Shoaib Saleem et al. *Journal of Inequalities and Applications* (2016) 2016:265
DOI 10.1186/s13660-016-1204-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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Reverse Poincaré-type inequalities for the weak solution of system of partial differential inequalities

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Abstract

In this work we develop the square integral estimates for the functions of n variables which are subharmonic with respect to some variables and for other remaining variables are superharmonic. It is in a sense a generalization of reverse Poincaré type inequalities for the difference of superharmonic functions developed in (J. Inequal. Appl. 2015: doi:10.1186/s13660-015-0916-9, 2015).

Keywords: subharmonic functions; superharmonic functions; system of differential inequalities; weak solution

1 Introduction

The second order partial differential equations represent a large number of practical problems. One of the most important classes of linear second order partial differential equations is elliptic equations. A second order partial differential equation is uniformly elliptic if the matrix of higher order coefficients is positive definite. The particular and important case of a second order uniformly elliptic equation is the Laplace equation.

The Laplace equation not only emerges in a variety of physical problems but also arises in the study of analytic functions and probabilistic investigations of Brownian motion.

Let Δ be the second order Laplace operator of n variables and $B(x_o, r)$ is a ball in $\mathbb{R}^n$, with center x_o, and radius r. A function $u(x) \in C^2(B(x_o, r)) \cap C(\overline{B})$ is subharmonic if $\Delta u(x) \geq 0$, and it is said to be superharmonic if $\Delta u(x) \leq 0$. The subharmonic functions attain their maximum and superharmonic functions attain their minimum on the boundary (see *e.g.* Evans Section 6.4 [2]).

Subharmonic and superharmonic functions play a key role in classical as well as in modern potential theory. These functions are most familiar in partial differential equations in the construction of solutions to the Dirichlet problem [3, 4].

There is a lot of information on subharmonic and superharmonic functions and also on their properties in [3–6].

The weighted square integral inequality for convex functions of one variable was developed by Hussain, Pečarić, and Shashiashvili [7]. Such kinds of inequalities are widely used in finance and physical problems. The function of n variables is convex if its Hessian matrix is positive definite. The natural generalization of convex functions for n variables is a subharmonic function and similarly of concave functions it is a superharmonic function. It is also clear that a function which is convex (concave) w.r.t. each of its variables may not be convex (concave) as a whole but such kinds of functions are a subclass of subharmonic functions (superharmonic functions).

The weighted square integral inequalities for superharmonic functions are developed in [1]. There is also another important class of functions which are convex w.r.t. some variables but concave w.r.t. other remaining variables. The generalization of such functions is subharmonic for some variables and superharmonic for the other variables. In this research our notations are standard.

Let $u(x)$, $x \in \mathbb{R}^n$ be a solution of the following system of partial differential inequalities:

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_j^2} \geq 0, \\ \frac{\partial^2 u}{\partial x_{j+1}^2} + \frac{\partial^2 u}{\partial x_{j+2}^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \leq 0, \end{aligned}\right\} \tag{1.1}$$

where $1 \leq j < n$, $n \geq 2$.

The bounded measurable function $u(x)$ is a weak solution of the system (1.1) if $\forall \phi(x) \in C_c^2(B)$, for the space of twice continuously differentiable functions having compact support, the following holds:

$$\left.\begin{aligned} \int_B u(x) \Delta_{1,j} \phi(x)\, dx \geq 0, \\ \int_B u(x) \Delta_{j+1,n} \phi(x)\, dx \leq 0, \end{aligned}\right\} \tag{1.2}$$

where

$$\Delta_{1,j} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_j^2} \tag{1.3}$$

and

$$\Delta_{j+1,n} = \frac{\partial^2}{\partial x_{j+1}^2} + \frac{\partial^2}{\partial x_{j+2}^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1.4}$$

It is trivial that $\Delta = \Delta_{1,j} + \Delta_{j+1,n}$ where $\Delta_{1,j}$ and $\Delta_{j+1,n}$ are both operators that are self adjoint operators.

$\operatorname{grad} u(x)$ is an n-dimensional vector given by

$$\operatorname{grad} u(x) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right). \tag{1.5}$$

We also introduce

$$\left.\begin{aligned} \operatorname{grad}_{1,j} u(x) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_j}\right), \\ \operatorname{grad}_{j+1,n} u(x) = \left(\frac{\partial u}{\partial x_{j+1}}, \frac{\partial u}{\partial x_{j+2}}, \ldots, \frac{\partial u}{\partial x_n}\right), \end{aligned}\right\} \tag{1.6}$$

where $1 < j < n$.

It is trivial that

$$\operatorname{grad} u(x) = \left(\operatorname{grad}_{1,j} u(x), \operatorname{grad}_{j+1,n} u(x) \right).$$

We will organize the paper in following way. In the second section we prove the inequality for the smooth solution of system (1.1) and also approximate the weak solution of the system (1.1) by smooth ones. In the last section we prove that the continuous weak solutions possess first order weak derivatives and also we will prove the inequality for a weak solution of system (1.1).

2 The reverse Poincaré inequalities for smooth subsolution and approximation of weak subsolution by smooth ones

The following lemmas for superharmonic functions and subharmonic functions are proved in [1].

Lemma 2.1 ([1]) *Consider two arbitrary smooth superharmonic functions $u_i(x)$, $i=1,2$ over Domain D, $D\subset\mathbb{R}^n$ (the domain is bounded and has a smooth boundary) i.e. $u_i(x)\in C^2(\overline{D})$, $i=1,2$, and $\Delta u_i(x)\leq 0$ if $x\in D$, $i=1,2$.*

Then by equation (2.5) *in Theorem* 2.1 *of* [1] *we have*

$$\int_D \bigl|\operatorname{grad} u_2(x)-\operatorname{grad} u_1(x)\bigr|^2 w(x)\,dx \leq \int_D\biggl[\frac{(u_2(x)-u_1(x))^2}{2}-\bigl\|u_2(x)-u_1(x)\bigr\|_{L^\infty}\bigl(u_2(x)+u_1(x)\bigr)\biggr]\Delta w(x)\,dx,$$

where $w(x)$ is the non-negative weight function satisfying

$$w(x)=\frac{\partial w(x)}{\partial x_i}=0,\quad i=1,2,\dots,n, x\in\partial D. \tag{2.1}$$

Lemma 2.2 ([1]) *Consider two arbitrary smooth subharmonic functions $u_i(x)$, $i=1,2$, over Domain D, $D\subset R^n$ (the domain is bounded and has a smooth boundary) i.e. $u_i(x)\in\overline{C^2}(D)$, $i=1,2$, and $\Delta u_i(x)\leq 0$ if $x\in D$, $i=1,2$. Then the following holds*:

$$\int_D \bigl|\operatorname{grad} u_2(x)-\operatorname{grad} u_1(x)\bigr|^2 w(x)\,dx \leq \int_D\biggl[\frac{(u_2(x)-u_1(x))^2}{2}+\bigl\|u_2(x)-u_1(x)\bigr\|_{L^\infty}\bigl(u_2(x)+u_1(x)\bigr)\biggr]\Delta w(x)\,dx,$$

where $w(x)$ is a non-negative weight function satisfying (2.1).

We will start by the following theorem.

Theorem 2.3 *Let $u_i(x)$, $i=1,2$ be the two smooth solutions of system* (1.1) *over the domain $D\subseteq\mathbb{R}^n$, having a smooth boundary and let $w(x)$ be the arbitrary non-negative smooth function on the domain D satisfying* (2.1); *then the following estimate holds*:

$$\begin{aligned}\int_D \bigl|\operatorname{grad} u_2(x)-\operatorname{grad} u_1(x)\bigr|^2 w(x)\,dx &\leq \|u_2-u_1\|_{L^\infty}\bigl(\|u_1\|_{L^\infty}+\|u_2\|_{L^\infty}\bigr)\\ &\quad\times\int_D\bigl|\widetilde{\Delta} w(x)\bigr|\,dx+\frac{1}{2}\|u_2-u_1\|_{L^\infty}^2\int_D\bigl|\Delta w(x)\bigr|\,dx,\end{aligned} \tag{2.2}$$

where Δ is a Laplace operator and $\widetilde{\Delta}=\Delta_{1,j}-\Delta_{j+1,n}$.

Proof Let

$$u(x) = u_2(x) - u_1(x). \tag{2.3}$$

Take

$$\begin{aligned}
&\int_D \left|\operatorname{grad} u(x)\right|^2 w(x)\,dx \\
&\quad= \int_D \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_j}\right)^2 + \left(\frac{\partial u}{\partial x_{j+1}}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2\right] w(x)\,dx \\
&\quad= \left(\int_D \left(\frac{\partial u}{\partial x_1}\right)^2 w(x)\,dx + \cdots + \int_D \left(\frac{\partial u}{\partial x_j}\right)^2 w(x)\,dx\right) \\
&\qquad+ \left(\int_D \left(\frac{\partial u}{\partial x_{j+1}}\right)^2 w(x)\,dx + \cdots + \int_D \left(\frac{\partial u}{\partial x_n}\right)^2 w(x)\,dx\right).
\end{aligned} \tag{2.4}$$

Using (1.6), in equation (2.4) we obtain the following:

$$= \int_D \left|\operatorname{grad}_{1,j} u(x)\right|^2 w(x)\,dx + \int_D \left|\operatorname{grad}_{j+1,n} u(x)\right|^2 w(x)\,dx. \tag{2.5}$$

Now using Lemma 2.2 on the first integral and Lemma 2.1 on the second integral we obtain

$$\begin{aligned}
&\int_D \left|\operatorname{grad} u(x)\right|^2 w(x)\,dx \\
&\quad\le \int_D \left[\frac{(u_2-u_1)^2}{2} + \|u_2-u_1\|_{L^\infty}(u_2+u_1)\right] \Delta_{1,j} w(x)\,dx \\
&\qquad+ \int_D \left[\frac{(u_2-u_1)^2}{2} - \|u_2-u_1\|_{L^\infty}(u_2+u_1)\right] \Delta_{j+1,n} w(x)\,dx \\
&\quad\le \int_D \frac{(u_2-u_1)^2}{2}\big(\Delta_{1,j} w(x) + \Delta_{j+1,n} w(x)\big)\,dx \\
&\qquad+ \int_D \|u_2-u_1\|_{L^\infty}(u_2+u_1)\big(\Delta_{1,j} w(x) - \Delta_{j+1,n} w(x)\big)\,dx,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\int_D \left|\operatorname{grad} u(x)\right|^2 w(x)\,dx \\
&\quad\le \int_D \frac{(u_2-u_1)^2}{2}\Delta w(x)\,dx + \int_D \|u_2-u_1\|_{L^\infty}(u_2+u_1)\widetilde{\Delta} w(x)\,dx,
\end{aligned} \tag{2.7}$$

where $\widetilde{\Delta} = \Delta_{1,j} - \Delta_{j+1,n}$.

Taking the infinite norm on (2.7) we get the result (2.2). □

Remark 2.4 *The above theorem is also true for an arbitrary ball B, $B = B(x_o, r)$ with center x_o and radius r,*

$$\begin{aligned}
&\int_{B(x_o,r)} \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 w(x)\,dx \\
&\quad\le \|u_2-u_1\|_{L^\infty_{(B)}}\big(\|u_1\|_{L^\infty_{(B)}} + \|u_2\|_{L^\infty_{(B)}}\big)\int_{B(x_o,r)} \left|\widetilde{\Delta} w(x)\right|dx \\
&\qquad+ \frac{1}{2}\|u_2-u_1\|^2_{L^\infty_{(B)}}\int_{B(x_o,r)} \left|\Delta w(x)\right|dx.
\end{aligned} \tag{2.8}$$

From now onward we will use $B(x_o, r)$ as a domain and the following particular weight function:

$$w(x) = \left[r^2 - (x - x_o)^2\right]^2.$$

It is trivial that

$$\frac{\partial w}{\partial x_i}(x) = w(x) = 0 \quad \text{if } x \in \partial B \ \forall i = 1, 2, \ldots, n.$$

Now we prove that for a weak solution of system of inequality (1.1), we may approximate it by a system of smooth solutions. For this purpose, we will make use of the mollification technique [2].

Define

$$\varphi(x) = \begin{cases} c \exp \frac{1}{x^2 - 1}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2.9}$$

where $x \in \mathbb{R}^n$, $c > 0$ such that

$$\int_{\mathbb{R}^n} \varphi(x) = 1. \tag{2.10}$$

Now we define the mollifier of a bounded measurable solution $u(x)$ in the following way:

$$u_h(x) = h^{-n} \int_{B(x_o, r)} \varphi\left(\frac{x - y}{h}\right) u(y)\, dy. \tag{2.11}$$

Denote

$$\varphi_h(x - y) = h^{-n} \cdot \varphi\left(\frac{x - y}{h}\right). \tag{2.12}$$

It is trivial that

$$\frac{\partial^2}{\partial x_i^2} \varphi_h(x - y) = \frac{\partial^2}{\partial y_i^2} \varphi_h(x - y) \quad \forall i = 1, 2, \ldots, n. \tag{2.13}$$

So

$$\Delta_x u_h(x) = h^{-n} \int_{B(x_o, r)} u(y) \Delta_y \varphi_h(x - y)\, dy, \tag{2.14}$$

where Δ_x and Δ_y are the Laplace operators w.r.t. x and y, respectively.

We will define the smaller balls B_k, $k = 1, 2, \ldots,$ in the form

$$B_k = B(x_o, r_k) \quad \text{where } r_k = \frac{k+1}{k+2} r, k = 1, 2, \ldots,$$

and the corresponding weight functions are

$$w_k(x) = \left[r_k^2 - (x - x_o)^2\right]^2.$$

The next theorem tells us that the function $u_h(x)$ defined above are smooth solutions of the system of inequality (2.2) over the ball B_k for sufficiently small h.

Theorem 2.5 *Let $u(x)$ be the weak solution of system* (1.1) *on the ball B, $B = B(x_o, r)$. Then, for any $k = 1, 2, \dots$, there exists $\widehat{h} > 0$, such that if $0 < h < \widehat{h}$, each $u_h(x)$ is a smooth solution of the system* (1.1) *over the ball B_k.*

Proof For fixed $k = 1, 2, \dots$, let

$$\widehat{h} = \frac{r}{2(k+2)}.$$

It is clear that for arbitrary $h > 0$ the function $u_h(x)$ is infinitely differentiable.

Now we check that, for arbitrary $x \in B_k$, $\varphi_h(x-y)$ has compact support in the ball $B(x_o, r)$. Take the ball $\widehat{B}_k$ in the following way:

$$\widehat{B}_k = B\left(x_o, \frac{2k+3}{2k+4} r\right). \tag{2.15}$$

If $y \notin \widehat{B}_k$, then

$$|y - x| > \left|\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\right| = \frac{1}{2(k+2)} r > h \quad \Rightarrow \quad \varphi_h(x-y) = 0. \tag{2.16}$$

Hence $\varphi_h(x-y)$ has compact support in ball B as a function of y if $h < \widehat{h}$ and by the definition of a weak solution $u(x)$ we have

$$\begin{aligned} &\int_B u(y)(\Delta_y)_{1,j}\, \varphi_h(x-y)\, dy \geq 0, \\ &\int_B u(y)(\Delta_y)_{j+1,n}\, \varphi_h(x-y)\, dy \leq 0, \end{aligned} \tag{2.17}$$

which completes the proof. □

3 The existence and integrability of weak partial derivative and weighted square inequalities for the difference of weak subsolutions

The following theorem tells that a continuous weak subsolution of system (1.1) possesses all first order weak partial derivatives and also they are square integrable.

Theorem 3.1 *Every continuous weak solution $u(x)$ of system* (1.1) *has weak partial derivatives $\frac{\partial u}{\partial x_i}$ $i = 1, 2, \dots, n$, in the ball $B(x_o, r) \subseteq \mathbb{R}^n$ and also they are weighted square integrable i.e.*

$$\int_B \left|\operatorname{grad} u(x)\right|^2 w(x)\, dx < \infty, \tag{3.1}$$

where $w(x)$ is a non-negative weight function having compact support.

Proof The proof of the theorem can be done along similar lines to the proof of Theorem 3.1 of [1], using inequality (2.8) of the present paper instead of (3.5) of [1]. □

The next theorem will give us reverse Poincaré type inequalities for a weak subsolution of system (1.1).

Theorem 3.2 *For any two arbitrary continuous weak solutions $u_i(x)$, $i = 1, 2$, for the system* (1.1) *in the ball $B = B_(x_o, r)$, the following is valid*:

$$\begin{aligned}
&\int_B \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 w(x)\,dx \\
&\quad\leq \|u_2 - u_1\|_{L^\infty}\left(\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}\right) \\
&\qquad\times \int_{B(x_o,r)} \left|\widetilde{\Delta} w(x)\right|\,dx \\
&\qquad+ \frac{1}{2}\|u_2 - u_1\|_{L^\infty}^2 \int_{B(x_o,r)} \left|\Delta w(x)\right|\,dx,
\end{aligned} \tag{3.2}$$

where Δ is the Laplace operator and $\widetilde{\Delta} = \Delta_{1,j} - \Delta_{j+1,n}$.

Proof For the continuous weak sub solutions $u_i(x)$, $i = 1, 2$, for system (1.1), take a smooth approximation $u_{m,i}(x)$ $i = 1, 2$. In the ball B_{k+l} there exists an integer m_{k+l} s.t. the requirement that $u_{m,i}(x)$ is smooth in the ball B_{k+l} and $u_{m,i}(x)$ converges uniformly to $u_i(x)$ $i = 1, 2$ for $m \geq m_{k+l}$.

Let us write the inequality (2.8) for the functions $u_{m,1}(x)$ and $u_{m,2}(x)$ on the ball B_{k+l}:

$$\begin{aligned}
&\int_{B_{k+l}} \left|\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)\right|^2 w_{k+l}(x)\,dx \\
&\quad\leq \widetilde{c}_{k+l}\left\|u_{m,2}(x) - u_{m,1}(x)\right\|_{L^\infty_{(B)}}\left(\left\|u_{m,2}(x)\right\|_{L^\infty_{(B)}} + \left\|u_{m,1}(x)\right\|_{L^\infty_{(B)}}\right) \\
&\qquad+ \frac{1}{2}c_{k+l}\left\|u_{m,2}(x) - u_{m,1}(x)\right\|_{L^\infty_{(B)}}^2,
\end{aligned} \tag{3.3}$$

where

$$\widetilde{c}_{k+l} = \int_B \left|\widetilde{\Delta} w(x)\right|\,dx \tag{3.4}$$

and

$$c_{k+l} = \int_B \left|\Delta w(x)\right|\,dx. \tag{3.5}$$

Applying the limit $m \to \infty$, we get

$$\begin{aligned}
&\int_{B_{k+l}} \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 w_{k+l}(x)\,dx \\
&\quad\leq \widetilde{c}_{k+l}\left\|u_2(x) - u_1(x)\right\|_{L^\infty_{(B_{k+l})}}\left(\left\|u_2(x)\right\|_{L^\infty_{(B_{k+l})}} + \left\|u_1(x)\right\|_{L^\infty_{(B_{k+l})}}\right) \\
&\qquad+ \frac{1}{2}c_{k+l}\left\|u_2(x) - u_1(x)\right\|_{L^\infty_{(B_{k+l})}}^2.
\end{aligned} \tag{3.6}$$

Writing the left integral for the smaller ball $B_k \subseteq B_{k+l}$, and taking the limit as $l \to \infty$, we obtain

$$\begin{aligned}
&\int_{B_k} \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 w(x)\, dx \\
&\quad \leq \widetilde{c}_\infty \left\| u_2(x) - u_1(x) \right\|_{L^\infty_{(B)}} \left(\left\| u_2(x) \right\|_{L^\infty_{(B)}} + \left\| u_1(x) \right\|_{L^\infty_{(B)}} \right) \\
&\qquad + \frac{1}{2} c_\infty \left\| u_2(x) - u_1(x) \right\|^2_{L^\infty_{(B)}}.
\end{aligned} \tag{3.7}$$

By Theorem 3.1, we have

$$\int_B \left|\operatorname{grad} u_i(x)\right|^2 w(x)\, dx < \infty, \quad i = 1, 2. \tag{3.8}$$

Taking the limit as $k \to \infty$, we obtain (3.2). □

4 Conclusion

From our results we conclude that if a weak solution of system (1.1) is closed in a supremum norm then their weak derivatives are also closed in a weighted L^2 norm.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first theorem for generalized domain and generalized weight function was given by Josip Pečarić. Muhammad Shoaib Saleem and the M. Phil. student Mr. Abid Hussain developed the convolution technique and also gave the existence of the Sobolov gradient. Hamood Ur Rehman developed the inequality for weak solution of system (1.1). Mr. Abdul Rauf Nizami developed the last inequality. All authors read and approved the final manuscript.

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Acknowledgements
We are grateful to the reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper.

Received: 29 January 2016 Accepted: 12 October 2016 Published online: 24 October 2016

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