i = 1,3,4,6)$ and $G_j(\zeta, t, \mu)$ $(j = 2,5)$, which we define having the following two important properties:

1. $G_i(0,0,t,\mu) = O(\mu^{n+2})$, $G_j(0,t,\mu) = O(\mu^{n+2})$;
2. $G_i(\eta,w,t,\mu)$ is a contraction operator with contraction coefficient of order $O(\mu)$ for η and w of order $O(\mu)$; $G_j(\zeta,t,\mu)$ is a contraction operator with contraction coefficient of order $O(\mu)$ for ζ of order $O(\mu)$.

We consider the problem on the interval $[0,\sigma]$. For the subsequent analysis, we need to do some deformations on G_1 and G_2. We have the following identical equations:

$$\begin{aligned} A\big(\bar{U}_{n+1}+w,\theta(t-\sigma),t\big) &\equiv A\big(\bar{U}_{n+1},\theta(t-\sigma),t\big)+A_\mu\big(\bar{U}_{n+1},\theta(t-\sigma),t\big)w \\ &\quad + q_1(w,t,\mu), \\ f\big(\bar{Z}_{n+1}+\zeta,\rho(t-\sigma),t\big) &\equiv f\big(\bar{Z}_{n+1},\rho(t-\sigma),t\big)+f_\mu\big(\bar{Z}_{n+1},\rho(t-\sigma),t\big)\zeta \\ &\quad + q_2(\zeta,t,\mu), \end{aligned}$$

where $q_1(w,t,\mu)$, $q_2(\zeta,t,\mu)$ are contraction operators with contraction coefficient of order $O(\mu)$ for w and ζ of order $O(\mu)$, and $q_{1,2}(0,t,\mu) = 0$. As $C(\bar{U}_{n+1}+w,\theta(t-\sigma),t)$ is expressed in an analog form (corresponding to $q_1(w,t,\mu)$ there is a contraction operator which we denote by $q_3(w,t,\mu)$). In the same way, after doing some deformations on $B(\bar{U}_{n+1}+w,\theta(t-\sigma),t)$ and $D(\bar{U}_{n+1}+w,\theta(t-\sigma),t)$, the functions G_i ($i=1,3$) and G_2 can be written in the form

$$\begin{aligned} G_i(\eta,w,t,\mu) &= \mu a_i(t,\mu)\eta+\mu b_i(t,\mu)w+c_i(t,\mu)\eta w \\ &\quad + q_i(w,t,\mu)\bar{Y}_0(t,u)+Q_i(\eta,w,t,\mu), \\ G_2(\zeta,t,\mu) &= \mu a_2(t,\mu)\zeta+q_2(\zeta,t,\mu)+Q_2(\zeta,t,\mu), \end{aligned}$$

where a_i, b_i and c_i ($i=1,2,3$) are certain bounded functions or matrices, $Q_i(\eta,w,t,\mu)$ are contraction operators with contraction coefficient of order $O(\mu)$ for η and w of order $O(\mu)$, and $Q_i(0,0,t,\mu) = O(\mu^{n+2})$. For brevity, here and in what follows, a function or matrix is represented by the symbol ω.

Let $w = \lambda + \frac{C(\bar{U}_0,\theta(t-\sigma),t)}{A(\bar{U}_0,\theta(t-\sigma),t)}\zeta$. After exchanging w of (3.3)-(3.5) into λ, we have

$$\begin{aligned} \mu\frac{d\zeta}{dt} &= \frac{A_\mu(\bar{U}_0,\theta(t-\sigma),t)C(\bar{U}_0,\theta(t-\sigma),t)}{A(\bar{U}_0,\theta(t-\sigma),t)}\zeta + A\big(\bar{U}_0,\theta(t-\sigma),t\big)\eta \\ &\quad + \left(\omega\bar{Y}_0(t,\mu)\lambda + G_1\left(\eta,\lambda+\frac{C(\bar{U}_0,\theta(t-\sigma),t)}{A(\bar{U}_0,\theta(t-\sigma),t)}\zeta,t,\mu\right)\right), \end{aligned} \tag{3.9}$$

$$\mu\frac{d\eta}{dt} = f_\mu\big(\bar{Z}_0,\rho(t-\sigma),t\big)\zeta + G_2(\zeta,t,\mu), \tag{3.10}$$

$$\mu\frac{d\lambda}{dt} = h(t,\mu)\lambda + \big(G(\lambda,\eta,\zeta,t,\mu)+q(\lambda,\zeta,t,\mu)\bar{Y}_0(t,\mu)+Q(\lambda,\eta,\zeta,t,\mu)\big), \tag{3.11}$$

where

$$h(t,\mu) = O\left(\mu+\exp\left(\frac{-\sigma_0 t}{\mu}\right)+\exp\left(\frac{-\sigma_0(\sigma-t)}{\mu}\right)\right), \tag{3.12}$$

$$G(\lambda,\eta,\zeta,t,\mu) = \mu\omega\eta+\mu\omega\zeta+\omega\lambda\eta+\omega\eta\zeta, \tag{3.13}$$

$$q(\lambda,\zeta,t,\mu) = -\frac{C(\bar{U}_0,\theta(t-\sigma),t)}{A(\bar{U}_0,\theta(t-\sigma),t)}q_1\left(\lambda+\frac{C(\bar{U}_0,\theta(t-\sigma),t)}{A(\bar{U}_0,\theta(t-\sigma),t)}\zeta,t,\mu\right)$$

$$+ q_2\left(\lambda + \frac{C(\bar{U}_0, \theta(t-\sigma), t)}{A(\bar{U}_0, \theta(t-\sigma), t)}\zeta, t, \mu\right), \tag{3.14}$$

$$Q(\lambda, \eta, \zeta, t, \mu) = -\frac{C(\bar{U}_0, \theta(t-\sigma), t)}{A(\bar{U}_0, \theta(t-\sigma), t)} Q_1\left(\eta, \lambda + \frac{C(\bar{U}_0, \theta(t-\sigma), t)}{A(\bar{U}_0, \theta(t-\sigma), t)}\zeta, t, \mu\right)$$
$$+ Q_2\left(\eta, \lambda + \frac{C(\bar{U}_0, \theta(t-\sigma), t)}{A(\bar{U}_0, \theta(t-\sigma), t)}\zeta, t, \mu\right). \tag{3.15}$$

The operator $q(\lambda, \zeta, t, \mu)$ is a contraction with a contraction coefficient of order $O(\mu)$ for λ and ζ of order $O(\mu)$, and $q(0, 0, t, \mu) = 0$; $Q(\lambda, \eta, \zeta, t, \mu)$ is for contraction operators with contraction coefficient of order $O(\mu)$ for λ, η and ζ of order $O(\mu^2)$, and $Q(0, 0, 0, t, \mu) = O(\mu^{n+2})$.

We will consider $G_2(\zeta, t, \mu)$ and the right-hand parentheses of (3.9) and (3.11) as non-homogeneous terms and write them into the equivalent integral equations. Let us denote the Green's matrix of (3.9)-(3.10) by $\boldsymbol{\gamma}(t, s, \mu)$. Under the boundary condition $\zeta(0, \mu) = \zeta(\sigma, \mu) = 0$, $\boldsymbol{\gamma}$ is satisfied with the estimate $\boldsymbol{\gamma}(t, s, \mu) = O(\exp(\frac{-\sigma_0|t-s|}{\mu}))$. By the conditions $\zeta(0, \mu) = O(\mu^{n+2})$ and $\zeta(\sigma, \mu) = O(\mu^{n+2})$, the corresponding homogeneous equation has a solution which has the same order of smallness as the boundary value. Therefore, replacing (3.9)-(3.10) by the integral equation, we have

$$\begin{pmatrix} \zeta(t, \mu) \\ \eta(t, \mu) \end{pmatrix} = O(\mu^{n+2}) + \int_0^\sigma \mu^{-1} \boldsymbol{\gamma}(t, s, \mu) \begin{pmatrix} \omega \bar{Y}_0(s, \mu)\lambda(s, \mu) + G_1 \\ G_2 \end{pmatrix} ds$$
$$= \begin{pmatrix} S_1(\lambda, \eta, \zeta, t, \mu) \\ S_2(\lambda, \eta, \zeta, t, \mu) \end{pmatrix}. \tag{3.16}$$

Denote the fundamental solution matrix of the homogeneous equations of (3.11) by $H(t, s, \mu)$. From (3.12), one can claim that $H(t, s, \mu)$ is bounded. Clearly, the initial value condition for $\lambda(t, \mu)$ is of the same type as that for $w(t, \mu)$, that is, $\lambda(0, \mu) = O(\mu^{n+2})$. Therefore, (3.11) can be rewritten as the integral equation

$$\lambda(t, \mu) = O(\mu^{n+2}) + \mu^{-1} \int_0^t H(t, s, \mu)\big(G(\lambda, \eta, \zeta, s, \mu) + q(\lambda, \zeta, s, \mu)\bar{Y}_0(t, \mu)$$
$$+ Q(\lambda, \eta, \zeta, t, \mu)\big)\, ds. \tag{3.17}$$

By virtue of the properties of Q, $R_1(\lambda, \eta, \zeta, t, \mu) \equiv \mu^{-1} \int_0^t H(t, s, \mu) Q\, ds$ is a contraction with contraction coefficient of order $O(\mu)$ for λ, η, and ζ of order $O(\mu)$; moreover, $R_1(0, 0, 0, t, \mu) = O(\mu)$. Since

$$\bar{Y}_0(t, \mu) = O\left(\exp\left(\frac{-\sigma_0 t}{\mu}\right) + \exp\left(\frac{-\sigma_0(\sigma - t)}{\mu}\right)\right),$$

and then

$$\int_0^t H(t, s, \mu)\bar{Y}_0(t, \mu) = \int_0^t O\left(\exp\left(\frac{-\sigma_0 t}{\mu}\right) + \exp\left(\frac{-\sigma_0(\sigma - t)}{\mu}\right)\right) = O(\mu),$$

the operator

$$R_2(\lambda, \eta, \zeta, t, \mu) \equiv \mu^{-1} \int_0^t H(t, s, \mu) q(\lambda, \zeta, s, \mu)\bar{Y}_0(s, \mu)\, ds$$

has the same properties as $R_1(\lambda, \eta, \zeta, t, \mu)$. Let $R(\lambda, \eta, \zeta, t, \mu) = R_1 + R_2 + O(\mu^{n+2})$. Replacing ζ and η in G by (3.16), (3.17) can be rewritten as

$$\lambda(t,\mu) = \int_0^t H(t,s,\mu)\left(\omega S_1 + \omega S_2 + \frac{\omega}{\mu}\lambda S_2 + \frac{\omega}{\mu} S_1 S_2\right) ds + R(\lambda, \eta, \zeta, t, \mu). \tag{3.18}$$

Taking into account the estimate of the Green's function, namely,

$$\boldsymbol{\gamma}(t,s,\mu) = O\left(\exp\left(\frac{-\sigma_0|t-s|}{\mu}\right)\right),$$

the estimate of $\bar{Y}_0(t,\mu)$, and

$$\int_0^t \int_0^\sigma \mu^{-1} \exp\left(\frac{-\sigma_0|s-p|}{\mu}\right)\left(\exp\left(\frac{-\sigma_0 p}{\mu}\right) + \exp\left(\frac{-\sigma_0(\sigma-p)}{\mu}\right)\right) dp\, ds = O(\mu),$$

it is not difficult to prove that the first term in the right-hand of (3.18) is a contraction operator of the same type as the second term $R(\lambda, \eta, \zeta, t, \mu)$. Then we obtain the following expression for $\lambda(t,\mu)$:

$$\lambda(t,\mu) = T_1(\lambda, \eta, \zeta, t, \mu), \tag{3.19}$$

where $T_1(\lambda, \eta, \zeta, t, \mu)$ is a contraction operator with contraction coefficient of order $O(\mu)$ for λ, η, and ζ of order $O(\mu)$, and $T_1(0,0,0,t,\mu) = O(\mu^{n+1})$.

Using (3.19) in (3.16), we get

$$\zeta(t,\mu) = S_1(\lambda, \eta, \zeta, t, \mu) \equiv T_2(\lambda, \eta, \zeta, t, \mu), \tag{3.20}$$

$$\eta(t,\mu) = S_2(\lambda, \eta, \zeta, t, \mu) \equiv T_3(\lambda, \eta, \zeta, t, \mu), \tag{3.21}$$

where the operators T_2 and T_3 are analogous to T_1.

By applying the method of successive approximation in the system (3.19)-(3.21), we can prove that for sufficiently small $\mu_0 > 0$ a unique solution exists in a certain $c\mu$-tube of the curve $\lambda = \eta = \zeta = 0$, and it satisfies estimates $\lambda = O(\mu^{n+1})$, $\eta = O(\mu^{n+1})$, $\zeta = O(\mu^{n+1})$, and $w = O(\mu^{n+1})$. Thus $\zeta = z - Z_{n+1}$, $\eta = y - Y_{n+1}$, and $w = u - U_{n+1}$ are all of order $O(\mu^{n+1})$.

The proof of the interval $[\sigma, T]$ is analogous to that of the interval $[0, \sigma]$. Thus, we obtain the inequality (3.2) because of $X_{n+1}(t,\mu) - X_n(t,\mu) = O(\mu^{n+1})$. This finishes the proof. □

4 Example

Consider the following system:

$$\mu\frac{dz}{dt} = y(t), \qquad \mu\frac{dy}{dt} = z(t) + 2z\left(t - \frac{1}{2}\right), \quad 0 \le t \le 1, \tag{4.1}$$

$$\mu\frac{du}{dt} = (2 \quad 1)u\left(t - \frac{1}{2}\right)u(t)y(t), \tag{4.2}$$

with the initial and boundary conditions

$$z(t,\mu) = -1, \qquad u(t,\mu) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \quad -\frac{1}{2} \le t \le 0, \tag{4.3}$$

$$z(1,\mu) = 3. \tag{4.4}$$

According to (1.1)-(1.5), we have $A = 1$, $B = D = 0$, $f = z(t) + 2z(t-\frac{1}{2})$, $C = (2\ 1)u(t-\frac{1}{2})u(t)y(t)$, $\rho(t) = -1$, $\theta(t) = \binom{\frac{1}{2}}{1}$ and $Z^T = 3$. It is easy to see that $Af_z = 1 > 0$. Then the condition (H_1') is satisfied.

By calculating, we have

$$F_x = \begin{pmatrix} 0 & 1 & 0 \\ f_z & 0 & 0 \\ 0 & C & 0 \end{pmatrix}.$$

Therefore F_x has $\lambda = 0$ as an eigenvalue of multiplicity 2 as well as two eigenvalues of opposite signs, namely, $\lambda_{1,2} = \pm 1$. Thus, it is a critical conditionally stable case.

By taking $\mu = 0$, we obtain the solution of the degenerated problem on the interval $[0,\frac{1}{2}]$, that is,

$$\bar{y}_0(t) = 0, \qquad \bar{z}_0(t) = 2, \qquad \bar{u}_0(t) = \bar{\boldsymbol{\gamma}}(t),$$

where $\bar{\boldsymbol{\gamma}}(t) = \binom{\bar{\gamma}_1(t)}{\bar{\gamma}_2(t)}$ is an arbitrary two-dimensional vector. For $L_0x(\tau_0)$ and $\bar{x}_1(t)$, we have

$$\frac{d}{d\tau_0}L_0 z = L_0 y(\tau_0), \qquad \frac{d}{d\tau_0}L_0 y = L_0 z(\tau_0), \tag{4.5}$$

$$\frac{d}{d\tau_0}L_0 u = 2\begin{pmatrix} \bar{\boldsymbol{\gamma}}_1(0) + L_{01}u(\tau_0) \\ \bar{\boldsymbol{\gamma}}_2(0) + L_{02}u(\tau_0) \end{pmatrix} L_0 y(\tau_0), \tag{4.6}$$

$$L_0 u(0) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{\gamma}}_1(0) \\ \bar{\boldsymbol{\gamma}}_2(0) \end{pmatrix}, \qquad L_0 x(+\infty) = 0 \tag{4.7}$$

and $\bar{y}_1(t) = 0$, $\bar{z}_1(t) = 0$, $\bar{\boldsymbol{\gamma}}'(t) = 0$. After calculating, we get

$$L_0 z(\tau_0) = -L_0 y(\tau_0) = -3e^{-\tau_0},$$

$$L_0 u(\tau_0) = -\bar{\boldsymbol{\gamma}}(0) + \bar{\boldsymbol{\gamma}}(0)e^{2L_0 z(\tau_0)},$$

$$\bar{\boldsymbol{\gamma}}(t) = \begin{pmatrix} \bar{\boldsymbol{\gamma}}_1(t) \\ \bar{\boldsymbol{\gamma}}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^6 \\ e^6 \end{pmatrix}.$$

It is easy to see that $L_0 u(+\infty) = 0$. Therefore, $L_0 x(\tau_0)$ satisfies an exponential decay estimation $\|L_0 x(\tau_0)\| \le C_1 e^{-\epsilon\tau_0}$, for C_1 and ϵ are positive constants. Then the conditions (H_2) and (H_3) are satisfied.

The system for $Q_0^{(-)}x(\tau_c)$ is

$$\frac{d}{d\tau_c}Q_0^{(-)}z = Q_0^{(-)}y(\tau_c), \qquad \frac{d}{d\tau_c}Q_0^{(-)}y = Q_0^{(-)}z(\tau_c), \tag{4.8}$$

$$\frac{d}{d\tau_c}Q_0^{(-)}u = 2\left(\bar{\boldsymbol{\gamma}}\left(\frac{1}{2}\right) + Q_0^{(-)}u(\tau_c)\right)Q_0^{(-)}y(\tau_c), \tag{4.9}$$

$$Q_0^{(-)}z(0) = p_0 - 2, \qquad Q_0^{(-)}x(-\infty) = 0. \tag{4.10}$$

Similar to $L_0x(\tau_0)$, we obtain

$$Q_0^{(-)}u(\tau_c) = -\bar{\gamma}\left(\frac{1}{2}\right) + \bar{\gamma}\left(\frac{1}{2}\right)e^{2L_0z(\tau_0)}, \qquad Q_0^{(-)}y(\tau_c) = \left(2\int_0^{Q_0^{(-)}z(\tau_c)} \xi\, d\xi\right)^{\frac{1}{2}}.$$

Considering the problem on the interval $[\frac{1}{2}, 1]$, by the zero approximation equation of the regular part, we have

$$\bar{\bar{z}}_0(t) = -4, \qquad \bar{\bar{y}}_0(t) = 0, \qquad \bar{\bar{u}}_0(t) = \bar{\bar{\gamma}}(t),$$

where $\bar{\bar{\gamma}}(t) = \binom{\bar{\bar{\gamma}}_1(t)}{\bar{\bar{\gamma}}_2(t)}$ is an arbitrary two-dimensional vector. For $Q_0^{(+)}x(\tau_c)$, we get

$$\frac{d}{d\tau_c}Q_0^{(+)}z = Q_0^{(+)}y(\tau_c), \qquad \frac{d}{d\tau_c}Q_0^{(+)}y = Q_0^{(+)}z(\tau_c) - 3e^{-\tau_c}, \tag{4.11}$$

$$\frac{d}{d\tau_c}Q_0^{(+)}u = (2 \quad 1)\begin{pmatrix} \bar{\gamma}_1(\frac{1}{2}) + L_{01}u \\ \bar{\gamma}_2(\frac{1}{2}) + L_{02}u \end{pmatrix}\left(\bar{\bar{\gamma}}\left(\frac{1}{2}\right) + Q_0^{(+)}u(\tau_c)\right)Q_0^{(+)}y(\tau_c), \tag{4.12}$$

$$Q_0^{(+)}z(0) = p_0 + 4, \qquad Q_0^{(+)}x(+\infty) = 0. \tag{4.13}$$

By calculating, we obtain

$$Q_0^{(+)}y(\tau_c) = -\left(2\int_0^{Q_0^{(+)}z(\tau_c)} \left(\xi - 3e^{-\tau_c}\right) d\xi\right)^{\frac{1}{2}}, \tag{4.14}$$

$$\begin{pmatrix} \bar{\bar{\gamma}}_1(\frac{1}{2}) \\ \bar{\bar{\gamma}}_2(\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} Q_{01}^{(+)}u(0)\{e^{[2\bar{\gamma}_1(\frac{1}{2})+\bar{\gamma}_2(\frac{1}{2})]e^{2L_0z}Q_0^{(+)}z(0)} - 1\}^{-1} \\ Q_{02}^{(+)}u(0)\{e^{[2\bar{\gamma}_1(\frac{1}{2})+\bar{\gamma}_2(\frac{1}{2})]e^{2L_0z}Q_0^{(+)}z(0)} - 1\}^{-1} \end{pmatrix}. \tag{4.15}$$

The equation for $\bar{\bar{x}}_1(t)$ is of the form $\bar{\bar{y}}_1(t) = 0$, $\bar{\bar{z}}_1(t) = 0$, $\bar{\bar{\gamma}}'(t) = 0$, which combined with the initial condition (4.15) yields $\bar{\bar{\gamma}}(t)$. Then the condition (H_6) is satisfied.

By the continuous condition of the zero times approximation, $\bar{y}_0(\sigma) + Q_0^{(-)}y(0) = \bar{\bar{y}}_0(\sigma) + Q_0^{(+)}y(0)$, *i.e.*,

$$\left(2\int_0^{p_0-2} \xi\, d\xi\right)^{\frac{1}{2}} = -\left(2\int_0^{p_0+4} (\xi - 3)\, d\xi\right)^{\frac{1}{2}},$$

we obtain $p_0 = 2$. Let $M(p_0) = (2\int_0^{p_0-2} \xi\, d\xi)^{\frac{1}{2}} = -(2\int_0^{p_0+4}(\xi - 3)\, d\xi)^{\frac{1}{2}}$; we have $\frac{dM}{dp_0}|_{p_0=2} = 1 + [(p_0+1)^2 - 9]^{-\frac{1}{2}}(p_0+1)|_{p_0=2} \neq 0$. Then the condition ($H_7$) is satisfied.

Bringing $p_0 = 2$ into (4.10) and considering (4.8)-(4.9), we have

$$Q_0^{(-)}y(\tau_c) = Q_0^{(-)}z(\tau_c) = 0, \qquad Q_0^{(-)}u(\tau_c) = 0. \tag{4.16}$$

It is easy to check that $Q_0^{(-)}x(\tau_0)$ satisfies an exponential decay estimation. Then the condition (H_4) is satisfied.

Next, substituting (4.14) into the first equation of (4.11) and in view of the initial value, we get

$$Q_0^{(+)}z(\tau_c) = -\frac{3}{2} \cdot \frac{1 - 2\,\text{LambertW}(-e^{-2\tau_c-1}) + \text{Lambert}\,W(-e^{-2\tau_c-1})^2}{e^{\tau_c}\,\text{LambertW}(-e^{-2\tau_c-1})},$$

where the LambertW function satisfies $\mathrm{LambertW}(x)e^{\mathrm{LambertW}(x)} = x$, x is a algebraic expression. Taking account of (4.14)-(4.15), $Q_0^{(+)}y(\tau_c)$ and $Q_0^{(+)}u(\tau_c)$ are both determined and obey an exponential decay estimation. Then the condition (H_5) is satisfied.

For $R_0x(\tau_1)$, we have

$$\frac{d}{d\tau_1}R_0z = R_0y(\tau_1), \qquad \frac{d}{d\tau_1}R_0y = R_0z(\tau_1), \tag{4.17}$$

$$\frac{d}{d\tau_1}R_0u = (2 \quad 1)\begin{pmatrix} \bar{\boldsymbol{\gamma}}_1(1) + Q_{01}^{(-)}u \\ \bar{\boldsymbol{\gamma}}_2(1) + Q_{02}^{(-)}u \end{pmatrix}\left(\bar{\bar{\boldsymbol{\gamma}}}(1) + R_0u(\tau_1)\right)R_0y(\tau_1), \tag{4.18}$$

$$R_0z(0) = 7, \qquad R_0x(-\infty) = 0, \tag{4.19}$$

whose solution is

$$R_0z(\tau_1) = R_0y(\tau_1) = 7e^{\tau_1},$$

$$\begin{pmatrix} R_{01}u \\ R_{02}u \end{pmatrix} = \begin{pmatrix} \bar{\bar{\boldsymbol{\gamma}}}_1(1)\{e^{[2(\bar{\gamma}_1(1)+Q_{01}^{(-)}u)+\bar{\gamma}_2(1)+Q_{02}^{(-)}u]R_0z} - 1\} \\ \bar{\bar{\boldsymbol{\gamma}}}_2(1)\{e^{[2(\bar{\gamma}_1(1)+Q_{01}^{(-)}u)+\bar{\gamma}_2(1)+Q_{02}^{(-)}u]R_0z} - 1\} \end{pmatrix}.$$

It is not difficult to check that $R_0u(-\infty) = 0$. Therefore, $R_0x(\tau_1)$ satisfies an exponential decay estimation. Hence we construct a zeroth asymptotic solution $x = (z\ y\ u)^T$:

$$x(t,\mu) = \begin{cases} \bar{x}_0(t) + L_0x(\tau_0) + Q_0^{(-)}x(\tau_c), & 0 \le t \le \frac{1}{2}, \\ \bar{\bar{x}}_0(t) + Q_0^{(+)}x(\tau_c) + R_0x(\tau_1), & \frac{1}{2} \le t \le 1, \end{cases} \tag{4.20}$$

where $\tau_0 = \frac{t}{\mu}$, $\tau_c = \frac{t-\frac{1}{2}}{\mu}$, $\tau_1 = \frac{t-1}{\mu}$.

5 Summary

Semiconductor device simulation heavily depends on the two-dimensional model of the space. However, with the complicated design of the electronic devices, the simulation of the three-dimensional space and the higher dimensional space becomes increasingly important. Then the question arises: how do we get the results of the n-dimensional case and extend these results to the three-dimensional case and the lower-dimensional case? In this paper, by using the boundary layer function method we consider a class of generalized high-dimensional delayed semiconductor equations. Under the critical conditionally stable situation, we obtain the approximate solution expression. In comparison with [4–8], the system which we study is more general. We not only increase the dimension of the system but also add the effect of a lag. The approximate analytic solution that we obtain can carry out differential and integral operations. Therefore, we can continue to find other related results. Thus the result is more simple, practical, and reliable.

Competing interests
The author declares that no conflict of competing interests exists.

Author's contributions
NW carried out all work of this article and the main theorem. The author read and approved the final manuscript.

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