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Extinction and decay estimates of solutions for a p-Laplacian evolution equation with nonlinear gradient source and absorption

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Abstract
We investigate the extinction properties of non-negative nontrivial weak solutions of the initial-boundary value problem for a p-Laplacian evolution equation with nonlinear gradient source and absorption terms.
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1 Introduction

We are concerned with the initial-boundary value problem for a p-Laplacian evolution equation with gradient source and absorption terms,

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda|\nabla u|^r - \beta u^q, \quad x \in \Omega, t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $1 < p < 2$, $0 < q \leq 1$, $\lambda, \beta, r > 0$, $\Omega \subset R^N$ $(N \geq 1)$ is a bounded domain with smooth boundary and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a non-negative function. The symbols $\|\cdot\|_s$, $\|\cdot\|_{1,s}$ denote $L^s(\Omega)$, $W^{1,s}(\Omega)$ norms, respectively (where $s \geq 1$), and $|\Omega|$ denotes the measure of Ω.

Equation (1.1) appears in the study of non-Newtonian fluids through porous media, combustion theory, and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo plastics. If $p = 2$, they are Newtonian fluids. The p-Laplacian operator also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p < 2$, see [1]), flow through porous media ($p = \frac{3}{2}$, see [2]) or glacial sliding ($p \in (1, \frac{4}{3}]$, see [3]). Many nonlinear diffusion phenomena are described by the cooperation and interaction between the nonlinear source term and absorption term during the diffusion. From a physical point of view, $\lambda|\nabla u|^r$ is called gradient source term and $-\beta u^q$ represents an absorption term.

The extinction phenomenon is an important property for solutions of many evolutionary equations, especially for fast diffusion equations. In 1974, Kalashnikov [4] considered the Cauchy problem of a semilinear equation with absorption term $u_t = \Delta u - u^q$ and firstly introduced the definition of extinction for its solution, that is, there exists a finite time $T > 0$ such that the solution is nontrivial on $(0, T)$ and then $u(x,t) \equiv 0$ for all $(x,t) \in \Omega \times [T, +\infty)$. In this case, T is called the extinction time. Later, many authors became interested in the extinction and nonextinction of all kinds of evolutionary equations. We have the following parabolic equation without gradient source term:

$$u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + \lambda u^r - \beta u^q, \quad x \in \Omega, t > 0, \tag{1.4}$$

where $r > 0$ and $0 < q \leq 1$. In the case $\lambda = \beta = 0$, Dibenedetto [5] and Yuan *et al.* [6] proved that the necessary and sufficient condition for the extinction to occur is $1 < p < 2$. For the case $\lambda = 0$, Gu [7] proved that if $1 < p < 2$ or $0 < q < 1$, the solutions of the problem vanish in finite time, but if $p \geq 2$ and $q \geq 1$, there is nonextinction. Tian [8] and Yin *et al.* [9] showed that $r = p - 1$ is the critical exponent of the weak solution for the case $\beta = 0$. But all the results are limited to a local range and a higher dimensional space, while a precise decay estimate has not been given. Recently, Fang and Li [10] considered equation (1.4) with $q = 1$, when the diffusion term was replaced by a doubly degenerate operator in the whole dimensional space, and they showed that the extinction of the weak solution is determined by competition of source and absorption terms. They also obtained the exponential decay estimates which depend on the initial data, coefficients, and domains. Thereafter, they obtained the same results for a class of nonlocal problems, see [11, 12].

Recently, many researchers have devoted studies to the occurrence of such a phenomenon for a class of nonlinear parabolic equations with gradient terms. For example, Benachour *et al.* [13] considered the semilinear heat equation with absorption term,

$$u_t = \Delta u - \lambda|\nabla u|^r, \quad x \in \Omega, t > 0, \tag{1.5}$$

subject to (1.2) and (1.3) and proved that the sufficient condition for the extinction to occur is $0 < r < 1$ by using the upper and lower solutions methods. Lagar *et al.* [14] studied the qualitative properties of non-negative solutions to the Cauchy problem for the fast diffusion equation with gradient absorption

$$u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) - |\nabla u|^q, \quad x \in R^N, t > 0, \tag{1.6}$$

with $1 < p < 2$, $q > 0$ and obtained the result that the solution of (1.6) either remains positive as $t \to \infty$ for $q > p - \frac{N}{N+1}$ or vanishes in finite time for $0 < q < \frac{p}{2}$. For the porous medium equation with gradient source term and without absorption term, the research of the extinction and nonextinction of solutions has also been performed (see [15]).

Motivated by the works mentioned above, and because there is little literature on the study of the extinction and nonextinction properties for parabolic equations with nonlinear gradient source and absorption terms, in this paper, our goal is to establish the sufficient conditions about the extinction and nonextinction of solutions for the problem (1.1)-(1.3) in the whole dimensional space. By combining the L^p-integral norm estimate method and the technique of differential inequalities, we find that the critical exponent of extinction for the non-negative weak solution is determined by the competition of nonlinear terms for $1 < p < 2$, and decay estimates depend on the choices of initial data, coefficients, and domain. More precisely, we obtain the following results.

Theorem 1 *Assume that* $1 < p < 2$, $q = 1$ *and* $p - 1 = r$.

(1) *If* $N \geq 2$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time for any non-negative initial data* u_0 *provided that* λ *is sufficiently small,and we have the following.*

(a) *If* $\frac{2N}{N+2} < p < 2$,

$$\|u(\cdot,t)\|_2 \leq \left[\left(\|u_0\|_2^{2-p} + \frac{C_1}{\beta}\right)e^{(p-2)\beta t} - \frac{C_1}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [0, T_1),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_1, +\infty).$$

(b) *If* $1 < p \leq \frac{2N}{N+2}$,

$$\|u(\cdot,t)\|_{l+1} \leq \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_2}{\beta}\right)e^{(p-2)\beta t} - \frac{C_2}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [0, T_2),$$

$$\|u(\cdot,t)\|_{l+1} \equiv 0, \quad t \in [T_2, +\infty).$$

Here $l = \frac{2N-(N+1)p}{p}$, C_1, T_1, C_2, *and* T_2 *are given by* (3.5), (3.6), (3.11), *and* (3.12), *respectively.*

(2) *If* $N = 1$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time for any non-negative initial data* u_0 *provided that* λ *is sufficiently small, and*

$$\|u(\cdot,t)\|_2 \leq \left[\left(\|u_0\|_2^{2-p} + \frac{C_3}{\beta}\right)e^{(p-2)\beta t} - \frac{C_3}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [0, T_3),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_3, +\infty),$$

where C_3 *and* T_3 *are given by* (3.15) *and* (3.16), *respectively.*

Theorem 2 *Assume that* $1 < p < 2$, $q = 1$ *and* $p - 1 < r < \frac{p}{2}$.

(1) *If* $N \geq 2$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time provided that* u_0 (*or* λ *or* $|\Omega|$) *is sufficiently small, and we have the following.*

(a) *If* $\frac{2N}{N+2} < p < 2$,

$$\|u(\cdot,t)\|_2 \leq \|u_0\|_2 e^{-\alpha_1 t}, \quad t \in [0, T_4),$$

$$\|u(\cdot,t)\|_2 \leq \left[\left(\|u(\cdot,T_4)\|_2^{2-p} + \frac{C_4}{\beta}\right)e^{(p-2)\beta(t-T_4)} - \frac{C_4}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [T_4, T_5),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_5, +\infty).$$

(b) *If* $1 < p \leq \frac{2N}{N+2}$,

$$\|u(\cdot,t)\|_{l+1} \leq \|u_0\|_{l+1} e^{-\alpha_2 t}, \quad t \in [0, T_6),$$

$$\|u(\cdot,t)\|_{l+1} \leq \left[\left(\|u(\cdot,T_6)\|_{l+1}^{2-p} + \frac{C_5}{\beta}\right)e^{(p-2)\beta(t-T_6)} - \frac{C_5}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [T_6, T_7),$$

$$\|u(\cdot,t)\|_{l+1} \equiv 0, \quad t \in [T_7, +\infty),$$

where $l = \frac{2N-(N+1)p}{p}$, C_4, T_5, C_5, *and* T_7 *are given by* (3.17)-(3.20), *respectively.*

(2) *If* $N = 1$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time provided that* u_0 (*or* λ *or* $|\Omega|$) *is sufficiently small, and*

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2 e^{-\alpha_3 t}, \quad t \in [0, T_8),$$

$$\|u(\cdot,t)\|_2 \le \left[\left(\|u(\cdot,T_8)\|_2^{2-p} + \frac{C_6}{\beta}\right) e^{(p-2)\beta(t-T_8)} - \frac{C_6}{\beta}\right]^{\frac{1}{2-p}}, \quad t \in [T_8, T_9),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_9, +\infty),$$

where C_6 *and* T_9 *are given by* (3.21) *and* (3.22), *respectively.*

Theorem 3 *Assume* $1 < p < 2$, $q = 1$ *and* $p - 1 > r$. *Then the non-negative weak solution of problem* (1.1)-(1.3) *cannot vanish in finite time for any non-negative initial data* u_0 *provided that* λ *is sufficiently large.*

Remark 1 According to Theorems 1-3, we observe that $p - 1 = r$ is still the critical exponent of extinction for the solution of (1.1)-(1.3) when $1 < p < 2$ and $q = 1$.

Remark 2 Assume that $p \ge 2$, $q = 1$, then the non-negative weak solution of problem (1.1)-(1.3) cannot vanish in finite time for any $r > 0$ and non-negative initial data. This result can be extended to the case $q \ge 1$ (the detailed proof can be found in [7]).

Theorem 4 *Assume that* $1 < p < 2$, $0 < q < 1$ *and* $p - 1 = r$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time for any non-negative initial data* u_0 *provided that* λ *is sufficiently small.*

Theorem 5 *Assume that* $1 < p < 2$, $0 < q < 1$.

(1) *If* $N \ge 2$, $\frac{2N}{N+2} < p < 2$ *with* $\frac{2p^2q+Np(p-q-1)}{2p(1+q)+2N(p-q-1)} < r < \frac{p}{2}$ *or* $1 < p \le \frac{2N}{N+2}$ *with* $\frac{p^2q(s+1)+Np(p-q-1)}{p(s+1)(1+q)+2N(p-q-1)} < r < \frac{p}{2}$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time provided that* u_0 (*or* λ *or* $|\Omega|$) *is sufficiently small or* β *is sufficiently large* (*where* $s > \frac{2N-(N+1)p}{p}$).

(2) *If* $N = 1$, $\frac{2p^2q+Np(p-q-1)}{2p(1+q)+2N(p-q-1)} < r < \frac{p}{2}$, *the non-negative weak solution of problem* (1.1)-(1.3) *vanishes in finite time provided that* u_0 (*or* λ *or* $|\Omega|$) *is sufficiently small or* β *is sufficiently large.*

Remark 3 If $p - 1 \le q < 1$, the conditions in Theorem 5 imply that $r > p - 1$.

Remark 4 Assume that $p = 2$, $0 < q < 1$; Theorem 4 and Theorem 5 are still established.

Remark 5 Assume that $p > 2$, $0 < q < 1$ and $\frac{2p^2q+Np(p-q-1)}{2p(1+q)+2N(p-q-1)} < r < \frac{p}{2}$; the non-negative weak solution of problem (1.1)-(1.3) vanishes in finite time provided that u_0 (or λ or $|\Omega|$) is sufficiently small or β is sufficiently large (as $\frac{2N}{N+2} < 2 < p$, the proof of this result is the same as the proof for the case $\frac{2N}{N+2} < p < 2$ in Theorem 5(1)).

Remark 6 Assume that $p > 2$, $0 < q < 1$, $\frac{pq}{1+q} < r \le \frac{p^2q+N(p-1)(p-q-1)}{p(q+1)+N(p-q-1)}$ and $r < \frac{p}{2}$, the non-negative weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data u_0 provided that λ (or $|\Omega|$) is sufficiently small or β is sufficiently large (the detailed proof can be referred to [16]).

Remark 7 If $r = \frac{p}{2}$, Theorem 2, Theorem 5, and Remark 5 will be still established and the choice of $|\Omega|$ will not affect the extinction behavior of solutions any longer.

Remark 8 Theorems 1-5 all require that u_0 or λ or $|\Omega|$ should be sufficiently small or β should be sufficiently large, and we will give more concrete conditions to satisfy in the later proofs.

The outline of the paper is as follows. In Section 2, we firstly give the definition of weak solutions for problem (1.1)-(1.3), and then give some preliminary lemmas. Then we prove Theorems 1-3 and Theorems 4-5 in Section 3 and Section 4, respectively.

2 Preliminary results

Due to the singularity of (1.1), problem (1.1)-(1.3) has no classical solutions in general, and hence it is reasonable to find a weak solution of the problem. To this end, we first give the following definition of a weak local solution.

Definition 1 We say that a non-negative nontrivial function $u(x,t)$ defined in $Q_T = \Omega \times (0,T)$ is a weak solution of problem (1.1)-(1.3) if the following conditions hold:

(i) $u \in C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$;

(ii) For any $0 < t_1 < t_2 < T$ and any test function $0 \le \varphi \in C_0^\infty(Q_T)$

$$\int_\Omega u(x,t_2)\varphi(x,t_2)\,dx = \int_\Omega u(x,t_1)\varphi(x,t_1)\,dx + \int_{t_1}^{t_2}\int_\Omega \bigl\{u\varphi_s - |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\bigr\}\,dx\,ds + \int_{t_1}^{t_2}\int_\Omega \bigl\{\lambda|\nabla u|^r - \beta u^q\bigr\}\varphi(x,s)\,dx\,ds;$$

(iii) $u(x,t) = u_0(x)$ a.e. $x \in \Omega$.

We can also define the weak lower solution and the weak upper solution of problem (1.1)-(1.3) in the same way except that the '=' in Definition 1 is replaced by '$\le$' and '$\ge$', respectively. Similar to the analysis in [17] and [14, Section 6], the existence in time of a non-negative weak local solution of problem (1.1)-(1.3) can be constructed by the usual vanishing viscosity method which would satisfy a comparison principle.

Before proving our main results, we show some preliminary lemmas and the Gagliardo-Nirenberg inequality which are very important in the following proofs of our results. As for the proofs of these lemmas, we will not repeat them again (see [10–12, 18]).

Lemma 1 *Let $y(t)$ be a non-negative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt} + \alpha y^k \le 0, \quad t \ge 0; \qquad y(0) \ge 0,$$

where $\alpha > 0$ is a constant and $k \in (0,1)$, then we have the decay estimate

$$y(t) \le \bigl[y^{1-k}(0) - \alpha(1-k)t\bigr]^{\frac{1}{1-k}}, \quad t \in [0,T_*),$$
$$y(t) \equiv 0, \quad t \in [T_*,+\infty),$$

where $T_* = \frac{y^{1-k}(0)}{\alpha(1-k)}$.

Lemma 2 *Let $y(t)$ be a non-negative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt}+\alpha y^k+\beta y\le 0,\quad t\ge T_0;\qquad y(T_0)\ge 0,$$

where $\alpha,\beta>0$ are constants and $k\in(0,1)$, then we have the decay estimate

$$y(t)\le\left[\left(y^{1-k}(T_0)+\frac{\alpha}{\beta}\right)e^{(k-1)\beta(t-T_0)}-\frac{\alpha}{\beta}\right]^{\frac{1}{1-k}},\quad t\in[T_0,T_*),$$
$$y(t)\equiv 0,\quad t\in[T_*,+\infty),$$

where $T_*=\frac{1}{(1-k)\beta}\ln(1+\frac{\beta}{\alpha}y^{1-k}(T_0))+T_0$.

Lemma 3 *Let $0<k<m\le 1$, $y(t)\ge 0$ be a solution of the differential inequality*

$$\frac{dy}{dt}+\alpha y^k+\beta y\le\gamma y^m,\quad t\ge 0;\qquad y(0)=y_0>0,$$

where $\alpha,\beta>0$, γ is a positive constant such that $\gamma<\alpha y_0^{k-m}$, then there exists $\eta>\beta$, such that

$$0\le y(t)\le y_0e^{-\eta t},\quad t\ge 0.$$

Lemma 4 *Let $\alpha,\beta,\gamma>0$ and $0<m<k<1$, then there exists at least one non-constant solution of the ODE problem*

$$\frac{dy}{dt}+\alpha y^k+\beta y=\gamma y^m,\quad t\ge 0;\qquad y(0)=y_0>0,\quad y(t)>0,t>0.$$

Lemma 5 (Gagliardo-Nirenberg inequality) *Suppose that $u\in W_0^{k,m}(\Omega)$, $1\le m\le+\infty$, $0\le j<k$, $1\ge\frac{1}{r}\ge\frac{1}{m}-\frac{k}{N}$, then we have*

$$\|D^ju\|_q\le C\|D^ku\|_m^{\theta}\|u\|_r^{1-\theta},$$

where C is a constant depending only on N,m,r,j,k,q and $\frac{1}{q}=\frac{j}{N}+\theta(\frac{1}{m}-\frac{k}{N})+\frac{1-\theta}{r}$, $0\le\theta<1$. If $m<\frac{N}{k-j}$, then $q\in[\frac{Nr}{N+rj},\frac{Nm}{N-(k-j)m}]$, if $m\ge\frac{N}{k-j}$, then $q\in[\frac{Nr}{N+rj},+\infty]$.

3 The case $1<p<2$, $q=1$

3.1 Proof of Theorem 1

(1) If $N\ge 2$, we have the following.

(a) If $\frac{2N}{N+2}<p<2$, multiplying (1.1) by u and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_p^p+\beta\|u\|_2^2=\lambda\int_\Omega u|\nabla u|^r\,dx,\tag{3.1}$$

since $p-1=r$, we can easily get $\frac{r}{p-r}=p-1<1$. By the Young inequality, we obtain

$$\begin{aligned}\int_\Omega u|\nabla u|^r\,dx &\le \varepsilon\|\nabla u\|_p^p + C(\varepsilon)\|u\|_{1+\frac{r}{p-r}}^{1+\frac{r}{p-r}} \\ &\le \varepsilon\|\nabla u\|_p^p + C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{1+\frac{r}{p-r}} \\ &= \varepsilon\|\nabla u\|_p^p + C(\varepsilon)|\Omega|^{\frac{2-p}{2}}\|u\|_2^p, \end{aligned} \tag{3.2}$$

substituting (3.2) into (3.1) leads to

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1-\lambda\varepsilon)\|\nabla u\|_p^p + \beta\|u\|_2^2 \le \lambda C(\varepsilon)|\Omega|^{\frac{2-p}{2}}\|u\|_2^p.$$

Here we can choose ε small enough such that $1-\lambda\varepsilon>0$. By the Hölder inequality and the Sobolev embedding inequality, we have

$$\|u\|_2 \le |\Omega|^{\frac{1}{2}-\frac{N-p}{Np}}\|u\|_{\frac{Np}{N-p}} \le C_0|\Omega|^{\frac{1}{2}-\frac{N-p}{Np}}\|\nabla u\|_p, \tag{3.3}$$

then we substitute (3.3) into (3.1) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left[(1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda C(\varepsilon)|\Omega|^{\frac{2-p}{2}}\right]\|u\|_2^p + \beta\|u\|_2^2 \le 0, \tag{3.4}$$

i.e.

$$\frac{d}{dt}\|u\|_2 + C_1\|u\|_2^{p-1} + \beta\|u\|_2 \le 0,$$

where

$$C_1 = (1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda C(\varepsilon)|\Omega|^{\frac{2-p}{2}}. \tag{3.5}$$

Once ε is fixed, we can choose λ small enough such that $C_1>0$. By Lemma 2, we can obtain the desired decay estimate for

$$T_1 = \frac{1}{(2-p)\beta}\ln\left(1+\frac{\beta}{C_1}\|u_0\|_2^{2-p}\right). \tag{3.6}$$

(b) If $1<p\le\frac{2N}{N+2}$, multiplying (1.1) by u^l ($l=\frac{2N-(N+1)p}{p}\ge 1$) and integrating over Ω yield

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \frac{lp^p}{(p+l-1)^p}\left\|\nabla u^{\frac{p+l-1}{p}}\right\|_p^p + \beta\|u\|_{l+1}^{l+1} = \lambda\int_\Omega u^l|\nabla u|^r\,dx. \tag{3.7}$$

By the Young inequality, we have

$$\begin{aligned}\int_\Omega u^l|\nabla u|^r\,dx &= \left(\frac{p+l-1}{p}\right)^r\int_\Omega u^{l-\frac{(l-1)r}{p}}\left|\nabla u^{\frac{p+l-1}{p}}\right|^r dx \\ &\le \varepsilon\left\|\nabla u^{\frac{p+l-1}{p}}\right\|_p^p + C(\varepsilon)\int_\Omega u^{l+\frac{r}{p-r}}\,dx\end{aligned}$$

$$\begin{aligned}&\le \varepsilon \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p^p + C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}} \|u\|_{l+1}^{l+\frac{r}{p-r}} \\ &= \varepsilon \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p^p + C(\varepsilon)|\Omega|^{\frac{1-r}{l+1}} \|u\|_{l+1}^{l+p-1},\end{aligned} \tag{3.8}$$

then we substitute (3.8) into (3.7) to get

$$\begin{aligned}&\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right] \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p^p + \beta\|u\|_{l+1}^{l+1} \\ &\quad\le \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{l+1}} \|u\|_{l+1}^{l+p-1}.\end{aligned} \tag{3.9}$$

Here we can choose ε small enough such that $\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon > 0$. By the Sobolev embedding inequality, we have

$$\left\| u^{\frac{p+l-1}{p}} \right\|_{\frac{Np}{N-p}} \le C_{00} \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p,$$

i.e.

$$C_{00}^{-p} \|u\|_{\frac{(p+l-1)N}{N-p}}^{p+l-1} \le \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p^p.$$

By the choice of l, we have

$$C_{00}^{-p} \|u\|_{l+1}^{p+l-1} \le \left\| \nabla u^{\frac{p+l-1}{p}} \right\|_p^p. \tag{3.10}$$

Substituting (3.10) into (3.9) leads to

$$\frac{d}{dt}\|u\|_{l+1} + C_2\|u\|_{l+1}^{p-1} + \beta\|u\|_{l+1} \le 0,$$

where

$$C_2 = C_{00}^{-p}\left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right] - \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{l+1}}. \tag{3.11}$$

Once ε is fixed, we can choose λ small enough such that $C_2 > 0$. By Lemma 2, we can obtain the desired decay estimate for

$$T_2 = \frac{1}{(2-p)\beta}\ln\left(1 + \frac{\beta}{C_2}\|u_0\|_{l+1}^{2-p}\right). \tag{3.12}$$

(2) If $N = 1$, multiplying (1.1) by u^k and integrating over Ω, and then using the Young inequality, we have

$$\begin{aligned}&\frac{1}{k+1}\frac{d}{dt}\|u\|_{k+1}^{k+1} + \left[\frac{kp^p}{(k+p-1)^p} - \lambda\varepsilon\right] \left\| \nabla u^{\frac{p+k-1}{p}} \right\|_p^p + \beta\|u\|_{k+1}^{k+1} \\ &\quad\le \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{k+1}} \|u\|_{k+1}^{k+p-1}.\end{aligned} \tag{3.13}$$

By the Sobolev embedding theorem, we have

$$\left\| u^{\frac{p+k-1}{p}} \right\|_d \le \gamma \left\| \nabla u^{\frac{p+k-1}{p}} \right\|_p,$$

i.e.

$$\gamma^{-p}\|u\|^{p+k-1}_{\frac{(p+k-1)d}{p}} \leq \left\|\nabla u^{\frac{p+k-1}{p}}\right\|^{p}_{p}, \tag{3.14}$$

where $\gamma > 0$, $d \geq p$. Here setting $k = 1$, $d = 2$ leads to

$$\frac{d}{dt}\|u\|_2 + C_3\|u\|_2^{p-1} + \beta\|u\|_2 \leq 0,$$

where

$$C_3 = (1-\lambda\varepsilon)\gamma^{-p} - \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2}}. \tag{3.15}$$

Once ε is fixed, we can choose λ small enough such that $C_3 > 0$. By Lemma 2, we can obtain the desired decay estimate for

$$T_3 = \frac{1}{(2-p)\beta}\ln\left(1 + \frac{\beta}{C_3}\|u_0\|_2^{2-p}\right). \tag{3.16}$$

3.2 Proof of Theorem 2

(1) If $N \geq 2$, we have the following.

(a) If $\frac{2N}{N+2} < p < 2$, multiplying (1.1) by u and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \beta\|u\|_2^2 = \lambda\int_\Omega u|\nabla u|^r\,dx,$$

and substituting (3.2) and (3.3) into the above equality gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\|u\|_2^p + \beta\|u\|_2^2 \leq \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{1+\frac{r}{p-r}},$$

i.e.

$$\frac{d}{dt}\|u\|_2 + (1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\|u\|_2^{p-1} + \beta\|u\|_2 \leq \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{\frac{r}{p-r}}.$$

By Lemma 3, there exists $\alpha_1 > \beta$ such that

$$0 \leq \|u\|_2 \leq \|u_0\|_2 e^{-\alpha_1 t}, \quad t \geq 0,$$

provided that

$$\|u_0\|_2 < \left[\frac{(1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}}{\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}}\right]^{\frac{1}{\frac{r}{p-r}-p+1}}.$$

Furthermore, there exists $T_4 > 0$ such that

$$\begin{aligned}&(1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{\frac{r}{p-r}-p+1}\\ &\quad\geq (1-\lambda\varepsilon)C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\left(\|u_0\|_2 e^{-\alpha_1 T_4}\right)^{\frac{r}{p-r}-p+1} = C_4 > 0\end{aligned} \tag{3.17}$$

holds for $t \in [T_4, +\infty)$. Therefore, when $t \in [T_4, +\infty)$, we have

$$\frac{d}{dt}\|u\|_2 + C_4\|u\|_2^{p-1} + \beta\|u\|_2 \leq 0.$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_5 = \frac{1}{(2-p)\beta}\ln\left(1 + \frac{\beta}{C_4}\left\|u(\cdot, T_4)\right\|_2^{2-p}\right) + T_4. \tag{3.18}$$

(b) If $1 < p \leq \frac{2N}{N+2}$, multiplying (1.1) by u^l ($l = \frac{2N-(N+1)p}{p} \geq 1$) and integrating over Ω yield

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \frac{lp^p}{(p+l-1)^p}\left\|\nabla u^{\frac{p+l-1}{p}}\right\|_p^p + \beta\|u\|_{l+1}^{l+1} = \lambda\int_\Omega u^l|\nabla u|^r\,dx.$$

Substitute (3.8) and (3.10) into the above equality we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right]C_{00}^{-p}\|u\|_{l+1}^{p+l-1} + \beta\|u\|_{l+1}^{l+1} \leq \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}}\|u\|_{l+1}^{l+\frac{r}{p-r}},$$

i.e.

$$\frac{d}{dt}\|u\|_{l+1} + \left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right]C_{00}^{-p}\|u\|_{l+1}^{p-1} + \beta\|u\|_{l+1} \leq \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}}\|u\|_{l+1}^{\frac{r}{p-r}}.$$

By Lemma 3, there exists $\alpha_2 > \beta$ such that

$$0 \leq \|u\|_{l+1} \leq \|u_0\|_{l+1}e^{-\alpha_2 t}, \quad t \geq 0,$$

provided that

$$\|u_0\|_{l+1} < \left[\frac{C_{00}^{-p}(\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon)}{\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}}}\right]^{\frac{1}{\frac{r}{p-r}-p+1}}.$$

Furthermore, there exists $T_6 > 0$ such that

$$\begin{aligned}
&\left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right]C_{00}^{-p} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}}\|u\|_{l+1}^{\frac{r}{p-r}-p+1} \\
&\quad\geq \left[\frac{lp^p}{(p+l-1)^p} - \lambda\varepsilon\right]C_{00}^{-p} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(l+1)(p-r)}}\left(\|u_0\|_{l+1}e^{-\alpha_2 T_6}\right)^{\frac{r}{p-r}-p+1} \\
&\quad= C_5 > 0
\end{aligned} \tag{3.19}$$

holds for $t \in [T_6, +\infty)$. Therefore, when $t \in [T_6, +\infty)$, we have

$$\frac{d}{dt}\|u\|_{l+1} + C_5\|u\|_{l+1}^{p-1} + \beta\|u\|_{l+1} \leq 0.$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_7 = \frac{1}{(2-p)\beta}\ln\left(1 + \frac{\beta}{C_5}\left\|u(\cdot, T_6)\right\|_{l+1}^{2-p}\right) + T_6. \tag{3.20}$$

(2) If $N = 1$, multiplying (1.1) by u^k and integrating over Ω yield

$$\frac{1}{k+1}\frac{d}{dt}\|u\|_{k+1}^{k+1} + \frac{kp^p}{(k+p-1)^p}\big\|\nabla u^{\frac{p+k-1}{p}}\big\|_p^p + \beta\|u\|_{k+1}^{k+1} = \lambda\int_\Omega u^k|\nabla u|^r\,dx.$$

If we substitute (3.8) and (3.14) into the above equality, we have as a result

$$\begin{aligned}&\frac{1}{k+1}\frac{d}{dt}\|u\|_{k+1}^{k+1} + \left[\frac{kp^p}{(k+p-1)^p} - \lambda\varepsilon\right]\gamma^{-p}\|u\|_{k+1}^{p+k-1} + \beta\|u\|_{k+1}^{k+1}\\ &\quad\le \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(k+1)(p-r)}}\|u\|_{k+1}^{k+\frac{r}{p-r}},\end{aligned}$$

i.e.

$$\frac{d}{dt}\|u\|_{k+1} + \left[\frac{kp^p}{(k+p-1)^p} - \lambda\varepsilon\right]\gamma^{-p}\|u\|_{k+1}^{p-1} + \beta\|u\|_{k+1} \le \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(k+1)(p-r)}}\|u\|_{k+1}^{\frac{r}{p-r}}.$$

Here setting $k = 1$ leads to

$$\frac{d}{dt}\|u\|_2 + (1-\lambda\varepsilon)\gamma^{-p}\|u\|_2^{p-1} + \beta\|u\|_2 \le \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{\frac{r}{p-r}}.$$

By Lemma 3, there exists $\alpha_3 > \beta$ such that

$$0 \le \|u\|_2 \le \|u_0\|_2 e^{-\alpha_3 t}, \quad t \ge 0,$$

provided that

$$\|u_0\|_2 < \left[\frac{(1-\lambda\varepsilon)\gamma^{-p}}{\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}}\right]^{\frac{1}{\frac{r}{p-r}-p+1}}.$$

Furthermore, there exists $T_8 > 0$ such that

$$\begin{aligned}&(1-\lambda\varepsilon)\gamma^{-p} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{\frac{r}{p-r}-p+1}\\ &\quad\ge (1-\lambda\varepsilon)\gamma^{-p} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\big(\|u_0\|_2 e^{-\alpha_3 T_8}\big)^{\frac{r}{p-r}-p+1} = C_6 > 0\end{aligned} \tag{3.21}$$

holds for $t \in [T_8, +\infty)$. Therefore, when $t \in [T_8, +\infty)$, we have

$$\frac{d}{dt}\|u\|_2 + C_6\|u\|_2^{p-1} + \beta\|u\|_2 \le 0.$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_9 = \frac{1}{(2-p)\beta}\ln\left(1 + \frac{\beta}{C_6}\big\|u(\cdot, T_8)\big\|_2^{2-p}\right) + T_8. \tag{3.22}$$

3.3 Proof of Theorem 3

Let $v(x,t) = g(t)\phi(x)$, where $\phi(x)$ is the first eigenfunction corresponding to the first eigenvalue λ_1 for the homogeneous Dirichlet boundary value problem,

$$-\operatorname{div}\big(|\nabla\phi|^{p-2}\nabla\phi\big) = \lambda|\phi|^{p-2}\phi, \quad x\in\Omega; \qquad \phi(x) = 0, \quad x \in \partial\Omega,$$

and let $\phi(x)$ satisfy $\phi(x) > 0$, $\max_{x\in\Omega}\phi(x) = 1$; by Lemma 4, there exists one non-constant solution $g(t)$ that satisfies the ODE problem

$$g'(t) + \lambda_1 g^{p-1}(t) + \beta g(t) = g^r(t), \quad t \ge 0; \qquad g(0) = 0, \quad g(t) > 0, t > 0.$$

Then for any test function $0 \le \varphi(x,t) \in C_0^\infty(Q_T)$, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \bigl\{v_s(x,s)\varphi(x,s) - |\nabla v|^{p-2}\nabla v\nabla\varphi + \beta v(x,s)\varphi(x,s) - \lambda|\nabla v|^r\varphi(x,s)\bigr\}\,dx\,ds \\
&\quad= \int_0^t \int_\Omega \bigl\{g'(s)\phi(x) + \lambda_1 g^{p-1}(s)\phi^{p-1}(x) + \beta g(s)\phi(x) - \lambda g^r(s)|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds \\
&\quad= \int_0^t \int_\Omega \bigl\{\bigl(-\lambda_1 g^{p-1}(s) - \beta g(s) + g^r(s)\bigr)\phi(x) + \lambda_1 g^{p-1}(s)\phi^{p-1}(x) \\
&\qquad+ \beta g(s)\phi(x) - \lambda g^r(s)|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds \\
&\quad\le \int_0^t \int_\Omega \bigl\{g^r(s)\phi(x) + \lambda_1 g^{p-1}(s)\phi^{p-1}(x) - \lambda g^r(s)|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds \\
&\quad\le \int_0^t \int_\Omega \bigl\{g^r(s) + \lambda_1 g^{p-1}(s) - \lambda g^r(s)|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds \\
&\quad= \int_0^t \int_\Omega g^r(s)\bigl\{1 + \lambda_1 g^{p-1-r}(s) - \lambda|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds.
\end{aligned}$$

For such a $v(x,t)$ to be a subsolution of problem (1.1)-(1.3), it suffices to show that

$$\int_0^t \int_\Omega g^r(s)\bigl\{1 + \lambda_1 g^{p-1-r}(s) - \lambda|\nabla\phi|^r\bigr\}\varphi(x,s)\,dx\,ds \le 0.$$

Here we only show that

$$\int_\Omega \bigl\{1 + \lambda_1 g^{p-1-r}(s) - \lambda|\nabla\phi|^r\bigr\}\,dx \le 0. \tag{3.23}$$

Since for any $s \in (0,t)$, $g(s)$ is bounded and for $p-1 > r$, we find that there exists a positive constant $M > 0$ such that

$$0 < g(s) < M,$$

and

$$0 < g^{p-1-r}(s) < M^{p-1-r}. \tag{3.24}$$

By choosing $\lambda \ge \frac{(1+\lambda_1 M^{p-1-r})|\Omega|}{\|\nabla\phi\|_r^r}$, we get

$$\int_\Omega \bigl\{1 + \lambda_1 M^{p-1-r} - \lambda|\nabla\phi|^r\bigr\}\,dx \le 0,$$

which together with (3.24) implies that (3.23) holds. Moreover, $v(x,0) = g(0)\phi(x) = 0 \le u_0(x)$, $x \in \Omega$; $v(x,t) = 0$, $x \in \partial\Omega$, $t > 0$. Therefore, $v(x,t)$ is a non-extinction subsolution of problem (1.1)-(1.3). By the comparison principle, we have

$$u(x,t) \geq v(x,t) > 0, \quad x \in \Omega, t > 0,$$

which implies that the weak solution $u(x,t)$ of problem (1.1)-(1.3) cannot vanish in finite time.

4 The case $1 < p < 2$, $0 < q < 1$

4.1 Proof of Theorem 4

(1) If $N \geq 2$, we have the following.

(a) If $\frac{2N}{N+2} < p < 2$, multiplying (1.1) by u and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \beta\|u\|_{1+q}^{1+q} = \lambda \int_\Omega u|\nabla u|^r\,dx. \tag{4.1}$$

Since $p - 1 = r$, we easily get $\frac{r}{p-r} = p - 1 < 1$. By the Young inequality, we obtain

$$\begin{aligned}\int_\Omega u|\nabla u|^r\,dx &\leq \varepsilon\|\nabla u\|_p^p + C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_{1+\frac{r}{p-r}}^{1+\frac{r}{p-r}} \\ &= \varepsilon\|\nabla u\|_p^p + C(\varepsilon)|\Omega|^{\frac{1-r}{2}}\|u\|_p^p. \end{aligned}\tag{4.2}$$

We substitute (4.2) into (4.1) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left[1 - \lambda\varepsilon - \frac{\lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2}}}{\lambda_1}\right]\|\nabla u\|_p^p + \beta\|u\|_{1+q}^{1+q} \leq 0. \tag{4.3}$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_2 \leq C(N,p,q)\|u\|_{1+q}^{1-\theta_1}\|\nabla u\|_p^{\theta_1},$$

where $\theta_1 = (\frac{1}{1+q} - \frac{1}{2})(\frac{1}{N} - \frac{1}{p} + \frac{1}{1+q})^{-1} = \frac{Np(1-q)}{2p(q+1)+2N(p-q-1)}$. Since $\frac{2N}{N+2} < p < 2$ and $0 < q < 1$, we easily get $0 < \theta_1 < 1$. By the Young inequality again, we have

$$\begin{aligned}\|u\|_2^{k_1} &\leq C(N,p,q)^{k_1}\|u\|_{1+q}^{k_1(1-\theta_1)}\|\nabla u\|_p^{k_1\theta_1}, \\ &\leq C(N,p,q)^{k_1}\left(\eta_1\|\nabla u\|_p^p + C(\eta_1)\|u\|_{1+q}^{\frac{k_1 p(1-\theta_1)}{p-k_1\theta_1}}\right),\end{aligned}$$

where $\eta_1 > 0$, $k_1 > 1$ will be determined later. Here we set $k_1 = \frac{p(1+q)}{p(1-\theta_1)+\theta_1(1+q)} = \frac{2p(q+1)+2N(p-q-1)}{2p+N(p-q-1)}$, then we have $1 < k_1 < 2$ and $\frac{k_1 p(1-\theta_1)}{p-k_1\theta_1} = 1 + q$, and

$$\frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1} \leq \frac{\eta_1\beta}{C(\eta_1)}\|\nabla u\|_p^p + \beta\|u\|_{1+q}^{1+q}, \tag{4.4}$$

and we now substitute (4.4) into (4.3) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left[1 - \lambda\varepsilon - \frac{\lambda C(\varepsilon,r,p)|\Omega|^{\frac{1-r}{2}}}{\lambda_1} - \frac{\eta_1\beta}{C(\eta_1)}\right]\|\nabla u\|_p^p + \frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1} \leq 0.$$

Here we can choose λ small enough such that $1-\lambda\varepsilon-\frac{\lambda C(\varepsilon,r,p)|\Omega|^{\frac{1-r}{2}}}{\lambda_1}-\frac{\eta_1\beta}{C(\eta_1)}\geq 0$. Here setting $C_7=\frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)}$ leads to

$$\frac{d}{dt}\|u\|_2+C_7\|u\|_2^{k_1-1}\leq 0,$$

By Lemma 1, we have

$$\|u\|_2\leq\left[\|u_0\|_2^{2-k_1}-C_7(2-k_1)t\right]^{\frac{1}{2-k_1}},\quad t\in[0,T_{10}),$$
$$\|u\|_2\equiv 0,\quad t\in[T_{10},+\infty),$$

where $T_{10}=\frac{\|u_0\|_2^{2-k_1}}{C_7(2-k_1)}$.

(b) If $1<p\leq\frac{2N}{N+2}$, multiplying (1.1) by u^s (where $s>\frac{2N-(N+1)p}{p}\geq 1$) and integrating over Ω yield

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1}+\frac{sp^p}{(p+s-1)^p}\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^p+\beta\|u\|_{s+q}^{s+q}=\lambda\int_\Omega u^s|\nabla u|^r\,dx. \tag{4.5}$$

By the Young inequality, we have

$$\begin{aligned}\int_\Omega u^s|\nabla u|^r\,dx&=\left(\frac{p+s-1}{p}\right)^r\int_\Omega u^{s-\frac{(s-1)r}{p}}\left|\nabla u^{\frac{p+s-1}{p}}\right|^r dx\\&\leq\varepsilon\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^p+C(\varepsilon)\int_\Omega u^{s+\frac{r}{p-r}}\,dx\\&=\varepsilon\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^p+C(\varepsilon)\|u\|_{s+p-1}^{s+p-1}.\end{aligned}$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{s+1}\leq C(N,p,q,s)\|u\|_{s+q}^{1-\theta_2}\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^{\frac{\theta_2 p}{p+s-1}},$$

where $\theta_2=\frac{p+s-1}{p}(\frac{1}{s+q}-\frac{1}{s+1})(\frac{1}{N}-\frac{1}{p}+\frac{p+s-1}{p}\cdot\frac{1}{s+q})^{-1}=\frac{N(p+s-1)(1-q)}{(s+1)[p(q+s)+N(p-q-1)]}$. By the choice of s, we get $0<\theta_2<1$. By the Young inequality again, we obtain

$$\begin{aligned}\|u\|_{s+1}^{k_2}&\leq C(N,p,q,s)^{k_2}\|u\|_{s+q}^{k_2(1-\theta_2)}\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^{\frac{k_2\theta_2 p}{p+s-1}}\\&\leq C(N,p,q,s)^{k_2}\left(\eta_2\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^p+C(\eta_2)\|u\|_{s+q}^{\frac{k_2(1-\theta_2)(p+s-1)}{p+s-1-k_2\theta_2}}\right),\end{aligned}$$

where $\eta_2>0$, $k_2>0$ will be determined later. Here we set $k_2=\frac{(p+s-1)(s+q)}{(p+s-1)(1-\theta_2)+\theta_2(s+q)}=\frac{(s+1)[p(q+s)+N(p-q-1)]}{p(s+1)+N(p-q-1)}$, then we have $s<k_2<s+1$ and $\frac{k_2(1-\theta_2)(p+s-1)}{p+s-1-k_2\theta_2}=s+q$, and

$$\frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2}\leq\frac{\eta_2\beta}{C(\eta_2)}\left\|\nabla u^{\frac{p+s-1}{p}}\right\|_p^p+\beta\|u\|_{s+q}^{s+q}, \tag{4.6}$$

and we now substitute (4.6) into (4.5) to obtain

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1}+\left[\frac{sp^p}{(p+s-1)^p}-\lambda\varepsilon-\frac{\lambda C(\varepsilon)}{\lambda_1}-\frac{\eta_2\beta}{C(\eta_2)}\right]\|\nabla u^{\frac{p+s-1}{p}}\|_p^p$$
$$+\frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2}\le 0.$$

Here we can choose λ small enough such that $\frac{sp^p}{(p+s-1)^p}-\lambda\varepsilon-\frac{\lambda C(\varepsilon)}{\lambda_1}-\frac{\eta_2\beta}{C(\eta_2)}\ge 0$. Setting $C_8=\frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)}>0$, thus we have

$$\frac{d}{dt}\|u\|_{s+1}+C_8\|u\|_{s+1}^{k_2-s}\le 0.$$

By Lemma 1, we can obtain the desired decay estimate.

(2) If $N=1$, the proof is similar to the proof of (1)(a) except for using the Gagliardo-Nirenberg inequality in the lower dimensional space, and we omit it here.

4.2 Proof of Theorem 5

(1) If $N\ge 2$, we have the following.

(a) If $\frac{2N}{N+2}<p<2$, multiplying (1.1) by u and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_p^p+\beta\|u\|_{1+q}^{1+q}=\lambda\int_\Omega u|\nabla u|^r\,dx.$$

Substituting (4.4) into the above equality gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\left[1-\lambda\varepsilon-\frac{\eta_1\beta}{C(\eta_1)}\right]\|\nabla u\|_p^p+\frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1}\le\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{1+\frac{r}{p-r}}.$$

Here we can choose ε (or η_1) small enough such that $1-\lambda\varepsilon-\frac{\eta_1\beta}{C(\eta_1)}\ge 0$, thus we get

$$\frac{d}{dt}\|u\|_2+\|u\|_2^{k_1-1}\left[\frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)}-\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u\|_2^{\frac{r}{p-r}-k_1+1}\right]\le 0.$$

Therefore,

$$\frac{d}{dt}\|u\|_2+C_9\|u\|_2^{k_1-1}\le 0,$$

provided that

$$\|u_0\|_2<\left[\frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}}\right]^{\frac{1}{\frac{r}{p-r}-k_1+1}},$$

and

$$\frac{r}{p-r}>k_1-1,$$

i.e.

$$r > \frac{2p^2q + Np(p-q+1)}{2p(q+1) + 2N(p-q+1)},$$

where

$$C_9 = \frac{C(N,p,q)^{-k_1}\beta}{C(\eta_1)} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{2(p-r)}}\|u_0\|_2^{\frac{r}{p-r}-k_1+1} > 0.$$

By Lemma 1, we can obtain the desired decay estimate. Since $p > \frac{2N}{N+2}$, we have $2p > N(2-p)$. Therefore, if $q \geq p-1$, then $r > p-1$.

(b) If $1 < p \leq \frac{2N}{N+2}$, multiplying (1.1) by u^s (where $s > \frac{2N-(N+1)p}{p} \geq 1$) and integrating over Ω yield

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \frac{sp^p}{(p+s-1)^p}\|\nabla u^{\frac{p+s-1}{p}}\|_p^p + \beta\|u\|_{s+q}^{s+q} = \lambda\int_\Omega u^s|\nabla u|^r\,dx.$$

Substituting (4.6) into the above equality gives

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \left[\frac{sp^p}{(p+s-1)^p} - \lambda\varepsilon - \frac{\eta_2\beta}{C(\eta_2)}\right]\|\nabla u^{\frac{p+s-1}{p}}\|_p^p + \frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2}$$
$$\leq \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(s+1)(p-r)}}\|u\|_{s+1}^{s+\frac{r}{p-r}}.$$

Here we can choose ε (or η_1) small enough such that $\frac{sp^p}{(p+s-1)^p} - \lambda\varepsilon - \frac{\eta_2\beta}{C(\eta_2)} \geq 0$, thus we get

$$\frac{d}{dt}\|u\|_{s+1} + \|u\|_{s+1}^{k_2-s}\left[\frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)} - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(s+1)(p-r)}}\|u_0\|_{s+1}^{\frac{r}{p-r}-k_2+s}\right] \leq 0.$$

Therefore,

$$\frac{d}{dt}\|u\|_{s+1} + C_{10}\|u\|_{s+1}^{k_2-s} \leq 0,$$

provided that

$$\|u_0\|_{s+1} < \left[\frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2)\lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(s+1)(p-r)}}}\right]^{\frac{1}{\frac{r}{p-r}-k_2+s}},$$

and

$$\frac{r}{p-r} > k_2 - s,$$

i.e.

$$r > \frac{p^2q(s+1) + Np(p-q-1)}{p(s+1)(1+q) + 2N(p-q-1)},$$

where

$$C_{10} = \frac{C(N,p,q,s)^{-k_2}\beta}{C(\eta_2) - \lambda C(\varepsilon)|\Omega|^{\frac{p-2r}{(s+1)(p-r)}}}\|u_0\|_{s+1}^{\frac{r}{p-r}-k_2+s} > 0.$$

By Lemma 1, we can obtain the desired decay estimate. Since $s > \frac{2N-(N+1)p}{p}$, it follows that $p(s+1) > N(2-p)$. Therefore, if $q \geq p-1$, then $r > p-1$.

(2) If $N = 1$, the proof is similar to the proof of (1)(a) except for using the Gagliardo-Nirenberg inequality in the lower dimensional space, and we omit it here.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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