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A new iterative scheme for equilibrium problems, fixed point problems for nonexpansive mappings and zero points of maximal monotone operators

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Abstract

In this article, we introduce a new iterative scheme for finding a common element of the set of fixed points of strongly relatively nonexpansive mapping, the set of solutions for equilibrium problems and the set of zero points of maximal monotone operators in a uniformly smooth and uniformly convex Banach space. Consequently, we obtain new strong convergence theorems in the frame work of Banach spaces. Our theorems extend and improve the recent results of Wei et al., Takahashi and Zembayashi, and some recent results.

1 Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let C be a nonempty closed convex subset of E. Let E^* be the dual space of E and $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. We consider the problem for finding:

$$v \in E \text{ such that } 0 \in Av, \tag{1.1}$$

where A is an operator from E into E^*, such that $v \in E$ is called a *zero point* of A, i.e., $A^{-1}0 = \{v \in E : Av = 0\}$. Such a problem contains numerous problems in economics, optimization and physics. Many authors studied this problem see, for example [1-5] and references therein.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. *The equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of is denote by $EP(F)$. The above formulation (1.2) was shown in [6] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(F)$. In other words, the $EP(F)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, see, for example [6-9] and references therein.

In 2009, Takahashi and Zembayashi [10], proposed the iteration in a uniformly smooth and uniformly convex Banach space: as sequence $\{x_n\}$ generated by $x_1 \in E$,

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n) JSu_n), \end{cases} \tag{1.3}$$

for every $n \in \mathbb{N}$, where J is duality mappings form E to E^*, S is a relatively nonexpansive self mapping on C and $\{\alpha_n\}$ is appropriate positive real sequence. They proved that if J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to some element in $EP(F) \cap F(S)$, where $F(S)$ is the fixed point set of S i.e., $F(S) := \{x \in C : Sx = x\}$.

In 2010, Wei et al. [5], constructed the following iterative scheme to approximate the common element of the set of fixed points of a relatively nonexpansive mapping $S : E \to E$ and the set of zero points of a maximal monotone operator $A : E \to 2^{E^*}$:

$$\begin{cases} x_1 \in E, \\ y_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JQ_\lambda^A x_n\right), \\ x_{n+1} = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JSy_n\right), \end{cases} \tag{1.4}$$

for every $n \in \mathbb{N}$, where $Q_\lambda^A : E \to E^*$ define by $Q_\lambda^A x = (J + \lambda A)^{-1} Jx$ for all $x \in E$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequence in [0,1). They proved that if J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to some element in $F(S) \cap A^{-1}0$.

Recently, Nilsrakoo [11], proved a strong convergence theorem for finding a common element of the fixed points set of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a uniformly convex and uniformly smooth Banach space. In this article, motivated by the above results and the iterative schemes considered of Wei, et al. [5], Takahashi and Zembayashi [10], we present a new iterative scheme for approximation of a common element in the intersection of the set of solutions for equilibrium problems, the set of zero points of maximal monotone operators and set of fixed points for relatively nonexpansive mapping in a uniformly smooth and uniformly convex Banach space. We prove a strong convergence theorem under some mind conditions. The results presented in this article extend and improve the results of Wei et al. [5], Takahashi and Zembayashi [10], and some authors.

2 Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\right\}$$

for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually write $J_2 = J$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in U$ and $x \neq y$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$ exists or all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, \quad y \in E, \quad \|x\| = 1, \quad \|y\| = \tau\right\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}\frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$.

A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

We note that E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

A multi-valued operator $A : E \to 2^{E*}$;

(i) The graph of A, $G(A) = \{(u, v) \mid u \in E \text{ and } v \in A(u)\}$;
(ii) A is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, $x_1, x_2 \in D(A)$, $y_1 \in Ax_1$, $y_2 \in Ax_2$;
(iii) A is maximal monotone if it is monotone and its graph is maximal with respect to this property, i.e., it is not properly contained in the graph of any other monotone operator.

Example. The mapping $A : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$A(x) = \begin{cases} x - a, & x < 0, \\ [-a, a], & x = 0, \\ x + a, & x > 0. \end{cases} \tag{2.1}$$

Then, A is a monotone mapping.

Note. A is maximal monotone if and only if $A(x) = [-a, a]$, when $x = 0$.

Let E be smooth Banach space and J the normalized duality mapping from E to E^*. Alber [12] considered the following functional $\phi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in E. \tag{2.2}$$

It is obvious from the definition of the function ϕ that

$$\left(\|x\| - \|y\|\right)^2 \leq \varphi(x, y) \leq \left(\|x\| + \|y\|\right)^2$$

and

$$\varphi\left(x, J^{-1}\left(tJy + (1-t)Jz\right)\right) \leq t\varphi(x, y) + (1-t)\varphi(x, z) \tag{2.3}$$

for all $t \in [0,1]$ and $x, y, z \in E$. The following lemma is an analogue of Xu's inequality with respect to ϕ.

Lemma 2.1. [11]*Let E be a uniformly smooth Banach space and $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\varphi\left(x, J^{-1}\left(tJy + (1-t)Jz\right)\right) \leq t\varphi(x, y) + (1-t)\varphi(x, z) - t(1-t)g\left(\|Jy - Jz\|\right) \tag{2.4}$$

for all $t \in [0,1]$, $x \in E$ and $y, z \in B_r$.

Let C be a closed convex subset of E, and S be a mapping from C into itself. Let $F(S) = \{x \in C : Sx = x\}$ be the set of fixed points of S. A point $p \in C$ is said to be an asymptotic fixed point S if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Sx_n) = 0$. The set of asymptotic fixed points of S will denoted by $\widehat{F(s)}$.

A mapping S from C into itself is said to be *relatively nonexpansive* if

(C1) $F(s) \neq \emptyset$;

(C2) $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$;

(C3) $\widehat{F(S)} = F(S)$.

A mapping S from C into itself is said to be *strongly relatively nonexpansive* if

(D1) S is relatively nonexpansive;

(D2) $\phi(Sx_n, x_n) \to 0$ whenever $\{x_n\}$ is bounded sequence in C such that $\phi(p, x_n) - \phi(p, Sx_n) \to 0$ for some $p \in F(S)$.

Lemma 2.2. [13]*The duality mapping J has the following properties:*

(i) If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is single-valued;
(ii) For all $x \in E$ and $\lambda > 0$, $J(\lambda x) = \lambda Jx$;
(iii) If E is strictly convex, then J is one to one and strictly monotone, that is, $\langle x - y, x^* - y^*\rangle > 0$ hold for all $x^* \in Jx$ and $y^* \in Jy$ with $x \neq y$;
(iv) If E is a real uniformly convex and uniformly smooth Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping. Moreover, both J and J^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively.

Lemma 2.3. [8,12]*Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E, and $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

In this case, the mapping Π_C of E onto C defined by $\Pi_{C^x} = x_0$ for all $x \in E$ is called the *generalized projection operator*.

Lemma 2.4. [14]*Let E and C be the same as those in Lemma* 2.3. *Let $x \in E$ and $\hat{x} \in C$. Then,*

(a) $\hat{x} = \Pi_{C^x}$ if and only if $\langle y - \hat{x}, Jx - J\hat{x}\rangle \leq 0$, for all $y \in C$;
(b) $\varphi\left(y, \Pi_{C^x}\right) + \varphi\left(\Pi_{C^x}, x\right) \leq \varphi(y, x)$, for all $y \in C$.

Lemma 2.5. [14]*Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.6. [8]*Let E be a real smooth and strictly convex Banach space and let C be a closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself. Then $F(S)$ is convex and closed.*

Lemma 2.7. [14]*Let E be a real smooth Banach space, let C be a convex subset of E, let $x \in E$ and $x_0 \in C$. Then $\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx\rangle \geq 0$ for all $z \in C$.*

Lemma 2.8. [13,15]*Let E be a real smooth and uniformly convex Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone operator. Then $A^{-1}0$ is a closed and convex* *subset of E and the graph $G(A)$ of A, is demi-closed in the following sense: for all $\{x_n\} \subset D(A)$ with $x_n \rightharpoonup x \in E$ and $y_n \in Ax_n$ with $y_n \to y \in E^*$, we have $x \in D(A)$ and $y \in Ax$.*

Definition 2.9. Let E and A be the same as these in Lemma 2.8. For all $\lambda > 0$, define the operator $Q_\lambda^A : E \to E$ by $Q_\lambda^A x = (J + \lambda A)^{-1} Jx$ for all $x \in E$.

Lemma 2.10. [16]*Let E be a real reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}0$. Then, for all $x \in E$, $y \in A^{-1}0$ and $\lambda > 0$, we have $\varphi\left(y, Q_\lambda^A x\right) + \varphi\left(Q_\lambda^A x, x\right) \le \varphi(y, x)$.*

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We make use of the following mapping $V : E \times E \to \mathbb{R}$ studied in Alber [12]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma.

Lemma 2.11. [11]*Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.5). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \le V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

For solving the equilibrium problem, let us give the following assumptions for the bifunction $F : C \times C \to \mathbb{R}$, satisfies the following conditions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$, $\forall x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F\left(tz + (1 - t)x, y\right) \le F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

In what follows, we shall make use of the following lemmas.

Lemma 2.12. [10]*Let C be a closed convex subset of smooth, strictly convex and reflexive Banach space E, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A4) and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.13. [10]*Let C be nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C \right\},$$

for all $x \in C$. Then, the following conclusions hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y \rangle \le \langle T_r x - T_r y, Jx - Jy \rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.14. [17]*Let C be a nonempty closed convex subset of a Banach space E, let $F : C \times C \to \mathbb{R}$. be a bifunction satisfying conditions (A1)-(A4) and $z \in C$. Then $z \in EP(F)$ if and only if $F(y, z) \leq 0$, $\forall y \in C$.*

Remark 2.15. [18] Let C be a nonempty subset of a smooth Banach space E. If $S : C \to E$ is firmly nonexpansive-type mapping, then

$$\varphi(z, Sx) \leq \varphi(z, Sx) + \varphi(Sx, x) \leq \varphi(z, x)$$

for all $x \in C$ and $z \in F(S)$. In particular, S satisfies condition (C2).

Lemma 2.16. [19]*Let $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \quad n \geq 0$$

where $\{b_n\}$ is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} b_n = \infty$, $\lim_{n\to\infty} b_n = 0$;

(2) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.17. [20]*Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}, \tag{2.6}$$

for all $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3 Main result

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions for equilibrium problems, the set of zero points of maximal monotone operators and set of fixed points for strongly relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, F: $C \times C \to \mathbb{R}$ bea bifunction satisfying conditions (A1)-(A4), $S : C \to C$ be a strongly relatively nonexpansive mapping, and let $A : E \to 2^{E^*}$ be a maximal monotone operator with $\Omega =: F(S) \cap EP(F) \cap A^{-1}(0) \neq \emptyset$. For a positive number λ, let the sequence $\{x_n\}$ be generated by $x_0 \in C$, $x_1 \in E$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\left\langle y - u_n, Ju_n - Jx_n \right\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}\left(\beta_n Jx_0 + (1 - \beta_n)JQ_\lambda^A u_n\right), \\ x_{n+1} = J^{-1}\left(\alpha_n Ju_n + (1 - \alpha_n)JSy_n\right), \end{cases} \tag{3.1}$$

for every $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1)$, and $\{r_n\} \subset (0, \infty)$. If *the control sequences satisfy the following restrictions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$*and* $\lim_{n\to\infty} \beta_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p = \Pi_\Omega(x_0)$, where Π_Ω is the generalized projection from E onto Ω.

Proof. First, taking $p = \Pi_\Omega(x_0)$, since $\Pi_C, T_{r_n}, Q_\lambda^A$ and S satisfy the condition (C2) and (2.4), $u_n = T_{r_n}x_n$, it follows that

$$\begin{aligned}
\varphi(p, y_n) &= \varphi\left(p, \Pi_C J^{-1}\left(\beta_n Jx_0 + (1-\beta_n)JQ_\lambda^A u_n\right)\right)\\
&\le \varphi\left(p, J^{-1}\left(\beta_n Jx_0 + (1-\beta_n)JQ_\lambda^A u_n\right)\right)\\
&\le \beta_n\varphi(p, x_0) + (1-\beta_n)\,\varphi\left(p, Q_\lambda^A u_n\right)\\
&\le \beta_n\varphi(p, x_0) + (1-\beta_n)\varphi(p, u_n)\\
&= \beta_n\varphi(p, x_0) + (1-\beta_n)\varphi(p, T_{r_n}x_n)\\
&\le \beta_n\varphi(p, x_0) + (1-\beta_n)\varphi(p, x_n)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(p, x_{n+1}) &= \varphi\left(p, J^{-1}\left(\alpha_n Ju_n + (1-\alpha_n)JSy_n\right)\right)\\
&\le \alpha_n\varphi(p, u_n) + (1-\alpha_n)\varphi(p, Sy_n)\\
&= \alpha_n\varphi\left(p, T_{r_n}x_n\right) + (1-\alpha_n)\varphi(p, Sy_n)\\
&\le \alpha_n\varphi(p, x_n) + (1-\alpha_n)\varphi(p, y_n)\\
&\le \alpha_n\varphi(p, x_n) + (1-\alpha_n)\beta_n\varphi(p, x_0) + (1-\alpha_n)(1-\beta_n)\varphi(p, x_n)\\
&= (1-\alpha_n)\beta_n\varphi(p, x_0) + \left(1-\beta_n(1-\alpha_n)\right)\varphi(p, x_n)\\
&\le \max_{n\ge 1}\left\{\varphi(p, x_0), \varphi(p, x_n)\right\}\\
&\le \cdots\\
&\le \max_{n\ge 1}\left\{\varphi(p, x_0), \varphi(p, x_1)\right\}.
\end{aligned}$$

This show that $\{x_n\}$ is bounded. Hence $\{u_n\}$, $\{y_n\}$ and $\{Sy_n\}$ are also bounded.
Put $v_n = J^{-1}\left(\beta_n Jx_0 + (1-\beta_n)JQ_\lambda^A u_n\right)$.
Using Lemma 2.11 gives

$$\begin{aligned}
\varphi(p, y_n) &= \varphi(p, \Pi_C v_n)\\
&\le \varphi(p, v_n) = V(p, Jv_n)\\
&\le V\left(p, Jv_n - \beta_n\left(Jx_0 - Jp\right)\right) - 2\left\langle v_n - p, -\beta_n\left(Jx_0 - Jp\right)\right\rangle\\
&= \varphi\left(p, J^{-1}\left(\beta_n Jp + (1-\beta_n)JQ_\lambda^A u_n\right)\right) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\\
&\le \beta_n\varphi(p, p) + (1-\beta_n)\varphi\left(p, Q_\lambda^A u_n\right) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\\
&\le \beta_n\varphi(p, p) + (1-\beta_n)\varphi(p, u_n) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\\
&= \beta_n\varphi(p, p) + (1-\beta_n)\varphi\left(p, T_{r_n}x_n\right) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\\
&\le (1-\beta_n)\varphi(p, x_n) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle.
\end{aligned} \tag{3.2}$$

Let $g : [0, 2r] \to [0, \infty)$ be a function satisfying the properties of Lemma 2.1, where $r = \sup_{n\ge 1}\{\|u_n\|, \|Sy_n\|\}$.
By Lemma 2.1, Remark 2.15 and (3.2), we get

$$\begin{aligned}
\varphi(p, x_{n+1}) &= \varphi\left(p, J^{-1}\left(\alpha_n Ju_n + (1-\alpha_n)JSy_n\right)\right)\\
&\le \alpha_n\varphi(p, u_n) + (1-\alpha_n)\varphi(p, Sy_n) - \alpha_n(1-\alpha_n)g\left(\|Ju_n - JSy_n\|\right)\\
&\le \alpha_n\left(\varphi(p, x_n) - \varphi(u_n, x_n)\right) + (1-\alpha_n)\varphi(p, y_n) - \alpha_n(1-\alpha_n)g(\|Ju_n - JSy_n\|)\\
&\le \alpha_n\left(\varphi(p, x_n) - \varphi(u_n, x_n)\right) + (1-\alpha_n)\left((1-\beta_n)\varphi(p, x_n) + 2\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\right)\\
&\quad - \alpha_n(1-\alpha_n)g\left(\|Ju_n - JSy_n\|\right)\\
&= \left(1-(1-\alpha_n)\beta_n\right)\varphi(p, x_n) + 2(1-\alpha_n)\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle\\
&\quad - \alpha_n\varphi(u_n, x_n) - \alpha_n(1-\alpha_n)g\left(\|Ju_n - JSy_n\|\right)
\end{aligned} \tag{3.3}$$

$$\le \left(1-(1-\alpha_n)\beta_n\right)\varphi(p, x_n) + 2(1-\alpha_n)\beta_n\left\langle v_n - p, Jx_0 - Jp\right\rangle. \tag{3.4}$$

We divide the proof into two parts:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(p, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(p, x_n)\}_{n=n_0}^{\infty}$ is convergent. Then

$$\varphi(p, x_n) - \varphi(p, x_{n+1}) \to 0, \quad \text{as} \ n \to \infty. \tag{3.5}$$

From (3.3) and $\beta_n \to 0$, we have

$$\alpha_n \varphi(u_n, x_n) + \alpha_n(1 - \alpha_n) g\left(\|Ju_n - JSy_n\|\right) \to 0, \quad \text{as } n \to \infty.$$

Since $\{\alpha_n\} \subset [a, b] \subset (0,1)$, we get

$$\varphi(u_n, x_n) \to 0 \text{ and } g\left(\|Ju_n - JSy_n\|\right) \to 0, \quad \text{as } n \to \infty.$$

By Lemma 2.5, we have

$$u_n - x_n \to 0, \quad Ju_n - JSy_n \to 0 \ \text{ and } \ u_n - Sy_n \to 0, \quad \text{as } n \to \infty.$$

By (2.3) and $\beta_n \to 0$, we have

$$\begin{aligned}
\varphi(u_n, y_n) &\leq \varphi(u_n, v_n) \\
&= \varphi\left(u_n, J^{-1}\left(\beta_n Jx_0 + (1 - \beta_n) JQ_\lambda^A u_n\right)\right) \\
&\leq \beta_n \varphi(u_n, x_0) + (1 - \beta_n)\varphi\left(u_n, Q_\lambda^A u_n\right) \\
&\leq \beta_n \varphi(u_n, x_0) + (1 - \beta_n)\varphi(u_n, u_n) = \beta_n \varphi(u_n, x_0) \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

By Lemma 2.5, implies that $u_n - y_n \to 0$ and $u_n - v_n \to 0$ as $n \to \infty$.
Hence,

$$v_n - y_n \to 0, \quad y_n - Sy_n \to 0 \ \text{ and } \ y_n - x_n \to 0, \quad \text{as } n \to \infty. \tag{3.6}$$

Now let us show that $\omega(x_n) \subset EP(F) \cap F(S) \cap A^{-1}0$, where

$$\omega(x_n) := \left\{\bar{x} \in C : x_{n_i} \rightharpoonup \bar{x}, \quad \exists \{n_i\} \subset \{n\} \text{ with } \{n_i\} \to \infty\right\}.$$

Indeed, since $\{x_n\}$ is bounded and E is reflexive, we know that $\omega(x_n) \neq \emptyset$. Take $\bar{x} \in \omega(x_n)$ arbitrary, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$. Let us show that $\bar{x} \in A^{-1}0$.

From $T_{r_n} x_n = u_n$, Lemma 2.10 and (2.3), we have

$$\begin{aligned}
\varphi(p, x_{n+1}) &\leq \alpha_n \varphi(p, u_n) + (1 - \alpha_n)\varphi(p, Sy_n) \\
&\leq \alpha_n \varphi(p, u_n) + (1 - \alpha_n)\left[\beta_n \varphi(p, x_0) + (1 - \beta_n)\varphi\left(p, Q_\lambda^A u_n\right)\right] \\
&\leq \alpha_n \varphi(p, u_n) + (1 - \alpha_n)\beta_n \varphi(p, x_0) + (1 - \alpha_n)(1 - \beta_n)\left[\varphi(p, u_n) - \varphi\left(Q_\lambda^A u_n, u_n\right)\right] \\
&= (1 - (1 - \alpha_n)\beta_n)\varphi(p, u_n) + (1 - \alpha_n)\beta_n \varphi(p, x_0) - (1 - \alpha_n)(1 - \beta_n)\varphi\left(Q_\lambda^A u_n, u_n\right) \\
&= (1 - (1 - \alpha_n)\beta_n)\varphi(p, T_{r_n} x_n) + (1 - \alpha_n)\beta_n \varphi(p, x_0) - (1 - \alpha_n)(1 - \beta_n)\varphi\left(Q_\lambda^A u_n, u_n\right) \\
&\leq (1 - (1 - \alpha_n)\beta_n)\varphi(p, x_n) + (1 - \alpha_n)\beta_n \varphi(p, x_0) - (1 - \alpha_n)(1 - \beta_n)\varphi\left(Q_\lambda^A u_n, u_n\right) \\
&\leq \varphi(p, x_n) + (1 - \alpha_n)\beta_n \varphi(p, x_0) - (1 - \alpha_n)(1 - \beta_n)\varphi\left(Q_\lambda^A u_n, u_n\right).
\end{aligned}$$

It follows that

$$(1 - \alpha_n)(1 - \beta_n)\varphi\left(Q_\lambda^A u_n, u_n\right) \leq \varphi(p, x_n) - \varphi(p, x_{n+1}) + (1 - \alpha_n)\beta_n \varphi(p, x_0). \tag{3.7}$$

Since $\beta_n \to 0$, $\limsup_{n\to\infty} \alpha_n < 1$ and (3.5), we obtain $\varphi\left(Q_\lambda^A u_n, u_n\right) \to 0$ as $n \to \infty$.

From Lemma 2.5, implies that $Q_\lambda^A u_n - u_n \to 0$ as $n \to \infty$. Since $y_n - x_n \to 0$, then $y_{n_i} \rightharpoonup \bar{x}$ and from $Sy_n - y_n \to 0$ as as $n \to \infty$. Hence $\bar{x} \in \widehat{F(S)} = F(S)$.

Since J is uniformly continuous on bound subset of E, we have $JQ_\lambda^A u_n - Ju_n \to 0$ as $n \to \infty$.

Let $z_n = Q_\lambda^A u_n$. Then there exists $w_n \in Az_n$ such that

$$w_n = \frac{Ju_n - Jz_n}{\lambda} = \frac{Ju_n - JQ_\lambda^A u_n}{\lambda} \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Since $Q_\lambda^A u_n - u_n \to 0$, $u_n - x_n \to 0$, and $x_{n_i} \rightharpoonup \bar{x}$, then $Q_\lambda^A u_{n_i} = z_{n_i} \rightharpoonup \bar{x}$. By Lemma 2.8, $\bar{x} \in A^{-1}0$. Thus $\omega(x_n) \subset F(S) \cap A^{-1}0$.

Finally, let us show that $\bar{x} \in EP(F)$. Since $x_n - u_n \to 0$ and $x_{n_i} \rightharpoonup \bar{x}$, we obtain that $u_{n_i} \rightharpoonup \bar{x}$, and

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0.$$

From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jx_n\|}{r_n} = 0. \tag{3.9}$$

By the definition of $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C. \tag{3.10}$$

Replacing n by n_i, we have from (A2) that

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jx_{n_i}\rangle \geq -F\left(u_{n_i}, y\right) \geq F\left(y, u_{n_i}\right), \quad \forall y \in C. \tag{3.11}$$

Letting $i \to \infty$, from (3.9), (A4) and $u_{n_i} \rightharpoonup \bar{x}$ that

$$F(y, \bar{x}) \leq 0, \quad \forall y \in C. \tag{3.12}$$

From Lemma 2.14, we have $\omega(x_n) \subset EP(F) \cap F(S) \cap A^{-1}0$.

Since $\{y_n\}$ is bounded and E is reflexive, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle y_n - p, Jx_0 - Jp\rangle = \lim_{i\to\infty}\langle y_{n_i} - p, Jx_0 - Jp\rangle.$$

Since $y_{n_i} \rightharpoonup \bar{x} \in \Omega$. By Lemma 2.4(a), we obtain that

$$\limsup_{n\to\infty}\langle y_n - p, Jx_0 - Jp\rangle = \lim_{i\to\infty}\langle y_{n_i} - p, Jx_0 - Jp\rangle = \langle \bar{x} - p, Jx_0 - Jp\rangle \leq 0.$$

Since $v_n - y_n \to 0$, we have

$$\limsup_{n\to\infty}\langle v_n - p, Jx_0 - Jp\rangle = \limsup_{n\to\infty}\langle y_n - p, Jx_0 - Jp\rangle \leq 0. \tag{3.13}$$

By (3.4), it follows that

$$\varphi(p, x_{n+1}) \leq (1 - (1 - \alpha_n)\beta_n)\varphi(p, x_n) + 2(1 - \alpha_n)\beta_n\langle v_n - p, Jx_0 - Jp\rangle. \tag{3.14}$$

Set $b_n = (1 - \alpha_n)\beta_n$ and $c_n = 2\langle v_n - p, Jx_0 - Jp\rangle$. Then we have

$$\varphi(p, x_{n+1}) \leq (1 - b_n)\varphi(p, x_n) + b_n c_n. \tag{3.15}$$

From the condition (i) and (3.13), we see that $\sum_{n=0}^{\infty} b_n = \infty$ and $\limsup_{n\to\infty} c_n \leq 0$. Therefore, applying Lemma 2.16 to (3.15), we get that $\phi(p,x_n) \to 0$. Then $x_n \to p$ and since $u_n - x_n \to 0$, we have $u_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(p, x_{n_i}) < \varphi(p, x_{n_i+1}), \quad \forall i \in \mathbb{N}. \tag{3.16}$$

By Lemma 2.17, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\varphi(p, x_{m_k}) < \varphi(p, x_{m_k+1})$ and $\varphi(p, x_k) < \varphi(p, x_{m_k+1})$, for all $k \in \mathbb{N}$.

From (3.3), condition (i) and (ii), we have

$$\begin{aligned}
&\alpha_{m_k}\varphi(u_{m_k}, x_{m_k}) + \alpha_{m_k}(1-\alpha_{m_k})g(\|Ju_{m_k} - JSy_{m_k}\|)\\
&\leq \varphi(p, x_{m_k}) - \varphi(p, x_{m_k+1}) - (1-\alpha_{m_k})\beta_{m_k}\varphi(p, x_{m_k}) + 2(1-\alpha_{m_k})\beta_{m_k}\langle v_{m_k} - p, Jx_0 - Jp\rangle\\
&\leq -(1-\alpha_{m_k})\beta_{m_k}\varphi(p, x_{m_k}) + 2(1-\alpha_{m_k})\beta_{m_k}\langle v_{m_k} - p, Jx_0 - Jp\rangle.
\end{aligned}$$

Similary proof of Case 1, we obtain that

$$\limsup_{k\to\infty}\langle v_{m_k} - p, Jx_0 - Jp\rangle \leq 0. \tag{3.17}$$

From (3.3), we have

$$\varphi(p, x_{m_k+1}) \leq \varphi(p, x_{m_k}) - (1-\alpha_{m_k})\beta_{m_k}\varphi(p, x_{m_k}) + 2(1-\alpha_{m_k})\beta_{m_k}\langle v_{m_k} - p, Jx_0 - Jp\rangle. \tag{3.18}$$

Since $\varphi(p, x_{m_k}) \leq \varphi(p, x_{m_k+1})$, we have

$$\begin{aligned}
(1-\alpha_{m_k})\beta_{m_k}\varphi(p, x_{m_k}) &\leq \varphi(p, x_{m_k}) - \varphi(p, x_{m_k+1}) + 2(1-\alpha_{m_k})\beta_{m_k}\langle v_{m_k} - p, Jx_0 - Jp\rangle\\
&\leq 2(1-\alpha_{m_k})\beta_{m_k}\langle v_{m_k} - p, Jx_0 - Jp\rangle.
\end{aligned}$$

Since, $(1-\alpha_{m_k})\beta_{m_k} > 0$ and (3.17) we have

$$\varphi(p, x_{m_k}) \leq 2\langle v_{m_k} - p, Jx_0 - Jp\rangle \leq 0. \tag{3.19}$$

It follows that $\varphi(p, x_{m_k}) \to 0$. From (3.18), we gives $\varphi(p, x_{m_k+1}) \to 0$, as $k \to \infty$.

From $\varphi(p, x_k) \leq \varphi(p, x_{m_k+1})$ and Lemma 2.5, we obtain that $x_k \to p$ and $u_k \to p$, as $k \to \infty$.

From Cases 1 and 2, we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to p. This completes the proof.

Corollary 3.2. *Let C be nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, $S : C \to C$ be a strongly relatively nonexpansive mapping, and let $A : E \to 2^{E^*}$ be a maximal monotone operator with $F(S) \cap A^{-1}(0) \neq \emptyset$. For a positive number λ, let the sequence $\{x_n\}$ be generated by $x_0 \in C$, $x_1 \in E$ and*

$$\begin{cases} u_n = \Pi_C x_n, \\ y_n = \Pi_C J^{-1}(\beta_n Jx_0 + (1-\beta_n)JQ_\lambda^A u_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSy_n), \end{cases} \tag{3.20}$$

for every $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1). *If the control sequences satisfy the following restrictions:*

(i) $\sum_{n=0}^{\infty} \beta_n = \infty$*and* $\lim_{n\to\infty} \beta_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ converges strongly to $p = \Pi_{F(S)\cap A^{-1}(0)}(x_0)$, where $\Pi_{F(S)\cap A^{-1}(0)}$ is the generalized projection from E onto $F(S) \cap A^{-1}(0)$.

Proof. Put $F \equiv 0$ and $r_n \equiv 1$ in Theorem 3.1. Then the conclusion of Corollary 3.2 can be obtained the desired result easily.

Corollary 3.3. *Let C be nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E,* $F : C \times C \to \mathbb{R}$ *bea bifunction satisfying conditions (A1)-(A4) and* $S : C \to C$ *be a strongly relatively nonexpansive mapping, with* $F(S) \cap EP(F) \neq \emptyset$. *Let the sequence* $\{x_n\}$ *be generated by* $x_0 \in C$, $x_1 \in E$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}(\beta_n Jx_0 + (1-\beta_n)Ju_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSy_n), \end{cases} \tag{3.21}$$

for every $n \geq 1$, *where* $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0,1)$, *and* $\{r_n\} \subset (0, \infty)$. *If the control sequences satisfy the following restrictions:*

(i) $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $p = \Pi_{F(S)\cap EP(F)}(x_0)$, *where* $\Pi_{F(S)\cap EP(F)}$ *is the generalized projection from E onto* $F(S) \cap EP(F)$.

Proof. Put $A \equiv 0$, then Q_λ^A is an identity mapping, in Theorem 3.1. Then the conclusion of Corollary 3.3 can be obtained the desired result easily.

Corollary 3.4. *Let C be nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E,* F: $C \times C \to \mathbb{R}$ *bea bifunction satisfying conditions (A1)-(A4) and let* $A : E \to 2^{E^*}$ *be a maximal monotone operator with* $EP(F) \cap A^{-1}(0) \neq \emptyset$. *For a positive number* λ, *let the sequence* $\{x_n\}$ *be generated by* $x_0 \in C$, $x_1 \in E$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}(\beta_n Jx_0 + (1-\beta_n)JQ_\lambda^A u_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)Jy_n), \end{cases} \tag{3.22}$$

for every $n \geq 1$, *where* $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0,1)$, *and* $\{r_n\} \subset (0, \infty)$. If *the control sequences satisfy the following restrictions:*

(i) $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $p = \Pi_{EP(F)\cap A^{-1}(0)}(x_0)$, *where* $\Pi_{EP(F)\cap A^{-1}}(0)$ *is the generalized projection from E onto* $EP(F) \cap A^{-1}(0)$.

Proof. Put $S \equiv I$, is an identity mapping, in Theorem 3.1. Then the conclusion of Corollary 3.4 can be obtained the desired result easily.

Remark 3.5. Our main result extends and improves the recent results of Wei et al. [5], Takahashi and Zembayashi [10] and generalizes the result of Nilsakoo [11].

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Authors' contributions

All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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