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Superlinear gradient system with a parameter

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Abstract

In this paper, we study the multiplicity of nontrivial solutions for a superlinear gradient system with saddle structure at the origin. We make use of a combination of bifurcation theory, topological linking and Morse theory.

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Keywords: gradient system; superlinear; critical group; Morse theory; linking

1 Introduction

In this paper, we study the existence of multiple solutions to the gradient system

$$
\begin{cases}
-\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_u(x,u,v) \\ F_v(x,u,v) \end{pmatrix}, & x \in \Omega, \\
u = v = 0, & x \in \partial\Omega,
\end{cases}
\tag{$GS)_\lambda$}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with a smooth boundary $\partial\Omega$ and $N \geq 3$, λ is a real parameter and $A \in \mathcal{M}_2(\Omega)$ is fixed. $\mathcal{M}_2(\Omega)$ is the set of all continuous, cooperative and symmetric matrix functions on $\mathbb{R}^2$. A matrix function $A \in \mathcal{M}_2(\Omega)$ takes the form

$$
A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}
$$

with the functions $a, b, c \in C(\bar{\Omega}, \mathbb{R})$ satisfying the conditions that $b(x) \geq 0$ for all $x \in \bar{\Omega}$, which means A is cooperative and that $\max_{x \in \bar{\Omega}}\{a, c\} > 0$.

We impose the following assumptions on the function F:

(F_1) $F \in C^2(\Omega \times \mathbb{R}^2, \mathbb{R})$.

(F_2) $F(x, 0) = 0$, $\nabla F(x, 0) = 0$, $F''_z(x, 0) = 0$ for $x \in \Omega$.

(F_3) There is $C > 0$ and $2 < p < \frac{2N}{N-2} := 2^*$ such that

$$
\left| \nabla F(x, z) \right| \leq C\left(1 + |z|^{p-1}\right), \quad \text{for } x \in \Omega, z = (u, v) \in \mathbb{R}^2.
$$

(F_4) There is $\mu > 2$, $M > 0$ such that

$$
0 < \mu F(x, z) \leq \left(\nabla F(x, z), z\right)
$$

for all $x \in \Omega$, $|z| \geq M$.

(F_5) $F''_z(x, z) > 0$ for $|z| > 0$ small and $x \in \Omega$.

(F_6) $F_z''(x, z) < 0$ for $|z| > 0$ small and $x \in \Omega$.

Here and in the sequel, 0 is used to denote the origin in various spaces, $|\cdot|$ and $(\cdot, \cdot)$ denote the norm and the inner product in $\mathbb{R}^2$, Bz denotes the matrix product in $\mathbb{R}^2$ for a 2×2 matrix B and $z = (u, v) \in \mathbb{R}^2$. For two symmetric matrices B and C in $\mathbb{R}^2$, $B > C$ means that $B - C$ is positive definite.

Let E be the Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the inner product

$$\langle z, w \rangle = \big\langle (u, v), (\phi, \psi) \big\rangle = \int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi) \, dx, \quad z = (u, v), w = (\phi, \psi) \in E$$

and the associated norm

$$\|z\|^2 = \int_\Omega |\nabla z|^2 \, dx = \int_\Omega |\nabla u|^2 + |\nabla v|^2 \, dx, \quad z = (u, v) \in E.$$

By the compact Sobolev embedding $E \hookrightarrow L^p(\Omega) \times L^p(\Omega)$ for $p \in [2, 2^*)$, under the assumptions (F_1) and (F_3), the functional

$$\Phi(z) = \frac{1}{2} \int_\Omega |\nabla z|^2 - \lambda \big(A(x) z, z \big) \, dx - \int_\Omega F(x, z) \, dx, \quad z = (u, v) \in E \tag{1.1}$$

is well defined and is of class C^2 (see [1]) with derivatives

$$\big\langle \Phi'(z), w \big\rangle = \int_\Omega \nabla z \nabla w - \lambda \big(A(x) z, w \big) \, dx - \int_\Omega \big(\nabla F(x, z), w \big) \, dx,$$

$$\big\langle \Phi''(z) z_1, z_2 \big\rangle = \int_\Omega \nabla z_1 \nabla z_2 - \lambda \big(A(x) z_1, z_2 \big) \, dx - \int_\Omega \big(F_z''(x, z) z_1, z_2 \big) \, dx,$$

for $z = (u, v)$, $w = (\phi, \psi)$, $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in E$. Therefore, the solutions to $(GS)_\lambda$ are exactly critical points of Φ in E.

By (F_2) the system $(GS)_\lambda$ admits a trivial solution $z = 0$ for any fixed parameter $\lambda \in \mathbb{R}$. We are interested in finding nontrivial solutions to $(GS)_\lambda$. The existence of nontrivial solutions of $(GS)_\lambda$ depends on the behaviors of F near zero and infinity. The purpose of this paper is to find multiple nontrivial solutions to $(GS)_\lambda$ with superlinear term when the trivial solution $z = 0$ acts as a local saddle point of the energy functional Φ in the sense that the parameter λ is close to *a higher eigenvalue* of the linear gradient system with the given weight matrix A

$$\begin{cases} -\Delta \binom{u}{v} = \lambda A(x) \binom{u}{v}, & x \in \Omega, \\ u = v = 0, & x \in \partial \Omega. \end{cases} \tag{L_A}$$

It is known (see [2, 3]) that for a given matrix $A \in \mathcal{M}_2(\Omega)$, (L_A) admits a sequence of distinct eigenvalues of finite multiplicity

$$0 < \lambda_1^A < \lambda_2^A < \cdots < \lambda_k^A < \cdots$$

such that $\lambda_k^A \to \infty$ as $k \to \infty$.

Denote by F^- the negative part of F, *i.e.*, $F^-(x,z) = \max\{-F(x,z), 0\}$.

We will prove the following theorems.

Theorem 1.1 *Assume* (F_1)-(F_4), (F_5) *and let* $k \geq 1$ *be fixed. Then there is* $\delta > 0$ *such that when* $\sup_{(x,z) \in \Omega \times \mathbb{R}^2} F^-(x,z) \leq \delta$, *for all* $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A)$, $(GS)_\lambda$ *has at least three nontrivial solutions in* E.

Theorem 1.2 *Assume* (F_1)-(F_4), (F_6) *and let* $k \geq 1$ *be fixed. Then there is* $\delta > 0$ *such that when* $\sup_{(x,z) \in \Omega \times \mathbb{R}^2} F^-(x,z) \leq \delta$, *for all* $\lambda \in (\lambda_{k+1}^A, \lambda_{k+1}^A + \delta)$, $(GS)_\lambda$ *has at least three nontrivial solutions in* E.

Theorem 1.3 *Assume* (F_1)-(F_4) *and* $F \leq 0$ *for* $x \in \Omega$, $z \in \mathbb{R}^2$ *with* $|z|$ *small. Then there is* $\delta > 0$ *such that when* $\sup_{(x,z) \in \Omega \times \mathbb{R}^2} F^-(x,z) \leq \delta$, *for all* $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A]$, $(GS)_\lambda$ *has at least two nontrivial solutions in* E.

We give some comments and comparisons. The superlinear problems have been studied extensively *via* variational methods since the pioneering work of Ambrosetti and Rabinowitz [4]. Most known results on elliptic superlinear problems are contributed to a single equation with Dirichlet boundary data. Let us mention some historical progress on a single equation. When the trivial solution 0 acted as a local minimizer of the energy functional, one positive solution and one negative solution were obtained by using the mountain-pass theorem in [4] and the cut-off techniques; and a third solution was constructed in a famous paper of Wang [5] by using a two dimensional linking method and a Morse theoretic approach. When the trivial solution 0 acted as a local saddle point of the energy functional, the existence of one nontrivial solution was obtained by applying a critical point theorem, which is now well known as the generalized mountain-pass theorem, built by Rabinowitz in [6] under a global sign condition (see [1]). Some extensions were done in [7, 8] *via* local linking. More recently, in the work of Rabinowitz, Su and Wang [9], multiple solutions have been obtained by combining bifurcation methods, Morse theory and homological linking when 0 is a saddle point in the sense that the parameter λ is very close to a higher eigenvalue of the related linear operator.

In the current paper, we build multiplicity results for superlinear gradient systems by applying the ideas constructed in [9]. These results are new since, to the best of our knowledge, no multiplicity results for gradient systems have appeared in the literature for the case that $z = 0$ is a saddle point of Φ.

We give some explanations regarding the conditions and conclusions. The assumptions (F_1)-(F_4) are standard in the study of superlinear problems. (F_5) and (F_6) are used for bifurcation analysis. It sees that (F_5) implies that F is positive near zero, while (F_6) implies that F must be negative near zero. The local properties of F near zero are necessary for constructing homological linking. When $F \geq 0$, for any parameter λ in a bounded interval, say in $[\lambda_k^A, \lambda_{k+1}^A)$, one can use the same arguments as in [1] to construct linking starting from λ_{k+1}^A. In our theorems, we do not require the global sign condition $F \geq 0$. When the parameter λ is close to the eigenvalue λ_{k+1}^A, the homological linking will be constructed starting from λ_{k+2}^A and this linking is different from the one in [1]. This reveals the fact that when λ is close to λ_{k+1}^A from the right-hand side, the linking starting from λ_{k+2}^A can still be constructed even if F is negative somewhere. The conditions similar to (F_5) and (F_6) were first introduced in [10] where multiple periodic solutions for the second-order

Hamiltonian systems were studied *via* the ideas in [9]. Since we treat a different problem in the current paper, we need to present the detailed discussions although some arguments may be similar to those in [9, 10].

The paper is organized as follows. In Section 2, we collect some basic abstract tools. In Section 3, we get solutions by linking arguments and give partial estimates of homological information. In Section 4, we get solutions by bifurcation theorem and give the estimates of the Morse index. The final proofs of Theorems 1.1-1.3 are given in Section 5.

2 Preliminary

In this section, we give some preliminaries that will be used to prove the main results of the paper. We first collect some basic results on the Morse theory for a C^2 functional defined on a Hilbert space.

Let E be a Hilbert space and $\Phi \in C^2(E, \mathbb{R})$. Denote $\mathcal{K} = \{z \in E | \Phi'(z) = 0\}$, $\Phi^c = \{z \in E | \Phi(z) \leq c\}$, $\mathcal{K}_c = \{z \in \mathcal{K} | \Phi(z) = c\}$ for $c \in \mathbb{R}$. We say that Φ satisfies the $(PS)_c$ condition at the level $c \in \mathbb{R}$ if any sequence $\{z_n\} \subset E$ satisfying $\Phi(z_n) \to c$, $\Phi'(z_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Φ satisfies (PS) if Φ satisfies $(PS)_c$ at any $c \in \mathbb{R}$.

We assume that Φ satisfies (PS) and $\#\mathcal{K} < \infty$. Let $z_0 \in \mathcal{K}$ with $\Phi(z_0) = c \in \mathbb{R}$ and U be a neighborhood of z_0 such that $U \cap \mathcal{K} = \{z_0\}$. The group

$$C_q(\Phi, z_0) := H_q\big(\Phi^c \cap U, \Phi^c \cap U \setminus \{z_0\}\big), \quad q \in \mathbb{Z}$$

is called the qth critical group of Φ at z_0, where $H_*(A, B)$ denotes a singular relative homology group of the pair (A, B) with coefficients field $\mathbb{F}$ (see [11, 12]).

Let $a < \inf \Phi(\mathcal{K})$. The group

$$C_q(\Phi, \infty) := H_q\big(E, \Phi^a\big), \quad q \in \mathbb{Z}$$

is called the qth critical group of Φ at infinity (see [13]).

We call $M_q := \sum_{z \in \mathcal{K}} \dim C_q(\Phi, z)$ the qth Morse-type numbers of the pair (E, Φ^a) and $\beta_q := \dim C_q(\Phi, \infty)$ the Betti numbers of the pair (E, Φ^a). The core of the Morse theory [11, 12] is the following relations between M_q and β_q:

(Morse inequality) $\quad \sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \quad$ for $q \in \mathbb{Z}$;

(Morse equality) $\quad \sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q.$

If $\mathcal{K} = \emptyset$, then $\beta_q = 0$ for all q. Since $M_q \geq \beta_q$ for each $q \in \mathbb{Z}$, it follows that if $\beta_{q_*} \neq 0$ for some $q_* \in \mathbb{Z}$, then Φ must have a critical point z_* with $C_{q_*}(\Phi, z_*) \not\cong 0$. If $\mathcal{K} = \{z_*\}$, then $C_q(\Phi, \infty) \cong C_q(\Phi, z_*)$ for all q. Thus, if $C_q(\Phi, \infty) \not\cong C_q(\Phi, z_*)$ for some q, then Φ must have a new critical point. One can use critical groups to distinguish critical points obtained by other methods and use the Morse equality to find new critical points.

For the critical groups of Φ at an isolated critical point, we have the following basic facts (see [11, 12]).

Proposition 2.1 *Assume that z is an isolated critical point of $\Phi \in C^2(E, \mathbb{R})$ with a finite Morse index $m(z)$ and nullity $n(z)$. Then*

(1) *$C_q(\Phi, z) \cong \delta_{q,m(z)} \mathbb{F}$ if $n(z) = 0$;*

(2) *$C_q(\Phi, z) \cong 0$ for $q \notin [m(z), m(z) + n(z)]$ (Gromoll-Meyer [14]);*

(3) *if $C_{m(z)}(\Phi, z) \not\cong 0$ then $C_q(\Phi, z) \cong \delta_{q,m(z)} \mathbb{F}$;*

(4) *if $C_{m(z)+n(z)}(\Phi, z) \not\cong 0$ then $C_q(\Phi, z) \cong \delta_{q,m(z)+n(z)} \mathbb{F}$.*

Proposition 2.2 ([15, 16]) *Let 0 be an isolated critical point of $\Phi \in C^2(E, \mathbb{R})$ with a finite Morse index m_0 and nullity n_0. Assume that Φ has a local linking at 0 with respect to a direct sum decomposition $E = E^- \oplus E^+$, $\kappa = \dim E^-$, i.e., there exists $r > 0$ small such that*

$$\Phi(z) > 0 \quad \text{for } z \in E^+, 0 < \|z\| \leq r, \qquad \Phi(z) \leq 0 \quad \text{for } z \in E^-, \|z\| \leq r.$$

Then $C_q(\Phi, 0) \cong \delta_{q,\kappa} \mathbb{Z}$ for either $\kappa = m_0$ or $\kappa = m_0 + n_0$.

The concept of local linking was introduced in [7]. In [15] a partial result was given for a C^1 functional. The above result was obtained in [16].

Now, we recall an abstract linking theorem which is from [1, 12, 15].

Proposition 2.3 ([1, 12, 15]) *Let E be a real Banach space with $E = X \oplus Y$ and $\ell = \dim X$ be finite. Suppose that $\Phi \in C^1(E, \mathbb{R})$ satisfies (PS) and*

(Φ_1) *there exist $\rho > 0$ and $\alpha > 0$ such that*

$$\Phi(z) \geq \alpha, \quad z \in S_\rho = Y \cap \partial B_\rho, \tag{2.1}$$

where $B_\rho = \{z \in E | \|z\| \leq \rho\}$,

(Φ_2) *there exist $R > \rho > 0$ and $e \in Y$ with $\|e\| = 1$ such that*

$$\Phi(z) < \alpha, \quad z \in \partial Q, \tag{2.2}$$

where

$$Q = \{z = y + se | \|z\| \leq R, y \in X, 0 \leq s \leq R\}.$$

Then Φ has a critical point z_* with $\Phi(z_*) = c_* \geq \alpha$ and

$$C_{\ell+1}(\Phi, z_*) \neq 0. \tag{2.3}$$

We note here that under the framework of Proposition 2.3, S_ρ and ∂Q *homotopically link* with respect to the direct sum decomposition $E = X \oplus Y$. S_ρ and ∂Q are also *homologically linked*. The conclusion (2.3) follows from Theorems 1.1′ and 1.5 of Chapter II in [12]. (See also [15].)

We finally collect some properties of the eigenvalue problem (L_A). Associated with a matrix $A \in \mathcal{M}_2(\Omega)$, there is a compact self-adjoint operator $T_A : E \to E$ such that

$$\langle T_A z, w \rangle = \int_\Omega (A(x)z, w) \, dx, \quad z, w \in E.$$

The compactness of T_A follows from the compact embedding $E \hookrightarrow L^2(\Omega) \times L^2(\Omega)$. The operator T_A possesses the property that λ^A is an eigenvalue of (L_A) if and only if there is nonzero $z \in E$ such that

$$\lambda^A T_A z = z.$$

(L_A) has the sequence of distinct eigenvalues

$$0 < \lambda_1^A < \lambda_2^A < \cdots < \lambda_k^A < \cdots \to \infty,$$

and each eigenvalue λ^A of (L_A) has a finite multiplicity. For $j \in \mathbb{N}$, denote

$$E\left(\lambda_j^A\right) = \ker\left(-\Delta - \lambda_j^A A\right), \qquad E_j = \bigoplus_{i=1}^{j} E\left(\lambda_i^A\right), \qquad \ell_j = \dim E_j.$$

Set

$$Q_\lambda(z) = \int_\Omega |\nabla z|^2 - \lambda\left(A(x)z, z\right) dx, \quad z \in E.$$

Then the following variational inequalities hold:

$$Q_\lambda(z) \le \frac{\lambda_j^A - \lambda}{\lambda_j^A} \|z\|^2, \quad z \in E_j,$$

$$Q_\lambda(z) \ge \frac{\lambda_{j+1}^A - \lambda}{\lambda_{j+1}^A} \|z\|^2, \quad z \in E_j^\perp.$$

We refer to [2, 3] for more properties related to the eigenvalue problem (L_A) and the operator T_A.

3 Solutions *via* homological linking

In this section, we give the existence a nontrivial solution of $(GS)_\lambda$ by applying homological linking arguments and then give some estimate of its Morse index. The following lemmas are needed.

Lemma 3.1 *Assume that F satisfies (F_1)-(F_4), then for any fixed $\lambda \in \mathbb{R}$, the functional Φ satisfies the (PS) condition.*

Proof By (F_3) and the compact embedding $E \hookrightarrow L^p(\Omega) \times L^p(\Omega)$ for $1 \le p < 2^*$, it is enough to show that any sequence $\{z_n\} \subset E$ with

$$\left|\Phi(z_n)\right| \le C, \quad n \in \mathbb{N}, \qquad \Phi'(z_n) \to 0, \quad n \to \infty \tag{3.1}$$

is bounded in E. Here and below, we use C to denote various positive constants. We modify the arguments in [1]. Choosing a positive number $\beta \in (1/\mu, 1/2)$ for n large, we have that

$$C + \beta \|z_n\| \ge \Phi(z_n) - \beta\left\langle \Phi'(z_n), z_n\right\rangle.$$

By (F_4) we deduce that

$$F(x,z) \geq C|z|^{\mu} - C, \quad \text{for } z \in \mathbb{R}^2, x \in \Omega. \tag{3.2}$$

Therefore,

$$C + \beta \|z_n\|$$

$$\geq \Phi(z_n) - \beta \langle \Phi'(z_n), z_n \rangle$$

$$= \left(\frac{1}{2} - \beta\right) \int_{\Omega} |\nabla z_n|^2 - \lambda \left(A(x)z_n, z_n\right) dx - \int_{\Omega} \left(F(x,z_n) - \beta \left(\nabla F(x,z_n), z_n\right)\right) dx$$

$$\geq \left(\frac{1}{2} - \beta\right) \|z_n\|^2 - \left(\frac{1}{2} - \beta\right) |\lambda| \Lambda \|z_n\|_2^2 + C(\mu\beta - 1) \|z_n\|_{\mu}^{\mu} - C,$$

where $\Lambda = \max_{x \in \bar{\Omega}} \|A(x)\|_{\mathbb{R}^2}$. By the Hölder inequality and the Young inequality, we get for any $\epsilon > 0$ that

$$\|z_n\|_2^2 \leq |\Omega|^{\frac{\mu-2}{\mu}} \|z_n\|_{\mu}^2 \leq \frac{|\Omega|(\mu-2)}{\mu} \epsilon^{\frac{2}{2-\mu}} + \frac{2}{\mu} \epsilon \|z_n\|_{\mu}^{\mu}. \tag{3.3}$$

Thus, for a fixed $\epsilon > 0$ small enough, we have by (3.3) that

$$C + \beta \|z_n\| \geq \left(\frac{1}{2} - \beta\right) \|z_n\|^2 + \frac{1}{2} C(\mu\beta - 1) \|z_n\|_{\mu}^{\mu}.$$

Therefore, $\{z_n\}$ is bounded in E. The proof is complete. $\qquad\square$

Now, we construct a homological linking with respect to the direct sum decomposition of E for $k \geq 1$:

$$E = E_{k+1} \oplus E_{k+1}^{\perp}, \qquad \dim E_{k+1} = \ell_{k+1}.$$

Take an eigenvector ϕ_{k+2} corresponding to the eigenvalue λ_{k+2}^A of (L_A) with $\|\phi_{k+2}\| = 1$. Set

$$V_{k+1} := E_{k+1} \oplus \text{span}\{\phi_{k+2}\}.$$

Lemma 3.2 *Assume that F satisfies (F_1)-(F_4) and $k \geq 1$. Then there exist constants $\rho > 0$ small, $\alpha > 0$ such that for all $\lambda \leq \lambda_* := \frac{\lambda_{k+2}^A + \lambda_{k+1}^A}{2}$, such that*

$$\Phi(z) \geq \alpha, \quad \text{for } z \in E_{k+1}^{\perp} \text{ with } \|z\| = \rho. \tag{3.4}$$

Proof By the conditions (F_2) and (F_3), for $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$F(x,z) \leq \frac{1}{2S_2} \epsilon |z|^2 + C_\epsilon |z|^p,$$

where S_2 is the constant for the embedding $E \hookrightarrow L^2(\Omega) \times L^2(\Omega)$ such that $|z|_2^2 \leq S_2 \|z\|^2$ for $z \in E$. Since for $z \in E_{k+1}^{\perp}$,

$$Q_\lambda(z) = \int_{\Omega} |\nabla z|^2 - \lambda \left(A(x)z, z\right) dx \geq \frac{\lambda_{k+2}^A - \lambda}{\lambda_{k+2}^A} \|z\|^2,$$

it follows that

$$
\begin{aligned}
\Phi(z) &= \frac{1}{2}Q_\lambda(z) - \int_\Omega F(x,z)\,dx \\
&\geq \frac{1}{2}\frac{\lambda_{k+2}^A - \lambda}{\lambda_{k+2}^A}\|z\|^2 - \frac{1}{2S_2}\epsilon \int_\Omega |z|^2\,dx - C_\epsilon \int_\Omega |z|^p\,dx \\
&\geq \frac{1}{2}\frac{\lambda_{k+2}^A - \lambda - \epsilon}{\lambda_{k+2}^A}\|z\|^2 - \widetilde{C}_\epsilon\|z\|^p \\
&= \frac{1}{2}\eta(\lambda,\epsilon)\|z\|^2 - \widetilde{C}_\epsilon\|z\|^p,
\end{aligned}
$$

where $\widetilde{C}_\epsilon$ is independent of λ and

$$
\eta(\lambda,\epsilon) = \frac{\lambda_{k+2}^A - \lambda - \epsilon}{\lambda_{k+2}^A}. \tag{3.5}
$$

Since $p > 2$ and the function $g(r) = \frac{1}{2}\eta(\lambda,\epsilon)r^2 - \widetilde{C}_\epsilon r^p$ achieves its maximum

$$
g_{\max} = \frac{p-2}{2p}(p\widetilde{C}_\epsilon)^{\frac{2}{2-p}}\eta(\lambda,\epsilon)^{\frac{p}{p-2}} := \alpha(\lambda,\epsilon) \tag{3.6}
$$

on $(0,\infty)$ at

$$
\rho_{\lambda,\epsilon} = \left(\frac{\eta(\lambda,\epsilon)}{p\widetilde{C}_\epsilon}\right)^{\frac{1}{p-2}}, \tag{3.7}
$$

we see that

$$
\Phi(z) \geq \alpha(\lambda,\epsilon), \quad \text{for } z \in E_{k+1}^\perp \text{ with } \|z\| = \rho_{\lambda,\epsilon}. \tag{3.8}
$$

Since $\eta(\lambda,\epsilon)$ is a decreasing function with respect to λ for any fixed $\epsilon > 0$ small, (3.4) holds for

$$
\alpha := \alpha(\lambda_*,\epsilon_*), \qquad \rho := \rho_{\lambda_*,\epsilon_*}, \quad \text{here } \epsilon_* = \frac{1}{4}(\lambda_{k+2}^A - \lambda_{k+1}^A).
$$

The constants α and ρ are independent of $\lambda \leq \lambda_*$. The proof is complete. $\qquad\square$

Lemma 3.3 *Assume that F satisfies (F_1), (F_4) and $k \geq 1$. Then there exist $R > 0$, $\delta > 0$ and $\sigma \in \mathbb{R}$, all independent of λ, such that when $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A + \delta)$ and $\sup_{(x,z)\in\Omega\times\mathbb{R}^2} F^-(x, z) \leq \delta$,*

$$
\Phi(z) \leq \sigma < \alpha, \quad \text{for } z \in \partial Q, \tag{3.9}
$$

where

$$
Q = \{z \in V_{k+1} \,|\, \|z\| \leq R, z = y + s\phi_{k+2}, y \in E_{k+1}, s \geq 0\}.
$$

Proof From (F_4) we deduce (3.2) with a positive constant C independent of λ. For $z \in V_{k+1}$, write $z = y + w + \phi$, $y \in E_k$, $w \in E(\lambda_{k+1}^A)$, $\phi \in \text{span}\{\phi_{k+2}\}$. Assume that $\lambda \in (\lambda_k^A, \lambda_{k+2}^A)$, then

$$
\begin{aligned}
\Phi(z) &= \frac{1}{2}Q_\lambda(y) + \frac{1}{2}Q_\lambda(w) + \frac{1}{2}Q_\lambda(\phi) - \int_\Omega F(x,z)\,dx \\
&\leq \frac{\lambda_k^A - \lambda}{2\lambda_k^A}\|y\|^2 + \frac{\lambda_{k+1}^A - \lambda}{2\lambda_{k+1}^A}\|w\|^2 + \frac{\lambda_{k+2}^A - \lambda}{2\lambda_{k+2}^A}\|\phi\|^2 - C\|z\|_\mu^\mu + C \\
&\leq \frac{\lambda_{k+2}^A - \lambda_k^A}{2\lambda_{k+1}^A}\|z\|^2 - C\|z\|_\mu^\mu + C.
\end{aligned}
\tag{3.10}
$$

Since $\mu > 2$ and $\dim V_{k+1} < \infty$, (3.10) shows that there exists $R > 0$ independent of λ such that

$$
\Phi(z) \leq 0, \quad \text{for } z \in V_{k+1} \text{ with } \|z\| = R. \tag{3.11}
$$

Now, fix such an $R > 0$ that $R > \rho$ with ρ given in Lemma 3.2. For $y \in E_{k+1}$ with $\|y\| \leq R$, we write $y = w + \phi$, $w \in E_k$, $\phi \in E(\lambda_{k+1}^A)$. Set $\Gamma := \sup_{(x,z)\in\Omega\times\mathbb{R}^2} F^-(x,z)$. Then we have that

$$
\begin{aligned}
\Phi(y) &= \frac{1}{2}Q_\lambda(w) + \frac{1}{2}Q_\lambda(\phi) - \int_\Omega F(x,y)\,dx \\
&\leq \frac{\lambda_k^A - \lambda}{2\lambda_k^A}\|w\|^2 + \frac{\lambda_{k+1}^A - \lambda}{2\lambda_{k+1}^A}\|\phi\|^2 + \int_{\{x\in\Omega: F\leq 0\}} F^-(x,y)\,dx \\
&\leq \frac{|\lambda_{k+1}^A - \lambda|}{2\lambda_{k+1}^A}R^2 + |\Omega|\Gamma.
\end{aligned}
\tag{3.12}
$$

Since $\partial Q = \{z = y + s\phi_{k+1} | \|z\| = R, y \in E_k, s \geq 0\} \cup \{y \in E_k | \|y\| \leq R\}$, taking

$$
\delta = \left(\frac{R^2}{\lambda_{k+1}^A} + 2|\Omega|\right)^{-1}\alpha, \qquad \sigma = \frac{\alpha}{2},
$$

then, when $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A + \delta)$ and $\Gamma \leq \delta$,

$$
\Phi(z) \leq \sigma < \alpha, \quad \text{for } z \in \partial Q.
$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we apply Proposition 2.3 to get the following existence result with partial homological information.

Theorem 3.4 *Let F satisfy (F_1)-(F_4) and $k \geq 1$. Then there is $\delta > 0$ such that when $\Gamma \leq \delta$, for each $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A + \delta)$, $(GS)_\lambda$ has one nontrivial solution z^* with a critical group satisfying*

$$
C_{\ell_{k+1}+1}(\Phi, z^*) \not\cong 0. \tag{3.13}
$$

Proof By Lemma 3.1, Φ satisfies (PS). By Lemmas 3.2 and 3.3, for each fixed $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1} + \delta)$, Φ satisfies (Φ_1) and (Φ_2) in the sense that

$$
\begin{cases}
(\Phi_1): \Phi(z) \geq \alpha, & z \in S_{k+1} := S_\rho \cap E_{k+1}; \\
(\Phi_2): \Phi(y) \leq \sigma < \alpha, & y \in \partial Q.
\end{cases}
\tag{3.14}
$$

Since S_{k+1} and ∂Q homotopically link with respect to the decomposition $E = E_{k+1} \oplus E_{k+1}^\perp$, and dim $V_{k+1} = \ell_{k+1} + 1$, it follows from Proposition 2.3 that Φ has a critical point $z^* \in E$ with positive energy $\Phi(z^*) \geq \alpha > 0$ and its critical group satisfying (3.13). The proof is complete. □

We give some remarks. The existence of one nontrivial solution in Theorem 3.4 is valid when F is of class C^1. From Lemma 3.2, one sees that the energy of the obtained solution is bounded away from 0 as λ is close to λ_{k+1}^A. A rough local sign condition on F is needed. If $F \geq 0$, then for any fixed $\lambda \in [\lambda_k^A, \lambda_{k+1}^A)$, a linking with respect to $E_k \oplus E_k^\perp$ can be constructed. Proposition 2.3 is applied again to get a nontrivial solution z_* satisfying

$$
C_{\ell_k+1}(\Phi, z_*) \ncong 0.
\tag{3.15}
$$

Therefore, when a global sign condition $F \geq 0$ is present, as λ is close to λ_{k+1}^A from the left-hand side, two linkings can be constructed and two nontrivial solutions can be obtained. The question is how to distinguish z^* from z_*. Theorem 3.4 includes the case that for λ close to λ_{k+1}^A from the right-hand side, the linking with respect to $E_{k+1} \oplus E_{k+1}^\perp$ is constructed provided the negative values of F are small. This phenomenon was first observed in [9].

4 Solutions *via* bifurcation

In this section, we get two solutions for $(GS)_\lambda$ *via* bifurcation arguments [1]. We first cite the bifurcation theorem in [1].

Proposition 4.1 (Theorem 11.35 in [1]) *Let E be a Hilbert space and $\Psi \in C^2(E, \mathbb{R})$ with*

$$
\nabla \Psi(u) = Lu + H(u),
$$

where $L \in \mathcal{L}(E, E)$ is symmetric and $H(u) = o(\|u\|)$ as $\|u\| \to 0$. Consider the equation

$$
Lu + H(u) = \lambda u.
\tag{4.1}
$$

Let $\mu \in \sigma(L)$ be an isolated eigenvalue of finite multiplicity. Then either
  (i) *$(\mu, 0)$ is not an isolated solution of (4.1) in $\{\mu\} \times E$, or*
  (ii) *there is a one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (4.1) has at least two distinct nontrivial solutions, or*
  (iii) *there is a neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (4.1) has at least one nontrivial solution.*

We apply Proposition 4.1 to get two nontrivial solutions of $(GS)_\lambda$ for λ close to an eigenvalue of (L_A) and then give the estimates of the Morse index.

Theorem 4.2 *Assume that F satisfies (F_1)-(F_3). Let $k \geq 1$ be fixed. Then there exists $\delta > 0$
such that $(GS)_\lambda$ has at least two nontrivial solutions for*

(1) *every $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A)$ if (F_5) holds;*

(2) *every $\lambda \in (\lambda_{k+1}^A, \lambda_{k+1}^A + \delta)$ if (F_6) holds.*

Furthermore, the Morse index $m(z_\lambda)$ and the nullity $n(z_\lambda)$ of such a solution z_λ satisfy

$$\ell_k \leq m(z_\lambda) \leq m(z_\lambda) + n(z_\lambda) \leq \ell_{k+1}, \quad for \ 0 < \left|\lambda - \lambda_{k+1}^A\right| < \delta. \tag{4.2}$$

Proof Under the assumptions (F_1)-(F_2), for each eigenvalue λ_j^A of (L_A), $(\lambda_j^A, 0)$ is a bifurcation point of $(GS)_\lambda$ (see [1]).

Let $(\lambda, z) \in \mathbb{R} \times E$ be a *solution* of $(GS)_\lambda$ near $(\lambda_{k+1}^A, 0)$ which satisfies

$$\begin{cases} -\Delta z = \lambda A(x)z + \nabla F(x, z), & x \in \Omega, \\ z = (u, v) = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

By (F_1) and (F_2), we have

$$\nabla F(x, z) = \nabla F(x, 0) + F_z''(t, \eta z)x = F_z''(x, \eta z)z, \quad \text{for some } 0 < \eta < 1. \tag{4.4}$$

Let (F_5) hold. By the elliptic regularity theory (see [17]), $\|z\| > 0$ small implies $\|z\|_C > 0$ small. Then by (F_5), we have that

$$F_z''\big(z, \eta z(x)\big) > 0, \quad x \in \Omega. \tag{4.5}$$

Now, consider the linear eigenvalue gradient system:

$$\begin{cases} -\Delta y - F_z''(x, \eta z(x))y = \mu A(x)y, & x \in \Omega, \\ y = 0, & x \in \partial\Omega. \end{cases} \tag{4.6}$$

We denote the distinct eigenvalues of (4.6) by $\mu_1(z) < \mu_2(z) < \cdots < \mu_i(z) < \cdots$ as $z \neq 0$. By (F_2), if we take $z = 0$, then for each $i \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $\mu_i(0) = \lambda_j^A$. By (4.5), the standard variational characterization of the eigenvalues of (4.6) shows that $\mu_i(z)$ is *less than* the corresponding jth ordered eigenvalue λ_j^A of (L_A). Furthermore, $\mu_i(z) \to \lambda_j^A$ as $z \to 0$ in E. By (4.3) and (4.4), we see that z is a solution of (4.6) with eigenvalue λ. It must be that $\lambda < \lambda_{k+1}^A$ since λ is close to λ_{k+1}^A. Therefore, the case (ii) of Proposition 4.1 occurs under the given conditions. This proves the case (1). The existence for the case (2) is proved in the same way.

Now, we estimate the Morse indices for the solutions obtained above. Let z_λ be a bifurcation solution of $(GS)_\lambda$. Then

$$\|z_\lambda\| \to 0 \quad \text{as } \lambda \to \lambda_{k+1}^A.$$

Applying the elliptic regularity theory, we have that

$$\|z_\lambda\|_C \to 0, \quad \lambda \to \lambda_{k+1}^A. \tag{4.7}$$

For each $y \in E$, we have

$$\langle \Phi''(z_\lambda)y, y \rangle = Q_\lambda(y) - \int_\Omega \left(F_z''(x, z_\lambda)y, y \right) dx.$$

Therefore, for $y \in E_k$,

$$\langle \Phi''(z_\lambda)y, y \rangle \leq \frac{\lambda_k^A - \lambda}{\lambda_k^A} \|y\|^2 + \int_\Omega \left| \left(F_z''(x, z_\lambda)y, y \right) \right| dx,$$

and for $\phi \in E_{k+1}^\perp$,

$$\langle \Phi''(z_\lambda)\phi, \phi \rangle \geq \frac{\lambda_{k+2}^A - \lambda}{\lambda_{k+2}^A} \|\phi\|^2 - \int_\Omega \left| \left(F_z''(x, z_\lambda)\phi, \phi \right) \right| dx.$$

By (F_2) and (4.7), there exists $\delta > 0$ such that when $0 < |\lambda - \lambda_{k+1}^A| < \delta$,

$$\langle \Phi''(z_\lambda)y, y \rangle < 0, \quad \forall y \in E_k \setminus \{0\},$$

$$\langle \Phi''(z_\lambda)\phi, \phi \rangle > 0, \quad \forall \phi \in E_{k+1}^\perp \setminus \{0\}.$$

Therefore, the Morse index $m(z_\lambda)$ and the nullity $n(z_\lambda)$ of z_λ satisfy (4.2). The proof is complete. □

5 Proofs of main theorems

In this section, we give the proof of main theorems in this paper. We first compute the critical groups of Φ at both infinity and zero.

Lemma 5.1 (see [5]) *Let F satisfy (F_1)-(F_4), then for any fixed $\lambda \in \mathbb{R}$,*

$$C_q(\Phi, \infty) \cong 0, \quad \text{for all } q \in \mathbb{Z}. \tag{5.1}$$

Proof The idea of the proof comes from the famous paper [5]. We include a sketched proof in an abstract version. Given $\lambda \in \mathbb{R}$, denote $B_1 = \{z \in E : \|z\| \leq 1\}$, $S_1 = \partial B_1$. Modifying the arguments in [5], we get the following facts:

$$\Phi(sz) \to -\infty \quad \text{as } s \to +\infty, \forall z \in S_1. \tag{5.2}$$

$$\text{For } a \ll -1, \quad \Phi(sz) \leq a \quad \Longrightarrow \quad \frac{d}{ds}\Phi(sz) < 0. \tag{5.3}$$

The following arguments are from [10]. As $\Phi(0) = 0$, it follows from (5.2) and (5.3) that for each $z \in S_1$, there is a *unique* $\tau(z) > 0$ such that

$$\Phi\left(\tau(z)z \right) = a. \tag{5.4}$$

By (5.4) and the implicit function theorem, we have that $\tau \in C(S_1, \mathbb{R})$. Define

$$\pi(z) = \begin{cases} 1, & \text{if } \Phi(z) \leq a, \\ \|z\|^{-1}\tau(\|z\|^{-1}z), & \text{if } \Phi(z) > a, z \neq 0. \end{cases}$$

Then $\pi \in C(E \setminus \{0\}, \mathbb{R})$. Define a map $\varrho : [0,1] \times E \setminus \{0\} \to E \setminus \{0\}$ by

$$\varrho(t,z) = (1-t)z + t\pi(z)z. \tag{5.5}$$

Clearly, ϱ is continuous, and for all $z \in E \setminus \{0\}$ with $\Phi(z) > a$, by (5.4),

$$\Phi\big(\varrho(1,z)\big) = \Phi\big(\pi\big(\|z\|^{-1}z\big)\|z\|^{-1}z\big) = a.$$

Therefore,

$$\varrho(1,z) \in \Phi^a \quad \text{for all } z \in E \setminus \{0\}, \qquad \varrho(t,z) = z \quad \text{for all } t \in [0,1], z \in \Phi^a,$$

and so Φ^a is a strong deformation retract of $E \setminus \{0\}$. Hence,

$$C_q(\Phi, \infty) := H_q\big(E, \Phi^a\big) \cong H_q\big(E, E \setminus \{0\}\big) \cong H_q(B_1, S_1) \cong 0, \quad q \in \mathbb{Z}$$

since S_1 is contractible, which follows from the fact that $\dim E = \infty$. $\qquad\square$

Lemma 5.2 *Let F satisfy* (F_1)-(F_3).
  (1) *For* $\lambda \in (\lambda_k^A, \lambda_{k+1}^A)$, $C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$.
  (2) *For* $\lambda \in (\lambda_{k+1}^A, \lambda_{k+2}^A)$, $C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$.
  (3) *For* $\lambda = \lambda_{k+1}^A$, *if* $F(x, z) \leq 0$ *for* $|z|$ *small, then* $C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$.
  (4) *For* $\lambda = \lambda_{k+1}^A$, *if* $F(x, z) \geq 0$ *for* $|z|$ *small, then* $C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$.

Proof By (F_2), we have

$$\big\langle \Phi''(0)y, y \big\rangle = Q_\lambda(y) = \int_\Omega |\nabla y|^2 - \lambda\big(A(x)y, y\big)\, dx, \quad y \in E.$$

(1) When $\lambda \in (\lambda_k^A, \lambda_{k+1}^A)$, $z = 0$ is a nondegenerate critical point of Φ with the Morse index $m_0 = \ell_k$, thus $C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$.

(2) When $\lambda \in (\lambda_{k+1}^A, \lambda_{k+2}^A)$, $z = 0$ is a nondegenerate critical point of Φ with the Morse index $m_0 = \ell_{k+1}$, thus $C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$.

(3) When $\lambda = \lambda_{k+1}^A$, $z = 0$ is a degenerate critical point of Φ with the Morse index $m_0 = \ell_k$ and the nullity $n_0 = \dim E(\lambda_{k+1}^A)$, $m_0 + n_0 = \ell_{k+1}$.

Assume that $F(x, z) \leq 0$ for $|z| \leq \sigma$ with $\sigma > 0$ small. We will show that Φ has a local linking structure at $z = 0$ with respect to $E = E_k \oplus E_k^\perp$. If this has been done, then by Proposition 2.2, we have $C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$.

Now, Φ can be written as

$$\Phi(z) = \frac{1}{2}Q_{\lambda_{k+1}^A}(z) - \int_\Omega F(x, z)\, dx.$$

By (F_2) and (F_3), for $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$F(x, z) \leq \frac{1}{2}\epsilon|z|^2 + C_\epsilon|z|^p, \quad z \in \mathbb{R}^2, x \in \Omega.$$

Hence, for $z \in E_k$, we have that

$$\Phi(z) \leq \frac{\lambda_k^A - \lambda_{k+1}^A}{2\lambda_k^A}\|z\|^2 + \frac{\epsilon}{2}|z|_2^2 + C_\epsilon |z|_p^p.$$

Since E_k is finite dimensional, all norms on E_k are equivalent, hence for $r > 0$ small,

$$\Phi(z) \leq 0, \quad \text{for } z \in E_k, \|z\| \leq r.$$

By (F_3), we have that for some $C > 0$,

$$\left|F(x,z)\right| \leq C|z|^p, \quad |z| \geq \sigma, x \in \Omega. \tag{5.6}$$

For $z \in E_k^\perp$, we write $z = y + w$ where $y \in E(\lambda_{k+1}^A)$ and $w \in E_{k+1}^\perp$. Then

$$\Phi(z) \geq \frac{\lambda_{k+2}^A - \lambda_{k+1}^A}{2\lambda_{k+2}^A}\|w\|^2 - \int_\Omega F(x,z)\,dx. \tag{5.7}$$

For $x \in \Omega$ with $|z(x)| \geq \sigma$, we have $|w(x)| \geq \frac{2}{3}|z(x)|$. Hence, by (5.6) and the Poincaré inequality, we have for various constants $C > 0$,

$$\int_{\{x:|z(x)|\geq\sigma\}} F(x,z)\,dx$$

$$\leq C \int_{\{x:|z(x)|\geq\sigma\}} \left|z(x)\right|^p dx$$

$$\leq C \int_\Omega \left|w(x)\right|^p dx$$

$$\leq C\|w\|^p. \tag{5.8}$$

For $x \in \Omega$ with $|z(x)| \leq \sigma$, $F(x, z(x)) \leq 0$. Therefore,

$$\Phi(z) \geq \frac{\lambda_{k+2}^A - \lambda_{k+1}^A}{2\lambda_{k+2}^A}\|w\|^2 - \int_{\{x:|z(x)|\leq\sigma\}} F\left(x, z(x)\right) dx - C\|w\|^p$$

$$\geq \frac{\lambda_{k+2}^A - \lambda_{k+1}^A}{2\lambda_{k+2}^A}\|w\|^2 - C\|w\|^p. \tag{5.9}$$

Since $p > 2$, for $r > 0$ small,

$$\Phi(z) > 0, \quad \text{for } z = y + w \text{ with } w \neq 0, \|z\| \leq r. \tag{5.10}$$

For $z = y \in E(\lambda_{k+1}^A)$, it must hold that

$$\Phi(y) = -\int_\Omega F\left(x, y(x)\right) dx > 0 \quad \text{if } 0 < \|y\| \leq r. \tag{5.11}$$

Here we use a potential convention that $(GS)_\lambda$ has finitely many solutions and then 0 is isolated. Otherwise, one would have that as $r > 0$ small, $\|y\| \leq r$ implies $|y(x)| \leq \delta$ for all

$x \in \Omega$, $\nabla F(x,y) \equiv 0$ for all $x \in \Omega$. Thus, 0 would not be an isolated critical point of Φ and
$(GS)_\lambda$ would have infinitely many nontrivial solutions. By (5.10) and (5.11), we verify that

$$\Phi(z) > 0, \quad \text{for } z \in E_k^\perp, 0 < \|z\| \le r.$$

Applying Proposition 2.2, we obtain

$$C_q(\Phi, 0) = \delta_{q,\ell_k}\mathbb{F}.$$

(4) When $F(x,z) \ge 0$ for $|z|$ small, a similar argument shows that Φ has a local link-
ing structure at $z = 0$ with respect to $E = E_{k+1} \oplus E_{k+1}^\perp$. By Proposition 2.2, it follows that
$C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$. □

Finally, we prove the theorems.

Proof of Theorem 1.1 It follows from (F_5) that $F(x,z) \ge 0$ for $|z| > 0$ small. By Theorem 3.4
for the part $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A)$, $(GS)_\lambda$ has a nontrivial solution z^λ satisfying

$$C_{\ell_{k+1}+1}(\Phi, z^\lambda) \ncong 0. \tag{5.12}$$

By Theorem 4.2(1), $(GS)_\lambda$ has two nontrivial solutions z_λ^i ($i = 1, 2$) with their Morse indices
satisfying

$$\ell_k \le m(z_\lambda^i) \le m(z_\lambda^i) + n(z_\lambda^i) \le \ell_{k+1}, \quad i = 1, 2.$$

From Proposition 2.1(2), we have that

$$C_q(\Phi, z_\lambda^i) \cong 0, \quad q \notin [\ell_k, \ell_{k+1}], i = 1, 2. \tag{5.13}$$

From (5.12) and (5.13), we see that $z^\lambda \ne z_\lambda^i$ ($i = 1, 2$). The proof is complete. □

Proof of Theorem 1.2 With the same argument as above, it follows from Theorem 4.2(2)
and Theorem 3.4 for the part $\lambda \in (\lambda_{k+1}^A, \lambda_{k+1}^A + \delta)$. We omit the details. □

Proof of Theorem 1.3 By Theorem 3.4 for the part $\lambda \in (\lambda_{k+1}^A - \delta, \lambda_{k+1}^A]$, $(GS)_\lambda$ has a solution
z^λ with its energy $\Phi(z^\lambda) \ge \alpha > 0$ and

$$C_{\ell_{k+1}+1}(\Phi, z^\lambda) \ncong 0. \tag{5.14}$$

By Lemma 5.1 and Lemma 5.2(3), we have that

$$C_q(\Phi, \infty) \cong 0, \quad \text{for } q \in \mathbb{Z}, \tag{5.15}$$

$$C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}. \tag{5.16}$$

Assume that $(GS)_\lambda$ has only two solutions 0 and z^λ. Choose $a, b \in \mathbb{R}$ such that $a < 0 < b <$
$\Phi(z^\lambda)$. Then by the deformation and excision properties of singular homology (see [12]),

we have

$$
\begin{cases}
C_q(\Phi, \infty) \cong H_q(E, \Phi^a); \\
C_q(\Phi, 0) \cong H_q(\Phi^b, \Phi^a); \\
C_q(\Phi, z^\lambda) \cong H_q(E, \Phi^b).
\end{cases}
\tag{5.17}
$$

By (5.17), the long exact sequences for the topological triple (E, Φ^b, Φ^a) read as

$$
\cdots \to C_{q+1}(\Phi, \infty) \to C_{q+1}(\Phi, z^\lambda) \to C_q(\Phi, 0) \to C_q(\Phi, \infty) \to \cdots .
\tag{5.18}
$$

We deduce by (5.15) and (5.18) that

$$
C_{q+1}(\Phi, z^\lambda) \cong C_q(\Phi, 0), \quad \text{for } q \in \mathbb{Z}.
\tag{5.19}
$$

Take $q = \ell_{k+1}$ in (5.19), then

$$
C_{\ell_{k+1}+1}(\Phi, z^\lambda) \cong C_{\ell_{k+1}}(\Phi, 0) \cong 0
$$

which contradicts (5.14). The proof is complete. $\qquad\square$

We finally remark that Theorem 1.1 is valid for $\lambda \in (\lambda_1^A - \delta, \lambda_1^A)$, from which one sees that $z = 0$ is a local minimizer of Φ.

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