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Fixed points of mappings satisfying contractive condition of integral type in modular spaces endowed with a graph

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available at the end of the article**Abstract**

Jachymski (Proc. Am. Math. Soc. 136:1359-1373, 2008) gave a modified version of a Banach fixed point theorem on a metric space endowed with a graph. The aim of this paper is to present fixed point results of mappings satisfying integral type contractive conditions in the framework of modular spaces endowed with a graph. Some examples are presented to support the results proved herein. Our results generalize and extend various comparable results in the existing literature.

MSC: 47H10; 54H25; 54E50**Keywords:** connected graph; modular space; Banach G-contraction; integral type contraction

1 Introduction

Fixed point theory for nonlinear mappings is an important subject of nonlinear functional analysis. One of the basic and the most widely applied fixed point theorem in all of analysis is the 'Banach (or Banach-Caccioppoli) contraction principle' due to Banach [1]. This Banach contraction principle [1] is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. Due to its applications in mathematics and other related disciplines, the Banach contraction principle has been generalized in many directions.

The existence of fixed points in ordered metric spaces has been discussed by Ran and Reurings [2]. Recently, many researchers have obtained fixed point and common fixed point results for single valued maps defined on partially ordered metric spaces (see, e.g., [3, 4]). Jachymski [5] investigated a new approach in metric fixed point theory by replacing an order structure with graph structure on a metric space. In this way, the results proved in ordered metric spaces are generalized (see for details [5] and the references therein). For further work in this direction, we refer to, e.g., [6–8].

In 1968, Kannan [9] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper led to the genesis for a multitude of fixed point papers over the next two decades. Since then, there have been many theorems dealing with mappings satisfying various types of contractive inequalities involving linear and nonlinear expressions. For a thorough survey, we refer to [10] and the references therein. On the other hand, Branciari [11] obtained a fixed point theorem

for a single valued mapping satisfying an analog of Banach’s contraction principle for an integral type inequality. Recently, Akram *et al.* [12] introduced a new class of contraction maps, called A -contractions, which is a proper generalization of Kannan’s mappings [9], Bianchini’s mappings [13], and Reich type mappings [14].

The theory of modular spaces was initiated by Nakano [15] in connection with the theory of ordered spaces which was further generalized by Musielak and Orlicz [16] (see also [17–21]). The study of fixed point theory in the context of modular function spaces was initiated by Khamsi *et al.* [22] (see also [22–26]). Also, some fixed point theorems have been proved for mappings satisfying contractive conditions of integral type in modular space [27, 28].

In this paper, we introduce three new classes of mappings satisfying integral type contractive conditions in the setup of modular space endowed with graphs. We study the existence, uniqueness, and iterative approximations of fixed points for such mappings. Our results extend, unify, and generalize the comparable results in [5, 11, 12].

2 Preliminaries

A mapping T from a metric space (X, d) into (X, d) is called a Picard operator (PO) if T has a unique fixed point $z \in X$ and $\lim_{n \rightarrow \infty} T^n x = z$ for all $x \in X$.

Define $\Phi = \{\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ : \varphi \text{ is a Lebesgue integral mapping which is summable, non-negative and satisfies } \int_0^\epsilon \varphi(t) dt > 0, \text{ for each } \epsilon > 0\}$.

Let $A = \{\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ : \alpha \text{ is continuous and } a \leq kb \text{ for some } k \in [0, 1) \text{ whenever } a \leq \alpha(a, b, b) \text{ or } a \leq \alpha(b, a, b) \text{ or } a \leq \alpha(b, b, a) \text{ for all } a, b\}$.

Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nondecreasing mapping which satisfies the following conditions:

- (ψ_1) $\psi(x) = 0$ if and only if $x = 0$;
- (ψ_2) for a sequence $\{x_n\}$ in \mathbb{R}_+ , we have $\psi(x_n) \rightarrow 0$ if and only if $x_n \rightarrow 0$ as $n \rightarrow \infty$;
- (ψ_3) for every $x, y \in \mathbb{R}_+$, we have $\psi(x + y) \leq \psi(x) + \psi(y)$.

The collection of all such mappings will be denoted by Ψ .

Define

$$\Phi_1 = \{\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ : \phi \text{ is increasing, upper semi-continuous and } \phi(t) < t, \forall t > 0\}.$$

Theorem 2.1 [11] *Let (X, d) be a complete metric space, $\eta \in [0, 1)$, and $T : X \rightarrow X$ a mapping. Suppose that*

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \eta \int_0^{d(x, y)} \varphi(t) dt$$

is satisfied for every $x, y \in X$, where $\varphi \in \Phi$. Then T has a unique fixed point $z \in X$ and for each $x \in X$, we have $\lim_{n \rightarrow \infty} T^n x = z$.

Lemma 2.2 [29] *Let (X, d) be a metric space, $\varphi \in \Phi$, and $\{x_n\}$ a nonnegative sequence. Then*

- (a) $\lim_{n \rightarrow \infty} x_n = x$ implies that $\lim_{n \rightarrow \infty} \int_0^{x_n} \varphi(t) dt = \int_0^x \varphi(t) dt$;
- (b) $\lim_{n \rightarrow \infty} \int_0^{x_n} \varphi(t) dt = 0$ if and only if $\lim_{n \rightarrow \infty} x_n = 0$.

Definition 2.3 [12] A selfmap T on a metric space X is called an A -contraction if for any $x, y \in X$ and for some $\alpha \in A$, the following condition holds:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)).$$

Now, we recall some basic facts and notations as regards modular spaces. For more details the reader may consult [16].

Definition 2.4 Let X be an arbitrary vector space. A functional $\rho : X \rightarrow [0, \infty]$ is called modular if for any arbitrary x, y in X :

- (m₁) $\rho(x) = 0$ if and only if $x = 0$;
- (m₂) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;
- (m₃) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$.

If (m₃) is replaced by $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ if $\alpha^s + \beta^s = 1$, where $s \in (0, 1], \alpha \geq 0, \beta \geq 0$ then we say that ρ is s -convex modular. If $s = 1$, then ρ is called convex modular.

$\rho : \mathbb{R} \rightarrow [0, \infty]$ defined by $\rho(x) = \sqrt{|x|}$ is a simple example of a modular functional.

The vector space X_ρ given by

$$X_\rho = \{x \in X; \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}$$

is called a modular space. In general the modular ρ is not sub-additive and therefore does not behave as a norm or a distance. One can associate to a modular an x -norm.

Remark 2.5 [28] The following are immediate consequences of condition (m₃):

- (r₁) if $a, b \in \mathbb{R}$ (set of all real numbers) with $|a| < |b|$, then $\rho(ax) < \rho(bx)$ for all $x \in X$;
- (r₂) if a_1, \dots, a_n are nonnegative real numbers with $\sum_{i=1}^n a_i = 1$, then we have

$$\rho\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{i=1}^n \rho(x_i) \quad (x_1, \dots, x_n \in X).$$

Define the ρ -ball, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \{h \in X_\rho; \rho(x - h) \leq r\}.$$

A point $x \in X_\rho$ is called a *fixed point* of $T : X_\rho \rightarrow X_\rho$ if $T(x) = x$.

A function modular is said to satisfy the Δ_2 -type condition if there exists $K > 0$ such that for any $x \in X_\rho$ we have $\rho(2x) \leq K\rho(x)$. A modular ρ is said to satisfy the Δ_2 -condition if $\rho(2x_n) \rightarrow 0$ as $n \rightarrow \infty$, whenever $\rho(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.6 Let X_ρ be a modular space. The sequence $\{x_n\} \subset X_\rho$ is said to be:

- (t₁) ρ -convergent to $x \in X_\rho$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$;
- (t₂) ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as n and $m \rightarrow \infty$.

X_ρ is ρ -complete if any ρ -Cauchy sequence is ρ -convergent. Note that ρ -convergence does not imply ρ -Cauchy since ρ does not satisfy the triangle inequality. In fact, one can show that this will happen if and only if ρ satisfies the Δ_2 -condition.

Proposition 2.7 [30] *Let X_ρ be a modular space. If $a, b \in \mathbb{R}_+$ with $b \geq a$, then $\rho(ax) \leq \rho(bx)$.*

Proposition 2.8 [30] *Suppose that X_ρ is a modular space, ρ satisfies the Δ_2 -condition and $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X_ρ . If $\rho(c(x_n - x_{n-1})) \rightarrow 0$, then $\rho(\alpha l(x_n - x_{n-1})) \rightarrow 0$, as $n \rightarrow \infty$, where $c, l, \alpha \in \mathbb{R}_+$ with $c > l$ and $\frac{l}{c} + \frac{1}{\alpha} = 1$. Now we give some basic definitions from graph theory needed in the sequel.*

Throughout this paper, $\Delta = \{(x, x) : x \in X\}$ denotes the diagonal of $X \times X$, where X is any nonempty set. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph such that $\Delta \subseteq E(G)$. Further assume that G has no parallel edge and G is a weighted graph in the sense that each edge is assigned a distance $d(x, y)$ between their vertices x and y and each vertex x is assigned a weight $d(x, x)$. The graph G is identified by the pair $(V(G), E(G))$.

If x and y are vertices of G , then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$, $n \in \{0, 1, 2, \dots, k\}$ of vertices such that $x = x_0, \dots, x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, \dots, k\}$.

Recall that a graph G is connected if there is a path between any two vertices and it is weakly connected if \tilde{G} is connected, where \tilde{G} denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of the edges. Thus

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{1}$$

Since it is more convenient to treat \tilde{G} as a directed graph for which the set of its edges is symmetric, under this convention we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{2}$$

Let G_x be the component of G consisting of all the edges and vertices which are contained in some path in G beginning at x . If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the equivalence class $[x]_G$ defined on $V(G)$ by the rule $R(xRy)$ if there is a path in G from x to y is such that $V(G_x) = [x]_G$.

Definition 2.9 [5, Definition 2.1] A mapping $T : X \rightarrow X$ is called a Banach G -contraction if and only if:

- (a) for each x, y in X with $(x, y) \in E(G)$, we have $(T(x), T(y)) \in E(G)$, that is, T preserves edges of G ;
- (b) there exists α in $(0, 1)$ such that for each $x, y \in X$ with $(x, y) \in E(G)$ implies

$$d(T(x), T(y)) \leq \alpha d(x, y). \tag{3}$$

That is, T decreases weights of edges of G .

For any $x, y \in V'$, $(x, y) \in E'$ such that $V' \subseteq V(G)$, $E' \subseteq E(G)$, (V', E') is called a subgraph of G .

3 Main results

In this section, we obtain several fixed point results in the setup of a modular space endowed with a graph. We start with the following definitions. Let X_ρ be a modular space

endowed with a graph G and let $T : X_\rho \rightarrow X_\rho$ be a mapping. Denote

$$X_T = \{x \in X : (x, Tx) \in E(G)\}.$$

Definition 3.1 Let $\{T^n x\}$ be a sequence, there exists $C > 0$ such that

$$\rho(C(T^n x - x^*)) \rightarrow 0 \quad \text{for } x^* \in X_\rho$$

and

$$(T^n x, T^{n+1} x) \in E(G) \quad \text{for all } n \in \mathbb{N}.$$

Then a graph G is called a C_ρ -graph if there exists a subsequence $\{T^{n_p} x\}$ of $\{T^n x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ for $p \in \mathbb{N}$.

Definition 3.2 A mapping $T : X_\rho \rightarrow X_\rho$ is called orbitally G_ρ -continuous for all $x, y \in X_\rho$ and any sequence $(n_p)_{p \in \mathbb{N}}$ of positive integers, if there exists $C > 0$ such that

$$\begin{aligned} \rho(C(T^{n_p} x - y)) \rightarrow 0, \quad (T^{n_p} x, T^{n_p+1} x) \in E(G) \quad \text{imply} \\ \rho(C(T(T^{n_p} x) - T(y))) \rightarrow 0, \end{aligned}$$

as $p \rightarrow \infty$.

Definition 3.3 A mapping T is called a $(G, A)_\rho$ -contraction if it satisfies the following conditions:

- (A₁) T preserves edges of G ;
- (A₂) there exist nonnegative numbers l, c with $l < c$ such that

$$\begin{aligned} & \int_0^{\rho(c(Tx - Ty))} \varphi(t) dt \\ & \leq \alpha \left(\int_0^{\rho(l(x-y))} \varphi(t) dt, \int_0^{\rho(l(x-Tx))} \varphi(t) dt, \int_0^{\rho(l(y-Ty))} \varphi(t) dt \right) \end{aligned}$$

holds for each $(x, y) \in E(G)$, and some $\alpha \in A$ and $\varphi \in \Phi$.

Remark 3.4 Let $T : X_\rho \rightarrow X_\rho$ be a $(G, A)_\rho$ -contraction. If there exists $x_0 \in X_\rho$ such that $Tx_0 \in [x_0]_{\tilde{G}}$, then

- (i) T is both a $(G^{-1}, A)_\rho$ -contraction and a $(\tilde{G}, A)_\rho$ -contraction,
- (ii) $[x_0]_{\tilde{G}}$ is T -invariant and $T|_{[x_0]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, A)_\rho$ -contraction.

Lemma 3.5 Let $T : X_\rho \rightarrow X_\rho$ be a $(G, A)_\rho$ -contraction. If $x \in X_T$, then there exists $r(x, Tx) \geq 0$ such that

$$\int_0^{\rho(c(T^n x - T^{n+1} x))} \varphi(t) dt \leq k^n r(x, Tx)$$

holds for all $n \in \mathbb{N}$, where $r(x, Tx) = \int_0^{\rho(c(x - Tx))} \varphi(t) dt$.

Proof Let $x \in X_T$, that is, $(x, Tx) \in E(G)$. Then by induction, we have $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Now, we have

$$\begin{aligned} & \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \\ & \leq \alpha \left(\int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt, \int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt, \int_0^{\rho(l(T^n x - T^{n+1}x))} \varphi(t) dt \right) \\ & \leq \alpha \left(\int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt, \int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt, \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \right). \end{aligned}$$

By the definition of α , we obtain

$$\int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \leq k \int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt \leq k \int_0^{\rho(c(T^{n-1}x - T^n x))} \varphi(t) dt$$

for some $k \in (0, 1)$. Thus we have

$$\begin{aligned} \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt & \leq k \int_0^{\rho(l(T^{n-1}x - T^n x))} \varphi(t) dt \\ & \leq k \int_0^{\rho(c(T^{n-1}x - T^n x))} \varphi(t) dt \\ & \leq k \cdot k \int_0^{\rho(l(T^{n-2}x - T^{n-1}x))} \varphi(t) dt \\ & \leq k^2 \int_0^{\rho(c(T^{n-2}x - T^{n-1}x))} \varphi(t) dt \\ & \vdots \\ & \leq k^n \int_0^{\rho(l(x - Tx))} \varphi(t) dt \\ & \leq k^n \int_0^{\rho(c(x - Tx))} \varphi(t) dt. \end{aligned}$$

That is, $\int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \leq k^n r(x, Tx)$ for all $n \in \mathbb{N}$, where $r(x, Tx) = \int_0^{\rho(c(x - Tx))} \varphi(t) dt$. □

Theorem 3.6 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a $(\tilde{G}, A)_\rho$ -contraction. If the set X_T is nonempty, the graph G is weakly connected and a C_ρ -graph, then T is a PO.*

Proof If $x \in X_T$, then $Tx \in [x]_{\tilde{G}}$ and $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ with $m > n$. Note that

$$\begin{aligned} & \rho\left(\frac{c}{m-n}(T^n x - T^m x)\right) \\ & = \rho\left(\frac{c}{m-n}(T^n x - T^{n+1}x) + \frac{c}{m-n}(T^{n+1}x - T^{n+2}x) + \dots + \frac{c}{m-n}(T^{m-1}x - T^m x)\right) \\ & \leq \rho(c(T^n x - T^{n+1}x)) + \rho(c(T^{n+1}x - T^{n+2}x)) + \dots + \rho(c(T^{m-1}x - T^m x)). \end{aligned}$$

Using Lemma 3.5, we have

$$\begin{aligned} \int_0^{\rho(\frac{c}{m-n}(T^n x - T^m x))} \varphi(t) dt &\leq \int_0^{\rho(c(T^n x - T^{n+1} x))} \varphi(t) dt + \int_0^{\rho(c(T^{n+1} x - T^{n+2} x))} \varphi(t) dt \\ &\quad + \dots + \int_0^{\rho(c(T^{m-1} x - T^m x))} \varphi(t) dt \\ &\leq k^n r(x, Tx) + k^{n+1} r(x, Tx) + \dots + k^{m-1} r(x, Tx) \\ &\leq \frac{k^n}{1-k} r(x, Tx) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It follows that $\{\frac{c}{m-n} T^n x\}$ is a ρ -Cauchy sequence in X_ρ . Since X_ρ is ρ -complete, there exists a point $x^* \in X_\rho$ such that $\rho(\frac{c}{m-n}(T^n x - x^*)) \rightarrow 0$. Consequently, $\rho(l(T^n x - x^*)) \rightarrow 0$.

Now we show that x^* is a fixed point of T . As $\rho(\frac{c}{m-n}(T^n x - x^*)) \rightarrow 0$, $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$ and G is a C_ρ -graph, there exists a subsequence $\{T^{n_p} x\}$ of $\{T^n x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ for each $p \in \mathbb{N}$. Since $(T^{n_p} x, x^*) \in E(\tilde{G})$ and T is a $(\tilde{G}, A)_\rho$ -contraction, it follows that

$$\begin{aligned} \int_0^{\rho(c(T^{n_p+1} x - Tx^*))} \varphi(t) dt \\ \leq \alpha \left(\int_0^{\rho(l(T^{n_p} x - x^*))} \varphi(t) dt, \int_0^{\rho(l(T^{n_p} x - T^{n_p+1} x))} \varphi(t) dt, \int_0^{\rho(l(x^* - Tx^*))} \varphi(t) dt \right), \end{aligned}$$

which on taking the limit as $p \rightarrow \infty$ gives

$$\int_0^{\rho(c(x^* - Tx^*))} \varphi(t) dt \leq \alpha \left(0, 0, \int_0^{\rho(l(x^* - Tx^*))} \varphi(t) dt \right) \leq \alpha \left(0, 0, \int_0^{\rho(c(x^* - Tx^*))} \varphi(t) dt \right).$$

By the definition of function α , we have

$$\int_0^{\rho(c(x^* - Tx^*))} \varphi(t) dt \leq k \cdot 0 = 0.$$

From Lemma 2.2, it follows that $\rho(c(x^* - Tx^*)) = 0$ and $Tx^* = x^*$.

Next, we prove that x^* is a unique fixed point. Suppose that T has another fixed point $y^* \in X_\rho - \{x^*\}$. Since G is a C_ρ -graph, there exists a subsequence $\{T^{n_p} x\}$ of $\{T^n x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ and $(T^{n_p} x, y^*) \in E(G)$ for each $p \in \mathbb{N}$. Furthermore, G is weakly connected, $(x^*, y^*) \in E(\tilde{G})$, and we have

$$\begin{aligned} \int_0^{\rho(c(x^* - y^*))} \varphi(t) dt &= \int_0^{\rho(c(Tx^* - Ty^*))} \varphi(t) dt \\ &\leq \alpha \left(\int_0^{\rho(l(x^* - y^*))} \varphi(t) dt, \int_0^{\rho(l(x^* - Tx^*))} \varphi(t) dt, \int_0^{\rho(l(y^* - Ty^*))} \varphi(t) dt \right) \\ &\leq \alpha \left(\int_0^{\rho(l(x^* - y^*))} \varphi(t) dt, 0, 0 \right) \\ &\leq \alpha \left(\int_0^{\rho(c(x^* - y^*))} \varphi(t) dt, 0, 0 \right). \end{aligned}$$

By the definition of α and Lemma 2.2, we have $\int_0^{\rho(c(x^*-y^*))} \varphi(t) dt \leq k \cdot 0 = 0$, $\rho(c(x^* - y^*)) = 0$, and $x^* = y^*$. \square

In Theorem 3.6, if we replace the condition that G is a C_ρ -graph with orbitally G_ρ -continuity of T , then we have the following theorem.

Theorem 3.7 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a $(\tilde{G}, A)_\rho$ -contraction and orbitally G_ρ -continuous. If the set X_T is nonempty and the graph G is weakly connected, then T is a PO.*

Proof If $x \in X_T$, then Theorem 3.6 implies that $\{\frac{c}{m-n}T^n x\}$ is a ρ -Cauchy sequence in X_ρ . Owing to ρ -completeness of X_ρ , there exists $x^* \in X_\rho$ such that $\rho(\frac{c}{m-n}(T^n x - x^*)) \rightarrow 0$. As $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$ and T is orbitally G_ρ -continuous, we have $\rho(\frac{c}{m-n}(T(T^n x) - T(x^*))) \rightarrow 0$, as $n \rightarrow \infty$. That is, $Tx^* = x^*$. Assume that y^* is another fixed point of T . Following arguments similar to those in the proof of Theorem 3.6, we obtain $y^* = x^*$. \square

Corollary 3.8 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be edge-preserving, the set X_T nonempty and graph G be weakly connected and a C_ρ -graph. If there exist nonnegative numbers l, c with $l < c$ such that*

$$\rho(c(Tx - Ty)) \leq \alpha(\rho(l(x - y)), \rho(l(x - Tx)), \rho(l(y - Ty)))$$

holds for all $(x, y) \in E(\tilde{G})$ and some $\alpha \in A$, then T is a PO.

Now, we introduce Hardy-Rogers type $(G)_\rho$ -contraction and obtain related fixed point results.

Definition 3.9 Let X_ρ be a modular space. A mapping $T : X_\rho \rightarrow X_\rho$ is called a Hardy-Rogers type $(G)_\rho$ -contraction if the following conditions hold:

- (H₁) T preserves edges of G ;
- (H₂) there exist nonnegative numbers l_i, c with $l_i < c$ for $i = 1, \dots, 5$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq \eta \int_0^{\rho(l_1(x-y))} \varphi(t) dt + \beta \int_0^{[\rho(l_2(x-Tx))+\rho(l_3(y-Ty))]} \varphi(t) dt + \gamma \int_0^{[\rho(\frac{l_4}{2}(x-Ty))+\rho(\frac{l_5}{2}(y-Tx))]} \varphi(t) dt$$

holds for each $(x, y) \in E(G)$ with nonnegative numbers η, β, γ such that $\eta + 2\beta + 2\gamma < 1$ and $\varphi \in \Phi$.

Remark 3.10 Let X_ρ be a modular space endowed with a graph G and let $T : X_\rho \rightarrow X_\rho$ be a Hardy-Rogers type $(G)_\rho$ -contraction. If there exists $x_0 \in X_\rho$ such that $Tx_0 \in [x_0]_{\tilde{G}}$, then

- (i) T is both a Hardy-Rogers type $(G^{-1})_\rho$ -contraction and a Hardy-Rogers type $(\tilde{G})_\rho$ -contraction,
- (ii) $[x_0]_{\tilde{G}}$ is T -invariant and $T|_{[x_0]_{\tilde{G}}}$ is a Hardy-Rogers type $(\tilde{G}_{x_0})_\rho$ -contraction.

Theorem 3.11 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a Hardy-Rogers type $(\tilde{G})_\rho$ -contraction. Assume that the set X_T is nonempty and the C_ρ -graph G is weakly connected. Then T is a PO.*

Proof If $x \in X_T$, then $Tx \in [x]_{\tilde{G}}$ and $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned} & \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \\ & \leq \eta \int_0^{\rho(l_1(T^{n-1}x - T^n x))} \varphi(t) dt + \beta \int_0^{[\rho(l_2(T^{n-1}x - T^n x)) + \rho(l_3(T^n x - T^{n+1}x))]} \varphi(t) dt \\ & \quad + \gamma \int_0^{[\rho(\frac{l_4}{2}(T^{n-1}x - T^{n+1}x)) + \rho(\frac{l_5}{2}(T^n x - T^{n+1}x))]} \varphi(t) dt \\ & \leq \eta \int_0^{\rho(c(T^{n-1}x - T^n x))} \varphi(t) dt + \beta \int_0^{[\rho(c(T^{n-1}x - T^n x)) + \rho(c(T^n x - T^{n+1}x))]} \varphi(t) dt \\ & \quad + \gamma \int_0^{[\rho(c(T^{n-1}x - T^n x)) + \rho(c(T^n x - T^{n+1}x))]} \varphi(t) dt. \end{aligned}$$

It follows that

$$\int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \leq h \int_0^{\rho(c(T^{n-1}x - T^n x))} \varphi(t) dt,$$

where $h = \frac{\eta + \beta + \gamma}{1 - \beta - \gamma} < 1$. Also,

$$\int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt \leq h^n \int_0^{\rho(c(x - Tx))} \varphi(t) dt. \tag{4}$$

Taking the limit as $n \rightarrow \infty$, and using Lemma 2.2, we get

$$\lim_n \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt = 0,$$

which implies that $\lim_n \rho(c(T^n x - T^{n+1}x)) = 0$.

Let $m, n \in \mathbb{N}$ with $m > n$. By (4) and Remark 2.5, we get

$$\begin{aligned} \int_0^{\rho(\frac{c}{m-n}(T^n x - T^m x))} \varphi(t) dt & \leq \int_0^{\rho(c(T^n x - T^{n+1}x))} \varphi(t) dt + \int_0^{\rho(c(T^{n+1}x - T^{n+2}x))} \varphi(t) dt \\ & \quad + \dots + \int_0^{\rho(c(T^{m-1}x - T^m x))} \varphi(t) dt \\ & \leq h^n \int_0^{\rho(c(x - Tx))} \varphi(t) dt + h^{n+1} \int_0^{\rho(c(x - Tx))} \varphi(t) dt \\ & \quad + \dots + h^{m-1} \int_0^{\rho(c(x - Tx))} \varphi(t) dt \\ & \leq \frac{h^n}{1 - h} \int_0^{\rho(c(x - Tx))} \varphi(t) dt \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, $\{\frac{c}{m-n}T^n x\}$ is a ρ -Cauchy sequence in X_ρ . Since X_ρ is ρ -complete, there exists a point $x^* \in X_\rho$ such that $\rho(\frac{c}{m-n}(T^n x - x^*)) \rightarrow 0$. As G is a C_ρ -graph, there exists a subsequence $\{T^{n_p} x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ for all $p \in \mathbb{N}$. Also, $(T^{n_p} x, x^*) \in E(G)$ for all $p \in \mathbb{N}$. Now we have

$$\begin{aligned} \int_0^{\rho(c(T^{n_p+1}x - Tx^*))} \varphi(t) dt &\leq \eta \int_0^{\rho(l_1(T^{n_p}x - x^*))} \varphi(t) dt \\ &+ \beta \int_0^{[\rho(l_2(T^{n_p}x - T^{n_p+1}x)) + \rho(l_3(x^* - Tx^*))]} \varphi(t) dt \\ &+ \gamma \int_0^{[\rho(\frac{l_4}{2}(T^{n_p}x - Tx^*)) + \rho(\frac{l_5}{2}(x^* - T^{n_p+1}x))]} \varphi(t) dt. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we have

$$\int_0^{\rho(c(x^* - Tx^*))} \varphi(t) dt \leq (\beta + \gamma) \int_0^{\rho(c(x^* - Tx^*))} \varphi(t) dt.$$

As $(\beta + \gamma) < 1$, so $\rho(c(x^* - Tx^*)) = 0$ and $x^* = Tx^*$.

Next, we prove that x^* is a unique fixed point. Suppose that T has another fixed point $y^* \in X_\rho - \{x^*\}$. Since G is a C_ρ -graph, there exists a subsequence $\{T^{n_p} x\}$ of $\{T^n x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ and $(T^{n_p} x, y^*) \in E(G)$ for each $p \in \mathbb{N}$. As G is weakly connected, we have $(x^*, y^*) \in E(\tilde{G})$, and

$$\begin{aligned} \int_0^{\rho(c(x^* - y^*))} \varphi(t) dt &= \int_0^{\rho(c(Tx^* - Ty^*))} \varphi(t) dt \\ &\leq \eta \int_0^{\rho(l_1(x^* - y^*))} \varphi(t) dt + \beta \int_0^{[\rho(l_2(x^* - Tx^*)) + \rho(l_3(y^* - Ty^*))]} \varphi(t) dt \\ &+ \gamma \int_0^{[\rho(\frac{l_4}{2}(x^* - Ty^*)) + \rho(\frac{l_5}{2}(y^* - Tx^*))]} \varphi(t) dt, \end{aligned}$$

which further implies that

$$\int_0^{\rho(c(x^* - y^*))} \varphi(t) dt \leq (\eta + 2\gamma) \int_0^{\rho(c(x^* - y^*))} \varphi(t) dt.$$

Since $(\eta + 2\gamma) < 1$, $\int_0^{\rho(c(x^* - y^*))} \varphi(t) dt = 0$. The result follows. \square

In Theorem 3.11, if we replace the condition that G is a C_ρ -graph with orbitally G_ρ -continuity of T , then we have the following theorem.

Theorem 3.12 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a Hardy-Rogers type $(\tilde{G})_\rho$ -contraction, which is orbitally G_ρ -continuous. Assume that the set X_T is nonempty and the graph G is weakly connected. Then T is a PO.*

In the following suppose that X_ρ is a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and $T : X_\rho \rightarrow X_\rho$ is edge-preserving such that the set X_T is nonempty.

Corollary 3.13 *Assume*

- (i) the C_ρ -graph G is weakly connected and
- (ii) there exist nonnegative numbers l, c with $l < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq \eta \int_0^{\rho(l(x-y))} \varphi(t) dt$$

holds for each $(x, y) \in E(\tilde{G})$ with $\eta \in (0, 1)$ and $\varphi \in \Phi$. Then T is a PO.

Corollary 3.14 *Assume*

- (i) the C_ρ -graph G is weakly connected and
- (ii) there exist nonnegative numbers l_1, l_2, c with $l_1, l_2 < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq \beta \int_0^{[\rho(l_1(x-Tx))+\rho(l_2(y-Ty))]} \varphi(t) dt$$

holds for each $(x, y) \in E(\tilde{G})$ with $\beta \in (0, \frac{1}{2})$ and $\varphi \in \Phi$. Then T is a PO.

Corollary 3.15 *Assume*

- (i) the C_ρ -graph G is weakly connected and
- (ii) there exist nonnegative numbers l_1, l_2, c with $l_1, l_2 < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq \gamma \int_0^{[\rho(\frac{l_1}{2}(x-Ty))+\rho(\frac{l_2}{2}(y-Tx))]} \varphi(t) dt$$

for each $(x, y) \in E(\tilde{G})$ with $\gamma \in (0, \frac{1}{2})$ and $\varphi \in \Phi$. Then T is a PO.

Now we introduce the $(G, \phi, \psi)_\rho$ -contraction and obtain some fixed point results.

Definition 3.16 A mapping $T : X_\rho \rightarrow X_\rho$ is called a $(G, \phi, \psi)_\rho$ -contraction if the following conditions hold:

- (Q₁) T preserves edges of G ;
- (Q₂) there exist nonnegative numbers l, c with $l < c$ such that

$$\int_0^{\psi(\rho(c(Tx-Ty)))} \varphi(t) dt \leq \phi \left(\int_0^{\psi(\rho(l(x-y)))} \varphi(t) dt \right)$$

holds for each $(x, y) \in E(G)$, where $\psi \in \Psi$, $\phi \in \Phi_1$, and $\varphi \in \Phi$.

Remark 3.17 Let $T : X_\rho \rightarrow X_\rho$ be a $(G, \phi, \psi)_\rho$ -contraction. If there exists $x_0 \in X_\rho$ such that $Tx_0 \in [x_0]_{\tilde{G}}$, then

- (i) T is both a $(G^{-1}, \phi, \psi)_\rho$ -contraction and a $(\tilde{G}, \phi, \psi)_\rho$ -contraction,
- (ii) $[x_0]_{\tilde{G}}$ is T -invariant and $T|_{[x_0]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, \phi, \psi)_\rho$ -contraction.

Theorem 3.18 Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a $(\tilde{G}, \phi, \psi)_\rho$ -contraction. If X_T is nonempty and C_ρ -graph G is weakly connected, then T is a PO.

Proof If $x \in X_T$, then $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$. First, we show that the sequence $\{\psi(\rho(c(T^n x - T^{n+1} x)))\}$ converges to 0. From Definition 3.16, we have

$$\begin{aligned} \int_0^{\psi(\rho(c(T^n x - T^{n+1} x)))} \varphi(t) dt &\leq \phi \left(\int_0^{\psi(\rho(l(T^{n-1} x - T^n x)))} \varphi(t) dt \right) \\ &< \int_0^{\psi(\rho(l(T^{n-1} x - T^n x)))} \varphi(t) dt \\ &< \int_0^{\psi(\rho(c(T^{n-1} x - T^n x)))} \varphi(t) dt. \end{aligned}$$

Thus,

$$\left\{ \int_0^{\psi(\rho(c(T^n x - T^{n+1} x)))} \varphi(t) dt \right\}$$

is decreasing and bounded from below and so

$$\left\{ \int_0^{\psi(\rho(c(T^n x - T^{n+1} x)))} \varphi(t) dt \right\}$$

converges to a nonnegative number L . If $L \neq 0$, we obtain

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \int_0^{\psi(\rho(c(T^n x - T^{n+1} x)))} \varphi(t) dt \leq \lim_{n \rightarrow \infty} \phi \left(\int_0^{\psi(\rho(l(T^{n-1} x - T^n x)))} \varphi(t) dt \right) \\ &\leq \lim_{n \rightarrow \infty} \phi \left(\int_0^{\psi(\rho(c(T^{n-1} x - T^n x)))} \varphi(t) dt \right), \end{aligned}$$

that is, $L \leq \phi(L)$, a contradiction. Hence $\int_0^{\psi(\rho(c(T^n x - T^{n+1} x)))} \varphi(t) dt \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\psi(\rho(c(T^n x - T^{n+1} x))) \rightarrow 0$. Suppose that

$$\limsup_{n \rightarrow \infty} \psi(\rho(c(T^n x - T^{n+1} x))) = \varepsilon > 0.$$

Then there exists a $v_\varepsilon \in \mathbb{N}$ and a sequence $\{T^{n_\nu} x\}_{\nu \geq v_\varepsilon}$ such that

$$\psi(\rho(c(T^{n_\nu} x - T^{n_\nu+1} x))) \rightarrow \varepsilon > 0, \quad \nu \rightarrow \infty, \tag{5}$$

$$\psi(\rho(c(T^{n_\nu} x - T^{n_\nu+1} x))) \geq \frac{\varepsilon}{2}, \quad \forall \nu \geq v_\varepsilon. \tag{6}$$

Hence we get the following:

$$0 = \lim_{\nu \rightarrow \infty} \int_0^{\psi(\rho(c(T^{n_\nu} x - T^{n_\nu+1} x)))} \varphi(t) dt \geq \int_0^{\frac{\varepsilon}{2}} \varphi(t) dt > 0.$$

Assume that there is an $\varepsilon > 0$ and there exist $m_\nu, n_\nu \in \mathbb{N}$ such that $m_\nu > n_\nu > \nu$ for each $\nu \in \mathbb{N}$ and

$$\psi(\rho(l(T^{m_\nu} x - T^{n_\nu} x))) \geq \varepsilon.$$

Then we choose the sequence $(m_\nu)_{\nu \in \mathbb{N}}$ and $(n_\nu)_{\nu \in \mathbb{N}}$ such that for each $\nu \in \mathbb{N}$, m_ν is minimal in the sense that $\psi(\rho(l(T^{m_\nu}x - T^{n_\nu}x))) \geq \varepsilon$ but $\psi(\rho(l(T^s x - T^{n_\nu}x))) < \varepsilon$, for all $s \in \{n_\nu + 1, \dots, m_\nu - 1\}$. Now, let $\delta \in \mathbb{R}_+$ be such that $\frac{l}{c} + \frac{1}{\delta} = 1$, then we have

$$\begin{aligned} \int_0^\varepsilon \varphi(t) dt &\leq \int_0^{\psi(\rho(l(T^{m_\nu}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\leq \int_0^{\psi(\rho(c(T^{m_\nu}x - T^{n_\nu+1}x)))} \varphi(t) dt + \int_0^{\psi(\rho(\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\leq \phi\left(\int_0^{\psi(\rho(l(T^{m_\nu-1}x - T^{n_\nu}x)))} \varphi(t) dt\right) + \int_0^{\psi(\rho(\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\leq \int_0^{\psi(\rho(l(T^{m_\nu-1}x - T^{n_\nu}x)))} \varphi(t) dt + \int_0^{\psi(\rho(\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\leq \int_0^\varepsilon \varphi(t) dt + \int_0^{\psi(\rho(\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt. \end{aligned}$$

Thus, taking the limit as $\nu \rightarrow \infty$, and Proposition 2.8, we have

$$\int_0^{\psi(\rho(\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \rightarrow 0.$$

Therefore,

$$\int_0^{\psi(\rho(l(T^{m_\nu}x - T^{n_\nu}x)))} \varphi(t) dt \rightarrow \varepsilon, \quad \nu \rightarrow \infty.$$

Now,

$$\begin{aligned} &\int_0^{\psi(\rho(l(T^{m_\nu}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\leq \int_0^{\psi(\rho(c(T^{m_\nu+1}x - T^{n_\nu+1}x)))} \varphi(t) dt + \int_0^{\psi(\rho(2\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\quad + \int_0^{\psi(\rho(2\delta l(T^{m_\nu+1}x - T^{m_\nu}x)))} \varphi(t) dt \\ &\leq \phi\left(\int_0^{\psi(\rho(l(T^{m_\nu}x - T^{n_\nu}x)))} \varphi(t) dt\right) + \int_0^{\psi(\rho(2\delta l(T^{n_\nu+1}x - T^{n_\nu}x)))} \varphi(t) dt \\ &\quad + \int_0^{\psi(\rho(2\delta l(T^{m_\nu+1}x - T^{m_\nu}x)))} \varphi(t) dt. \end{aligned}$$

If $\nu \rightarrow \infty$, then we have

$$\int_0^\varepsilon \varphi(t) dt \leq \phi\left(\int_0^\varepsilon \varphi(t) dt\right),$$

a contradiction for $\varepsilon > 0$. Hence, $\{lT^n x\}$ is a ρ -Cauchy sequence. By ρ -completeness of X_ρ , there exists $x^* \in X_\rho$ such that $\rho(l(T^n x - x^*)) \rightarrow 0$ as $n \rightarrow \infty$ and $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$ and G is a C_ρ -graph, then there exists a subsequence $\{T^{n_\nu} x\}$ such that $(T^{n_\nu} x, x^*) \in$

$E(G)$ for each $p \in \mathbb{N}$. Also, $(T^{n_p}x, x^*) \in E(G)$ for each $p \in \mathbb{N}$. From Remark 2.5 and (ψ_3) , it follows that

$$\begin{aligned} \psi\left(\rho\left(\frac{c}{2}(x^* - Tx^*)\right)\right) &= \psi\left(\rho\left(\frac{c}{2}(x^* - T^{n_p+1}x) + \frac{c}{2}(T^{n_p+1}x - Tx^*)\right)\right) \\ &\leq \psi(\rho(c(x^* - T^{n_p+1}x))) + \rho(c(T^{n_p+1}x - Tx^*)) \\ &\leq \psi(\rho(c(x^* - T^{n_p+1}x))) + \psi(\rho(c(T^{n_p+1}x - Tx^*))). \end{aligned}$$

Taking the limit as $p \rightarrow \infty$, we have

$$\begin{aligned} \int_0^{\psi(\rho(c(T^{n_p+1}x - Tx^*)))} \varphi(t) dt &\leq \phi\left(\int_0^{\psi(\rho(l(T^{n_p}x - x^*))} \varphi(t) dt\right) \\ &\leq \int_0^{\psi(\rho(l(T^{n_p}x - x^*))} \varphi(t) dt. \end{aligned}$$

Since $\rho(l(T^{n_p}x - x^*)) \rightarrow 0$ ($p \rightarrow \infty$), we obtain

$$\lim_{p \rightarrow \infty} \int_0^{\psi(\rho(l(T^{n_p}x - x^*))} \varphi(t) dt \leq 0,$$

which implies that $\psi(\rho(c(T^{n_p+1}x - Tx^*))) \rightarrow 0$, as $p \rightarrow \infty$. Thus,

$$\psi(\rho(c(x^* - T^{n_p+1}x))) + \psi(\rho(c(T^{n_p+1}x - Tx^*))) \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

Hence $\lim_{n \rightarrow \infty} \psi(\rho(\frac{c}{2}(x^* - Tx^*))) = 0$ and $x^* = Tx^*$.

Finally, we prove that x^* is a unique fixed point. Suppose that T has another fixed point $y^* \in X_\rho - \{x^*\}$. Since G is a C_ρ -graph, there exists a subsequence $\{T^{n_p}x\}$ of $\{T^n x\}$ such that $(T^{n_p}x, x^*) \in E(G)$ and $(T^{n_p}x, y^*) \in E(G)$ for each $p \in \mathbb{N}$. Furthermore, as G is weakly connected, $(x^*, y^*) \in E(\tilde{G})$. We have

$$\begin{aligned} \int_0^{\psi(\rho(c(x^* - y^*)))} \varphi(t) dt &= \int_0^{\psi(\rho(c(Tx^* - Ty^*))} \varphi(t) dt \\ &\leq \phi\left(\int_0^{\psi(\rho(l(x^* - y^*))} \varphi(t) dt\right) \\ &< \int_0^{\psi(\rho(l(x^* - y^*))} \varphi(t) dt \\ &< \int_0^{\psi(\rho(c(x^* - y^*))} \varphi(t) dt, \end{aligned}$$

a contradiction. Hence, $x^* = y^*$. □

In Theorem 3.18, if we replace the condition that G is a C_ρ -graph with orbitally G_ρ -continuity of T , then we have the following theorem.

Theorem 3.19 *Let X_ρ be a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a $(\tilde{G}, \phi, \psi)_\rho$ -contraction, which is orbitally G_ρ -continuous. Assume that the set X_T is nonempty and the graph G is weakly connected. Then T is a PO.*

In the following corollaries, suppose that X_ρ is a ρ -complete modular space endowed with a graph G , where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be edge-preserving and the set X_T be nonempty.

Corollary 3.20 *Assume*

- (i) *the C_ρ -graph G is weakly connected and*
- (ii) *there exist nonnegative numbers l, c with $l < c$ such that*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq \phi \left(\int_0^{\rho(l(x-y))} \varphi(t) dt \right)$$

hold for each $(x, y) \in E(\tilde{G})$ with $\phi \in \Phi_1$ and $\varphi \in \Phi$. Then T is a PO.

Corollary 3.21 *Assume*

- (i) *the C_ρ -graph G is weakly connected and*
- (ii) *there exist nonnegative numbers l, c with $l < c$ such that*

$$\int_0^{\psi(\rho(c(Tx-Ty)))} \varphi(t) dt \leq \eta \int_0^{\psi(\rho(l(x-y)))} \varphi(t) dt$$

for each $(x, y) \in E(\tilde{G})$ with $\eta \in (0, 1)$ and $\varphi \in \Phi$. Then T is a PO.

Now we provide examples in support of our results.

Example 3.22 Let $X_\rho = \{0, 1, 2, 3, 4, 5\}$ and $\rho(x) = |x|$, for all $x \in X_\rho$. Consider

$$E(\tilde{G}) = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\} \cup \{(0, x) : x \in X_\rho\},$$

and $Tx = 0, x \in X_\rho$. Then G is weakly connected and C_ρ -graph, X_T is nonempty and T is a $(\tilde{G}, A)_\rho$ -contraction where $c = \frac{7}{5}, l = \frac{6}{5}, \varphi(t) = 1$. Hence, T is a PO.

Example 3.23 Let $X_\rho = \{0, 1, 2, 3\}$ and $\rho(x) = |x|$, for all $x \in X_\rho$. Consider

$$E(\tilde{G}) = \{(0, 0), (0, 1), (1, 1), (1, 3), (2, 2), (0, 3), (2, 3), (3, 3)\}.$$

Define $T : X_\rho \rightarrow X_\rho$ as follows:

$$Tx = \begin{cases} 0, & x \in \{0, 1\}, \\ 1, & x \in \{2, 3\}. \end{cases}$$

Then G is weakly connected and C_ρ -graph, X_T is nonempty, and T is a Hardy-Rogers type $(\tilde{G})_\rho$ -contraction where $c = 4, l_1 = l_2 = l_3 = 3, l_4 = l_5 = \frac{1}{2}, \eta = \frac{1}{3}, \beta = \frac{1}{4}, \gamma = 0$, and $\varphi(t) = 1$. Moreover, 0 is a unique fixed point of T .

Example 3.24 Let $X_\rho = [0, 1]$ and $\rho(x) = |x|$, for all $x \in X_\rho$. Consider

$$E(G) = \{(0, 0)\} \cup \{(0, x) : x \geq 1/2\} \cup \{(x, y) : x, y \in (0, 1]\},$$

and $Tx = \frac{x}{2}$, $x \in X_\rho$. Then G is weakly connected and a C_ρ -graph, X_T is nonempty and T is a $(\tilde{G}, \phi, \psi)_\rho$ -contraction where $c = \frac{1}{2}$, $l = \frac{1}{3}$, $\varphi(t) = 1$, $\phi(\xi) = \frac{\xi}{1+\xi}$, and $\psi(\omega) = \frac{\omega}{3}$. Thus all conditions of Theorem 3.18 are satisfied. Moreover, T is a PO.

Remark 3.25 In the above examples, if we use $\rho(x) = x^2$, the conclusions remain the same.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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