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A mixed finite element method for the Reissner-Mindlin plate

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Abstract

In this paper, a new mixed variational form for the Reissner-Mindlin problem is given, which contains two unknowns instead of the classical three ones. A mixed triangle finite element scheme is constructed to get a discrete solution. A new method is put to use for proving the uniqueness of the solutions in both continuous and discrete mixed variational formulations. The convergence and error estimations are obtained with the help of different norms. Numerical experiments are given to verify the validity of the theoretical analysis.

Keywords: Reissner-Mindlin plate; mixed finite element; error estimates

1 Introduction

In the past few decades, many plate bending elements based on Reissner-Mindlin theory have been developed to construct numerical models for thick plate and shell structures. The existing literature, such as [1, 2], increases the understanding of the problem context. In [1], the theory of semigroups of linear operators is applied for proving the existence and uniqueness of solutions for the initial-boundary value problems in the thermoelasticity of micropolar bodies, and in [2], the theory of semigroups of operators is applied to obtain the existence and uniqueness of solutions for the mixed initial-boundary value problems in thermoelasticity of dipolar bodies.

Many works compute all three unknowns $(\theta, \omega, \upsilon)$ together, and some (see [3–6]) of them propose numerical techniques and effective formulations to eliminate shear locking when the thickness of the plate is thin. For instance, using discontinuous Galerkin techniques, [3] develops a locking-free nonconforming element, and in order to prove optimal error estimates, it uses penalty for θ. But in [4], in order to avoid the locking phenomenon, it presents a triangular mixed finite element method, which is based on a linked interpolation between deflections and rotations.

Moreover, [7] uses the ideas of discontinuous Galerkin methods to obtain and analyze two new families of locking-free finite element methods for approximation of the Reissner-Mindlin plate problem. Following their basic approach, but making different choices of finite element spaces, [8] develops and analyzes other families of locking-free finite elements, which can eliminate the need for the introduction of a reduction operator. A hybrid-mixed finite element model has been proposed in [9], and it is based on the Legendre polynomials. Duan [10] uses continuous linear elements (enriched with local bubbles) to approximate rotation and transverse displacement variables, and an L^2

projector is applied to the shear energy term onto the Raviart-Thomas H(div; Ω) finite element. Moreover, two first-order nonconforming rectangular elements are proposed in [11], and [12] generalizes these schemes to the quadrilateral mesh. For the first quadrilateral element, both components of the rotation are approximated by the usual conforming bilinear element and the modified nonconforming rotated Q_1 element enriched with the intersected term on each element to approximate the displacement, whereas the second one uses the enriched modified nonconforming rotated Q_1 element to approximate both the rotation and the displacement. Both elements employ a more complicated shear force space to overcome the shear force locking. In addition, [13] presents four quadrilateral elements for the Reissner-Mindlin plate model. The elements are the stabilized MITC4 element, the MIN4 element, the Q4BL element, and the FMIN4 element. All elements introduce a unifying variational formulation and prove the optimal H^1 error bounds to be uniform in the plate thickness except for the Q4BL element. The bending behaviors of composite plate with 3-D periodic configuration are considered in [14], and it designs a second-order two-scale (SOTS) computational method in a constructive way.

In this paper, the advantage is that only two unknowns (rotation θ and displacement ω) are computed. The existence and uniqueness of the solution of the variational formulation will be given. A low-degree mixed finite element method is adopted to solve the problem, which is based on the use of piecewise linear functions for both rotations and transversal displacements, and also a bubble ($\lambda_1\lambda_2\lambda_3$) is added to each component of rotations. The convergence and error estimation for the mixed finite element method are presented by the use of different norms.

The rest of the paper is organized as follows. In Section 2, the model of Reissner-Mindlin is be presented. In Section 3, a new mixed variational formulation is given, and also the existence and uniqueness of the solution are proved. In Section 4, the finite element spaces are constructed, and the corresponding discrete mixed variational formulation is presented. In Section 5, the error estimate is demonstrated. In Section 6, a numerical experiment is given to testify the accuracy of the theoretical analysis.

2 The Reissner-Mindlin problem

The Reissner-Mindlin problem with clamped boundary is to find (θ, ω, γ) such that

$$-\operatorname{div} \mathbf{C}\varepsilon(\theta) - \gamma = 0 \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \gamma = g \quad \text{in } \Omega, \tag{2}$$

$$\gamma = \lambda t^{-2}(\nabla\omega - \theta) \quad \text{in } \Omega, \tag{3}$$

$$\theta = 0, \qquad \omega = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where $\mathbf{C}$ is the tensor of bending moduli, θ represents the rotations, ω is the transversal displacement, and γ represents the scaled share stresses. Moreover, λ is the share correction factor, $g \in L^2(\Omega)$, t is the thickness, and ε is the usual symmetric gradient operator

$$\varepsilon(\theta) = \begin{pmatrix} \frac{\partial\theta_1}{\partial x_1} & \frac{1}{2}(\frac{\partial\theta_1}{\partial x_2} + \frac{\partial\theta_2}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial\theta_2}{\partial x_1} + \frac{\partial\theta_1}{\partial x_2}) & \frac{\partial\theta_2}{\partial x_2} \end{pmatrix}.$$

The above equations correspond to the minimization of the functional

$$J^t(\eta, \upsilon) = \frac{1}{2}a(\eta, \eta) + \frac{\lambda t^{-2}}{2}\|\nabla\upsilon - \eta\|^2_{0,\Omega} - (g, \upsilon), \tag{5}$$

where

$$a(\theta, \eta) = \int_\Omega \mathbf{C}\varepsilon(\theta) : \varepsilon(\eta)\,dx, \tag{6}$$

and $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. The operator : is defined as

$$\varepsilon(\theta) : \varepsilon(\eta) = \varepsilon_{11}(\theta)\varepsilon_{11}(\eta) + \varepsilon_{12}(\theta)\varepsilon_{12}(\eta) + \varepsilon_{21}(\theta)\varepsilon_{21}(\eta) + \varepsilon_{22}(\theta)\varepsilon_{22}(\eta).$$

3 New mixed variational formulation

The classical variational formulation of the Reissner-Mindlin problem is to find $(\theta, \omega, \gamma) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega) \times (L^2(\Omega))^2$ such that

$$a(\theta, \eta) - (\gamma, \eta) = 0, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2, \tag{7}$$

$$(\gamma, \nabla\upsilon) = (g, \upsilon), \quad \forall \upsilon \in H_0^1(\Omega), \tag{8}$$

$$\lambda^{-1}t^2(\gamma, \tau) - (\nabla\omega, \tau) + (\theta, \tau) = 0, \quad \forall \tau \in \left(L^2(\Omega)\right)^2. \tag{9}$$

In the former work on the Reissner-Mindlin problem, the three unknowns were just computed with this classical variational formulation (7)-(9). We will derive a new format, which contains only two unknowns.

In (9), instead of $\tau \in (L^2(\Omega))^2$, it suffices to take $\eta \in (H_0^1(\Omega))^2$, that is,

$$(\gamma, \eta) = \lambda t^{-2}(\nabla\omega, \eta) - \lambda t^{-2}(\theta, \eta), \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2. \tag{10}$$

Inserting (10) into (7), we have

$$\int_\Omega \mathbf{C}\varepsilon(\theta) : \varepsilon(\theta)\,dx + \lambda t^{-2}\int_\Omega \theta \cdot \theta\,dx - \lambda t^{-2}\int_\Omega \eta \cdot \nabla\omega\,dx = 0, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2. \tag{11}$$

Thus, $\nabla\upsilon \in (L^2(\Omega))^2$ for all $\upsilon \in H_0^1(\Omega)$. Let $\tau = \nabla\upsilon$ in (9). Then

$$(\gamma, \nabla\upsilon) - \lambda t^{-2}(\nabla\omega, \nabla\upsilon) + \lambda t^{-2}(\theta, \nabla\upsilon) = 0, \quad \forall \upsilon \in H_0^1(\Omega). \tag{12}$$

Inserting (12) into (8), we have

$$-\lambda t^{-2}(\theta, \nabla\upsilon) + \lambda t^{-2}(\nabla\omega, \nabla\upsilon) = (g, \upsilon), \quad \forall \upsilon \in H_0^1(\Omega). \tag{13}$$

Combining (11) with (13) and letting

$$a_1(\theta, \eta) = a(\theta, \eta) + \lambda t^{-2}(\theta, \eta) = \int_\Omega \mathbf{C}\varepsilon(\theta) : \varepsilon(\theta)\,dx + \lambda t^{-2}\int_\Omega \theta \cdot \theta\,dx,$$

$$b(\eta, \omega) = -\lambda t^{-2}\int_\Omega \eta \cdot \nabla\omega\,dx, \qquad c(\omega, \upsilon) = \lambda t^{-2}\int_\Omega \nabla\omega \cdot \nabla\upsilon\,dx,$$

$$g(\upsilon) = \int_\Omega g \cdot \upsilon \, dx,$$

we get the new mixed variational formulation: find $(\theta, \omega) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega)$ such that

$$a_1(\theta, \eta) + b(\eta, \omega) = 0, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2, \tag{14}$$

$$b(\theta, \upsilon) + c(\omega, \upsilon) = g(\upsilon), \quad \forall \upsilon \in H_0^1(\Omega). \tag{15}$$

For the new mixed variational formulation, it is obvious that the bilinear forms of $a_1(\cdot, \cdot)$ are $(H_0^1(\Omega))^2$-elliptic and continuous:

$$\begin{aligned} a_1(\theta, \theta) &= a(\theta, \theta) + \lambda t^{-2}(\theta, \theta) \\ &= \int_\Omega \mathbf{C}\varepsilon(\theta) : \varepsilon(\theta) \, dx + \lambda t^{-2} \int_\Omega \theta \cdot \theta \, dx \\ &\geq \alpha \|\theta\|_1^2, \quad \forall \theta \in \left(H_0^1(\Omega)\right)^2, \end{aligned}$$

where α is a positive constant, and this result follows by the Korn-inequality (see [15]). This means that $a_1(\cdot, \cdot)$ are $(H_0^1(\Omega))^2$-elliptic. Moreover,

$$\begin{aligned} a_1(\theta, \eta) &= a(\theta, \eta) + \lambda t^{-2}(\theta, \eta) \\ &= \int_\Omega \mathbf{C}\varepsilon(\theta) : \varepsilon(\eta) \, dx + \lambda t^{-2} \int_\Omega \theta \cdot \eta \, dx \\ &\leq \sigma \|\theta\|_1 \|\eta\|_1, \quad \forall \theta, \eta \in \left(H_0^1(\Omega)\right)^2, \end{aligned}$$

where σ is a positive constant, and this gives the continuity of $a_1(\cdot, \cdot)$ in $(H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$.

Differently from the former works on the Reissner-Mindlin problem, the pattern presented here contain only two variables θ and ω. Once θ and ω are found, γ can be obtained from (3). On the other hand, the new variational formulation (14)-(15) does not include to the classical mixed finite element model (see [16]), so we need to prove the existence and uniqueness of the solution of this new formulation.

Theorem 1 *The new mixed variational formulation* (14)-(15) *has a unique solution.*

Proof The new mixed variational formulation (14)-(15) can be derived from (7)-(9), so the solution of (7)-(9) is a solution of (14)-(15). Therefore, the remaining work is to verify the uniqueness of the solution.

In order to prove this, it suffices to prove that the homogenous problem

$$a_1(\theta, \eta) + b(\eta, \omega) = 0, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2, \tag{16}$$

$$b(\theta, \upsilon) + c(\omega, \upsilon) = 0, \quad \forall \upsilon \in H_0^1(\Omega), \tag{17}$$

has only the zero solution.

Equation (16) can be written as

$$a_1(\theta, \eta) = -b(\eta, \omega) = \lambda t^{-2}(\nabla \omega, \eta), \quad \forall \eta \in \left(H_0^1(\Omega)\right)^2.$$

Based on the former proof, the bilinear forms of $a_1(\cdot,\cdot)$ are $(H_0^1(\Omega))^2$-elliptic and continuous. In addition, for every fixed $\omega \in H_0^1(\Omega)$, $\lambda t^{-2}(\nabla\omega,\eta)$ can be seen as a continuous linear form in $(H_0^1(\Omega))^2$; in fact,

$$\left|\lambda t^{-2}(\nabla\omega,\eta)\right| \le \lambda t^{-2}\|\eta\|_{0,\Omega}|\omega|_{1,\Omega} \le C\|\eta\|_{1,\Omega}|\omega|_{1,\Omega}, \quad \forall\eta \in \left(H_0^1(\Omega)\right)^2.$$

By the Lax-Milgram lemma (see [17]), for every $\omega \in H_0^1(\Omega)$, there exists a unique $\theta = \theta(\omega) \in (H_0^1(\Omega))^2$ such that

$$a_1\big(\theta(\omega),\eta\big) = \lambda t^{-2}(\nabla\omega,\eta), \quad \forall\eta \in \left(H_0^1(\Omega)\right)^2.$$

It is easy to see that the function $\theta = \theta(\omega)$ linearly depends on ω.
Let $\eta = \theta(\omega)$ in (16). Then

$$a\big(\theta(\omega),\theta(\omega)\big) + \lambda t^{-2}\big(\theta(\omega),\theta(\omega)\big) = \lambda t^{-2}\big(\theta(\omega),\nabla\omega\big) \le \lambda t^{-2}\left\|\theta(\omega)\right\|_{0,\Omega}|\omega|_{1,\Omega},$$

and if $\omega \neq 0$, then $\theta(\omega) \neq 0$ since $\theta(\omega)$ linearly depends on ω. So we get

$$\left\|\theta(\omega)\right\|_{0,\Omega} \le |\omega|_{1,\Omega} - \frac{1}{\lambda t^{-2}}\frac{a(\theta(\omega),\theta(\omega))}{\|\theta(\omega)\|_{0,\Omega}}. \tag{18}$$

For (17), $\theta = \theta(\omega)$, so that

$$b\big(\theta(\omega),\upsilon\big) + c(\omega,\upsilon) = 0, \quad \forall\upsilon \in H_0^1(\Omega). \tag{19}$$

Equation (19) means $\omega = 0$. As a matter of fact, if $\omega \neq 0$, then letting $\upsilon = \nabla\omega$ in (19), we get the following estimate:

$$\begin{aligned}0 &= -\lambda t^{-2}\big(\theta(\omega),\nabla\omega\big) + \lambda t^{-2}(\nabla\omega,\nabla\omega)\\ &\ge \lambda t^{-2}|\omega|_{1,\Omega}^2 - \lambda t^{-2}\left\|\theta(\omega)\right\|_{0,\Omega}|\omega|_{1,\Omega}.\end{aligned}$$

Then, combining this inequality with (18), we have the estimate

$$\begin{aligned}0 &\ge \lambda t^{-2}|\omega|_{1,\Omega}^2 - \lambda t^{-2}\left\|\theta(\omega)\right\|_{0,\Omega}|\omega|_{1,\Omega}\\ &\ge \lambda t^{-2}|\omega|_{1,\Omega}^2 - \lambda t^{-2}\left(|\omega|_{1,\Omega} - \frac{1}{\lambda t^{-2}}\frac{a(\theta(\omega),\theta(\omega))}{\|\theta(\omega)\|_{0,\Omega}}\right)|\omega|_{1,\Omega}\\ &= \frac{a(\theta(\omega),\theta(\omega))}{\|\theta(\omega)\|_{0,\Omega}}|\omega|_{1,\Omega}\\ &\ge C\frac{\|\theta(\omega)\|_{1,\Omega}^2}{\|\theta(\omega)\|_{1,\Omega}}|\omega|_{1,\Omega}\\ &= C\left\|\theta(\omega)\right\|_{1,\Omega}|\omega|_{1,\Omega}, \quad \forall\omega \in H_0^1(\Omega).\end{aligned}$$

So, there must be $\omega = 0$ and thus $\theta(\omega) = 0$. This means that the homogenous equation system (16)-(17) has only the zero solution. That is to say, the new mixed variational formulation (14)-(15) has a unique solution. □

4 Mixed finite element discretion

We shall introduce a mixed finite element approximation of problem (1)-(4). Let $\{J_h\}$ be a series of regular triangle partitions of Ω. On a generic triangle $T \in J_h$, define the shape function spaces for approximating θ, ω as

$$P_\theta(T) = \bigl(P_1(T)\bigr)^2 \oplus \alpha_T \lambda_1 \lambda_2 \lambda_3,$$
$$P_\omega(T) = P_1(T),$$

where α_T is a vector, $P_1(T)$ denotes the set of polynomials of degree ≤ 1 on T, and λ_i $(i = 1, 2, 3)$ are the barycentric coordinates.

As is well known, a vector $\theta \in P_\theta(T)$ is uniquely determined by the four degrees of freedom

$$\Sigma'_T = \left\{\theta(a_i), i = 1, 2, 3, \frac{1}{|T|}\int_T \theta\, ds\right\}, \tag{20}$$

and a vector $\omega \in P_\omega(T)$ is uniquely determined by the three degrees of freedom

$$\Sigma''_T = \bigl\{\omega(a_i), i = 1, 2, 3\bigr\}, \tag{21}$$

where a_i, $i = 1, 2, 3$, are the vertices of the triangle T.

The finite element spaces are defined as follows:

$$H_h = \bigl\{\theta : \theta|_T \in P_\theta(T) \text{ defined by } \Sigma'_T, \theta|_{\partial\Omega} = 0\bigr\}, \tag{22}$$
$$W_h = \bigl\{\omega : \omega|_T \in P_\omega(T) \text{ defined by } \Sigma''_T, \omega|_{\partial\Omega} = 0\bigr\}. \tag{23}$$

For $\theta \in H_h$, obviously, $\theta \in C^0(\Omega)$, and hence $\theta \in (H_0^1(\Omega))^2$. Therefore, $H_h \subseteq (H_0^1(\Omega))^2$. Similarly, $\omega_h \subseteq H_0^1(\Omega)$. This illustrates that these are conforming element spaces.

In order to prove error estimates, we introduce the new norm

$$\|\theta\|_*^2 := \lambda t^{-2}\|\theta\|_0^2 + a(\theta, \theta). \tag{24}$$

Corresponding to the mixed variational formulation, the discrete problem is to find $(\theta_h, \omega_h) \in H_h \times W_h$ such that

$$a_1(\theta_h, \eta_h) + b(\eta_h, \omega_h) = 0, \quad \forall \eta_h \in H_h, \tag{25}$$
$$b(\theta_h, \upsilon_h) + c(\omega_h, \upsilon_h) = g(\upsilon_h), \quad \forall \upsilon_h \in W_h. \tag{26}$$

Similarly, for the discrete variational formulation, it is easy to prove that the bilinear forms of $a_1(\cdot,\cdot)$ are $\mathbf{V}^*$-elliptic and continuous in $H_h \times H_h$:

$$\begin{aligned} a_1(\theta_h, \eta_h) &= a(\theta_h, \eta_h) + \lambda t^{-2}(\theta_h, \eta_h) \\ &= \int_\Omega \mathbf{C}\varepsilon(\theta_h) : \varepsilon(\eta_h)\, dx + \lambda t^{-2}\int_\Omega \theta_h \cdot \eta_h\, dx \\ &\leq \|\theta_h\|_* \|\eta_h\|_*, \quad \forall \theta_h, \eta_h \in H_h, \end{aligned}$$

which proves the continuity of $a_1(\cdot,\cdot)$ in $H_h \times H_h$;

$$\begin{aligned} a_1(\theta_h,\theta_h) &= a(\theta_h,\theta_h) + \lambda t^{-2}(\theta_h,\theta_h) \\ &= \int_\Omega \mathbf{C}\varepsilon(\theta_h) : \varepsilon(\theta_h)\,dx + \lambda t^{-2}\int_\Omega \theta_h \cdot \theta_h\,dx \\ &\geq \alpha^* \|\theta_h\|_*^2, \quad \forall \theta_h \in H_h, \end{aligned}$$

where α^* is a positive constant, which means that $a_1(\cdot,\cdot)$ is $\mathbf{V}^*$-elliptic in $H_h \times H_h$.

Theorem 2 *The discrete mixed variational formulation* (25)-(26) *has a unique solution.*

Similarly to the previous arguments, proceeding in exactly the same way (see the proof of Theorem 1), the existence and uniqueness of the solution of the discrete problem can be obtained through proving that the homogenous problem has only the zero solution.

5 Error estimation

Subtracting (25) from (14) and subtracting (26) from (15), we obtain the error equations

$$a_1(\theta - \theta_h, \eta_h) + b(\eta_h, \omega - \omega_h) = 0, \quad \forall \eta_h \in H_h, \tag{27}$$

$$b(\theta - \theta_h, \upsilon_h) + c(\omega - \omega_h, \upsilon_h) = 0, \quad \forall \upsilon_h \in W_h. \tag{28}$$

First of all, the $\mathbf{V}^*$-ellipticity and linearity of $a_1(\cdot,\cdot)$ in $H_h \times H_h$ ensure the estimate

$$\begin{aligned} &\|\theta_h - \eta_h\|_*^2 \\ &\quad \leq a_1(\theta_h - \eta_h, \theta_h - \eta_h) \\ &\quad = a_1(\theta - \eta_h, \theta_h - \eta_h) + a_1(\theta_h - \theta, \theta_h - \eta_h). \end{aligned} \tag{29}$$

Then, for all $\theta_h - \eta_h \in H_h$, by (27) we have the equality

$$a_1(\theta_h - \theta, \theta_h - \eta_h) = b(\theta_h - \eta_h, \omega - \omega_h). \tag{30}$$

So, inserting (30) into the right-hand side of (29) yields

$$\begin{aligned} &\|\theta_h - \eta_h\|_*^2 \\ &\quad \leq a_1(\theta - \eta_h, \theta_h - \eta_h) + b(\theta_h - \eta_h, \omega - \omega_h). \end{aligned}$$

Then, using the continuity of $a_1(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we further get that

$$\|\theta_h - \eta_h\|_*^2 \leq \|\theta - \eta_h\|_* \|\theta_h - \eta_h\|_* + \lambda t^{-2}\|\theta_h - \eta_h\|_{0,\Omega} |\omega - \omega_h|_{1,\Omega}. \tag{31}$$

Based on the definition of the norm $\|\cdot\|_*$, we have

$$\lambda t^{-2}\|\theta_h - \eta_h\|_{0,\Omega} \leq \sqrt{\lambda t^{-2}}\|\theta_h - \eta_h\|_*.$$

Then, inserting this inequality into (31), we get

$$\|\theta_h - \eta_h\|_*^2 \le \|\theta - \eta_h\|_* \|\theta_h - \eta_h\|_* + \sqrt{\lambda t^{-2}} \|\theta_h - \eta_h\|_* |\omega - \omega_h|_{1,\Omega},$$

so,

$$\|\theta_h - \eta_h\|_* \le \|\theta - \eta_h\|_* + \sqrt{\lambda t^{-2}} |\omega - \omega_h|_{1,\Omega}. \tag{32}$$

Using the triangle inequality, we get the following estimate:

$$\begin{aligned}
\|\theta - \theta_h\|_* &\le \|\theta - \eta_h\|_* + \|\theta_h - \eta_h\|_* \\
&\le \|\theta - \eta_h\|_* + \|\theta - \eta_h\|_* + \sqrt{\lambda t^{-2}} |\omega - \omega_h|_{1,\Omega} \\
&= 2\|\theta - \eta_h\|_* + \sqrt{\lambda t^{-2}} |\omega - \omega_h|_{1,\Omega}.
\end{aligned} \tag{33}$$

Then, we first estimate the second term of (33). Then, for every $\upsilon_h \in W_h$,

$$c(\upsilon_h - \omega_h, \upsilon_h - \omega_h) \ge \lambda t^{-2} |\upsilon_h - \omega_h|_{1,\Omega}^2. \tag{34}$$

Moreover, the linearity of $c(\cdot,\cdot)$ and equation (28) ensure the estimate

$$\begin{aligned}
&c(\upsilon_h - \omega_h, \upsilon_h - \omega_h) \\
&\quad = c(\upsilon_h - \omega, \upsilon_h - \omega_h) + c(\omega - \omega_h, \upsilon_h - \omega_h) \\
&\quad = c(\upsilon_h - \omega, \upsilon_h - \omega_h) - b(\theta - \theta_h, \upsilon_h - \omega_h).
\end{aligned}$$

Using the Schwarz inequality (see [17]), we get the estimate

$$\begin{aligned}
&c(\upsilon_h - \omega_h, \upsilon_h - \omega_h) \\
&\quad \le \lambda t^{-2} |\omega - \upsilon_h|_{1,\Omega} |\upsilon_h - \omega_h|_{1,\Omega} + \lambda t^{-2} \|\theta - \theta_h\|_{0,\Omega} |\upsilon_h - \omega_h|_{1,\Omega}.
\end{aligned} \tag{35}$$

Combining (34) and (35) and dividing both sides of the inequalities by $|\upsilon_h - \omega_h|_{1,\Omega}$ yield the estimate

$$|\upsilon_h - \omega_h|_{1,\Omega} \le |\omega - \upsilon_h|_{1,\Omega} + \|\theta - \theta_h\|_{0,\Omega}, \quad \forall \upsilon_h \in W_h.$$

Then, inserting this inequality into the triangle inequality

$$|\omega - \omega_h|_{1,\Omega} \le |\omega - \upsilon_h|_{1,\Omega} + |\upsilon_h - \omega_h|_{1,\Omega},$$

we immediately get that

$$\begin{aligned}
|\omega - \omega_h|_{1,\Omega} &\le |\omega - \upsilon_h|_{1,\Omega} + \big(|\omega - \upsilon_h|_{1,\Omega} + \|\theta - \theta_h\|_{0,\Omega}\big) \\
&= 2|\omega - \upsilon_h|_{1,\Omega} + \|\theta - \theta_h\|_{0,\Omega}.
\end{aligned} \tag{36}$$

Inserting (36) into (33), we have

$$\begin{aligned}&\|\theta-\theta_h\|_*\\&\quad\le 2\|\theta-\eta_h\|_*+\sqrt{\lambda t^{-2}}\bigl(2|\omega-\upsilon_h|_{1,\Omega}+\|\theta-\theta_h\|_{0,\Omega}\bigr)\\&\quad=2\|\theta-\eta_h\|_*+2\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}+\sqrt{\lambda t^{-2}}\|\theta-\theta_h\|_{0,\Omega},\end{aligned}\tag{37}$$

and then subtracting $\sqrt{\lambda t^{-2}}\|\theta-\theta_h\|_{0,\Omega}$ from both sides of inequality (37), we get

$$\begin{aligned}&\|\theta-\theta_h\|_*-\sqrt{\lambda t^{-2}}\|\theta-\theta_h\|_{0,\Omega}\\&\quad\le 2\|\theta-\eta_h\|_*+2\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}.\end{aligned}\tag{38}$$

By rationalizing the numerator, from (38) it is easy to get the estimate

$$\frac{a(\theta-\theta_h,\theta-\theta_h)}{\|\theta-\theta_h\|_*+\sqrt{\lambda t^{-2}}\|\theta-\theta_h\|_{0,\Omega}}\le 2\|\theta-\eta_h\|_*+2\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}.$$

So

$$\begin{aligned}&a(\theta-\theta_h,\theta-\theta_h)\\&\quad\le\bigl[\|\theta-\theta_h\|_*+\sqrt{\lambda t^{-2}}\|\theta-\theta_h\|_{0,\Omega}\bigr]\bigl[2\|\theta-\eta_h\|_*+2\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}\bigr].\end{aligned}\tag{39}$$

Using the Korn and Poincaré inequalities (see[18]) in (39), we immediately get the estimate

$$|\theta-\theta_h|_{1,\Omega}^2\le C\|\theta-\theta_h\|_*\bigl[\|\theta-\eta_h\|_*+\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}\bigr].\tag{40}$$

Dividing by $|\theta-\theta_h|_{1,\Omega}$ and using the equivalence of the norms $\|\cdot\|_*$ and $|\cdot|_{1,\Omega}$, we reduce (40) to

$$|\theta-\theta_h|_{1,\Omega}\le C\bigl[\|\theta-\eta_h\|_*+\sqrt{\lambda t^{-2}}|\omega-\upsilon_h|_{1,\Omega}\bigr],\quad \eta_h\in H_h,\upsilon_h\in W_h.\tag{41}$$

Moreover, since $\eta_h\in H_h$ and $\upsilon_h\in W_h$ are arbitrary in (41), we derive

$$\begin{aligned}|\theta-\theta_h|_{1,\Omega}&\le C\Bigl[\inf_{\eta_h\in H_h}\|\theta-\eta_h\|_*+\sqrt{\lambda t^{-2}}\inf_{\upsilon_h\in W_h}|\omega-\upsilon_h|_{1,\Omega}\Bigr]\\&\le C\bigl[\|\theta-\Pi_h\theta\|_*+\sqrt{\lambda t^{-2}}|\omega-\Pi_h\omega|_{1,\Omega}\bigr]\\&\le C\bigl[|\theta-\Pi_h\theta|_{1,\Omega}+\sqrt{\lambda t^{-2}}\|\theta-\Pi_h\theta\|_0+\sqrt{\lambda t^{-2}}|\omega-\Pi_h\omega|_{1,\Omega}\bigr].\end{aligned}$$

Then, utilizing the standard interpolation theory and also the inverse inequality (see[17]) in this inequality, we get

$$\begin{aligned}|\theta-\theta_h|_{1,\Omega}&\le C\bigl[h|\theta|_{2,\Omega}+\sqrt{\lambda t^{-2}}h^2|\theta|_{2,\Omega}+\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}\bigr]\\&\le C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}.\end{aligned}\tag{42}$$

Inserting (42) into (36), we get

$$\begin{aligned}|\omega-\omega_h|_{1,\Omega} &\le 2|\omega-\upsilon_h|_{1,\Omega}+\|\theta-\theta_h\|_{0,\Omega}\\ &\le 2|\omega-\upsilon_h|_{1,\Omega}+C|\theta-\theta_h|_{1,\Omega}\\ &\le 2|\omega-\upsilon_h|_{1,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}. \qquad (43)\end{aligned}$$

Because $\upsilon_h \in W_h$ is arbitrary in (43),

$$\begin{aligned}&|\omega-\omega_h|_{1,\Omega}\\ &\quad\le C\inf_{\upsilon_h\in W_h}|\omega-\upsilon_h|_{1,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}\\ &\quad\le C|\omega-\Pi_h\omega|_{1,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C_2\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}. \qquad (44)\end{aligned}$$

Then, we immediately get the following estimate by using the interpolation theory:

$$\begin{aligned}|\omega-\omega_h|_{1,\Omega} &\le Ch|\omega|_{2,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega}\\ &\le C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}\bigr)h|\omega|_{2,\Omega}. \qquad (45)\end{aligned}$$

We finally obtain estimates (42) and (45) by the following convergence theorem.

Theorem 3 *Let* (θ,ω) *be the solution of the mixed variational formulation* (14)-(15), *and let* (θ_h,ω_h) *be that of the discrete problem* (25)-(26). *Then, the following estimates hold*:

$$\begin{aligned}&|\theta-\theta_h|_{1,\Omega}\le C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\sqrt{\lambda t^{-2}}h|\omega|_{2,\Omega},\\ &|\omega-\omega_h|_{1,\Omega}\le C\bigl(1+\sqrt{\lambda t^{-2}}h\bigr)h|\theta|_{2,\Omega}+C\bigl(1+\sqrt{\lambda t^{-2}}\bigr)h|\omega|_{2,\Omega}.\end{aligned}$$

In this paper, the constants C in all previous estimates are different from each other and also are independent of h.

6 Numerical experiments

In this section, we give an example to verify the theoretical analysis.

To check the convergence rate, we construct the following exact solutions for the two-dimensional Reissner-Mindlin model. Assume that the domain $\Omega=[0,1]^2$. Now let

$$\begin{aligned}\theta &= \bigl[y^3(y-1)^3x^2(x-1)^2(2x-1), x^3(x-1)^3y^2(y-1)^2(2y-1)\bigr]',\\ \omega &= \frac{1}{3}x^3y^3(x-1)^3(y-1)^3-\frac{2t^2}{5(1-k)}\bigl[y^3(y-1)^3x(x-1)\bigl(5x^2-5x+1\bigr)\\ &\quad + x^3(x-1)^3y(y-1)\bigl(5y^2-5y+1\bigr)\bigr].\end{aligned}$$

The corresponding $g(x,y)$ is

$$\begin{aligned}g(x,y) &= \frac{12\lambda}{5(1-k)}\bigl\{\bigl(5x^2-5x+1\bigr)y(y-1)\bigl[x(x-1)\bigl(5y^2-5y+1\bigr)+2y^2(y-1)^2\bigr]\\ &\quad + x(x-1)\bigl(5y^2-5y+1\bigr)\bigl[2x^2(x-1)^2+\bigl(5x^2-5x+1\bigr)y(y-1)\bigr]\bigr\},\end{aligned}$$

where $k=0.3$.

Table 1 Error results of the rotations $\boldsymbol{\theta}$

Step length	4^{-1}	8^{-1}	16^{-1}	32^{-1}	64^{-1}	128^{-1}
$\lvert\theta - \theta_h\rvert_{1,\Omega}$	0.0010	6.2835e–004	2.9050e–004	1.2380e–004	5.6950e–005	2.7729e–005
Order	–	0.6818	1.1131	1.2305	1.1202	1.0383

Table 2 Error results of transversal displacement $\boldsymbol{\omega}$

Step length	4^{-1}	8^{-1}	16^{-1}	32^{-1}	64^{-1}	128^{-1}
$\lvert\omega - \omega_h\rvert_{1,\Omega}$	2.0672e–004	1.2562e–004	5.6141e–005	2.3292e–005	1.0569e–005	5.1228e–006
Order	–	0.7186	1.1619	1.2692	1.1400	1.0449

Now for the regular triangle partitions of Ω, where the step lengths are $h = 4^{-1}$, $h = 8^{-1}$, $h = 16^{-1}$, $h = 32^{-1}$, $h = 64^{-1}$, $h = 128^{-1}$, we use the shape functions given in Section 4 to approximate θ, ω, and the errors and orders are given in Tables 1 and 2, where $\lambda = 3.5$, $t = 0.1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work is fulfilled in cooperation by Dr. Niu and Prof. Song. Both authors read and approved the final manuscript.

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