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Strong convergence theorems for the general split variational inclusion problem in Hilbert spaces

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Abstract

The purpose of this paper is to introduce and study a general split variational inclusion problem in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions, we prove that the sequence generated by the proposed new algorithm converges strongly to a solution of the general split variational inclusion problem. As a particular case, we consider the algorithms for a split feasibility problem and a split optimization problem and give some strong convergence theorems for these problems in Hilbert spaces.

Keywords: general split variational inclusion problem; split feasibility problem; split optimization problem; quasi-nonexpansive mapping; zero point; resolvent mapping

1 Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (*SFP*) is formulated as

$$\text{to find } x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [1] first introduced the *SFP* in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the *SFP* can also be used in various disciplines such as image restoration, computer tomography and radiation therapy treatment planning [3–5]. The *SFP* in an infinite-dimensional real Hilbert space can be found in [2, 4, 6–10]. For comprehensive literature, bibliography and a survey on SFP, we refer to [11].

Assuming that the *SFP* is consistent, it is not hard to see that $x^* \in C$ solves *SFP* if and only if it solves the fixed point equation

$$x^* = P_C\big(I - \gamma A^*(I - P_Q)A\big)x^*,$$

where P_C and P_Q are the metric projection from H_1 onto C and from H_2 onto Q, respectively, $\gamma > 0$ is a positive constant, and A^* is the adjoint of A.

A popular algorithm to be used to solves the *SFP* (1.1) is due to Byrne's *CQ-algorithm* [2]:

$$x_{k+1} = P_C\big(I - \gamma_k A^*(I - P_Q)A\big)x_k, \quad k \geq 1,$$

where $\gamma_k \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

On the other hand, let H be a real Hilbert space, and B be a set-valued mapping with domain $D(B) := \{x \in H : B(x) \neq \emptyset\}$. Recall that B is called *monotone*, if $\langle u - v, x, x - y\rangle \geq 0$ for any $u \in Bx$ and $v \in By$; B is *maximal monotone*, if its graph $\{(x, y) : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. An important problem for set-valued monotone mappings is to find $x^* \in H$ such that $0 \in B(x^*)$. Here, x^* is called a *zero point of B*. A well-known method for approximating a zero point of a maximal monotone mapping defined in a real Hilbert space H is *the proximal point algorithm* first introduced by Martinet [12] and generated by Rockafellar [13]. This is an iterative procedure, which generates $\{x_n\}$ by $x_1 = x \in H$ and

$$x_{n+1} = J^B_{\beta_n} x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\beta_n\} \subset (0, \infty)$, B is a maximal monotone mapping in a real Hilbert space, and J^B_r is the *resolvent mapping of B* defined by $J^B_r = (I + rB)^{-1}$ for each $r > 0$. Rockafellar [13] proved that if the solution set $B^{-1}(0)$ is nonempty and $\liminf_{n\to\infty} \beta_n > 0$, then the sequence $\{x_n\}$ in (1.2) converges weakly to an element of $B^{-1}(0)$. In particular, if B is the sub-differential ∂f of a proper convex and lower semicontinuous function $f : H \to \mathbb{R}$, then (1.2) is reduced to

$$x_{n+1} = \operatorname*{argmin}_{y\in H} \left\{ f(y) + \frac{1}{2\beta_n}\|y - x_n\|^2 \right\}, \quad \forall n \geq 1. \tag{1.3}$$

In this case, $\{x_n\}$ converges weakly to a minimizer of f. Later, many researchers have studied the convergence problems of the proximal point algorithm in Hilbert spaces (see [14–21] and the references therein).

Motivated by the works in [14–17] and related literature, the purpose of this paper is to introduce and consider the following *general split variational inclusion problem*.

Let H_1 and H_2 be two real Hilbert spaces, $B_i : H_1 \to H_1$ and $K_i : H_2 \to H_2$, $i = 1, 2, \ldots$ be two families of set-valued maximal monotone mappings, $A : H_1 \to H_2$ be a linear and bounded operator, and A^* be the adjoint of A. The so-called *general split variational inclusion problem* is

$$\text{to find } x^* \in H_1 \text{ such that } 0 \in \bigcap_{i=1}^{\infty} B_i\big(x^*\big) \text{ and } 0 \in \bigcap_{i=1}^{\infty} K_i\big(Ax^*\big). \tag{1.4}$$

The following examples are special cases of (GSVIP) (1.4).

Classical split variational inclusion problem. Let $B : H_1 \to H_1$ and $K : H_2 \to H_2$ be set-valued maximal monotone mappings. The so-called *classical split variational inclusion problem* (CSVIP) is

$$\text{to find } x^* \in H_1 \text{ such that } 0 \in B\big(x^*\big) \text{ and } 0 \in K\big(Ax^*\big), \tag{1.5}$$

which was introduced by Moudafi [17]. It is obvious that problem (1.5) is a special case of (GSVIP) (1.4). In [17], Moudafi proved that the iteration process

$$x_{n+1} = J_\lambda^B\bigl(x_n + \gamma A^*\bigl(J_\lambda^K - I\bigr)Ax_n\bigr)$$

converges weakly to a solution of problem (1.5), where λ and γ are given positive numbers.

Split optimization problem. Let $f : H_1 \to \mathbb{R}$, $g : H_2 \to \mathbb{R}$ be two proper convex and lower semicontinuous functions. The so-called *split optimization problem* (SOP) is

$$\text{to find } x^* \in H_1 \text{ such that } f\bigl(x^*\bigr) = \min_{y\in H_1} f(y) \text{ and } g\bigl(Ax^*\bigr) = \min_{z\in H_2} g(z). \tag{1.6}$$

Denote by $B = \partial(f)$ and $K = \partial(g)$, then B and K both are maximal monotone mappings, and problem (1.6) is equivalent to the following classical split variational inclusion problem, *i.e.*:

$$\text{to find } x^* \in H_1 \text{ such that } 0 \in \partial\bigl(f\bigl(x^*\bigr)\bigr) \text{ and } 0 \in \partial\bigl(g\bigl(Ax^*\bigr)\bigr). \tag{1.7}$$

Split feasibility problem. As in (1.1), let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and A be the same as above. The *split feasibility problem* is

$$\text{to find } x^* \in C \text{ such } Ax^* \in Q. \tag{1.8}$$

It is well known that this kind of problems was first introduced by Censor and Elfving [1] for modeling inverse problems arising from phase retrievals and in medical image reconstruction [2]. Also it can be used in various disciplines such as image restoration, computer tomography and radiation therapy treatment planning.

Let i_C (i_Q) be the indicator function of C (Q), *i.e.*,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C; \end{cases} \qquad i_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{if } x \notin Q. \end{cases} \tag{1.9}$$

Then i_C and i_Q both are proper convex and lower semicontinuous functions, and its subdifferentials ∂i_C and ∂i_Q are maximal monotone operators. Consequently problem (1.8) is equivalent to the following 'split optimization problem' and 'Moudafi's classical split variational inclusion problem', *i.e.*,

$$\begin{aligned} &\text{to find } x^* \in H_1 \text{ such that } i_C\bigl(x^*\bigr) = \min_{y\in H_1} i_C(y) \text{ and } i_Q\bigl(Ax^*\bigr) = \min_{z\in H_2} i_Q(z) \\ &\quad\Leftrightarrow\quad \text{to find } x^* \in H_1 \text{ such that } 0 \in \partial\bigl(i_C\bigl(x^*\bigr)\bigr) \text{ and } 0 \in \partial\bigl(i_Q\bigl(Ax^*\bigr)\bigr). \end{aligned} \tag{1.10}$$

For solving (GSVIP) (1.4), in our paper we propose the following iterative algorithms:

$$x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\beta_i}^{B_i}\bigl[x_n - \lambda_{n,i} A^*\bigl(I - J_{\beta_i}^{K_i}\bigr)Ax_n\bigr], \quad \forall n \ge 0, \tag{1.11}$$

where $f : H_1 \to H_1$ is a contraction mapping with a contractive constant $k \in (0,1)$, $\{\alpha_n\}$, $\{\xi_n\}$ and $\{\gamma_{n,i}\}$ are sequence in $[0,1]$ satisfying some conditions. Under suitable conditions, some strong convergence theorems for the sequence proposed by (1.11) to a solution for (GSVIP) (1.4) in Hilbert spaces are proved. As a particular case, we consider the algorithms for a split feasibility problem and a split optimization problem and give some strong convergence theorems for these problems in Hilbert spaces. Our results extend and improve the related results of Censor and Elfving [1], Byrne [2], Censor *et al.* [3–5], Rockafellar [13], Moudafi [14, 17], Eslamian and Latif [15], Eslamian [21], and Chuang [22].

2 Preliminaries

Throughout the paper, we denote by H a real Hilbert space, C be a nonempty closed and convex subset of H. $F(T)$ denote by the set of fixed points of a mapping T. Let $\{x_n\}$ be a sequence in H and $x \in H$. Strong convergence of $\{x_n\}$ to x is denoted by $x_n \to x$, and weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies.

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

The operator P_C is called the *metric projection*. The metric projection P_C is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \ge 0, \quad \forall x \in H, y \in C.$$

Recall that a mapping $T : C \to H$ is said to be *nonexpansive*, if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. T is said to be *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for every $x \in C$ and $p \in F(T)$. It is easy to see that $F(T)$ is a closed convex subset of C if T is a quasi-nonexpansive mapping. Besides, T is said to be a *firmly nonexpansive*, if

$$\begin{aligned} &\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle \quad \forall x, y \in C; \\ &\quad \Leftrightarrow \quad \|Tx - Ty\|^2 \le \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2 \quad \forall x, y \in C. \end{aligned}$$

Lemma 2.1 (demi-closed principle) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a nonexpansive mapping, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.*

Lemma 2.2 [23] *Let H be a* (*real*) *Hilbert space. Then for all $x, y \in H$,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.1}$$

Lemma 2.3 [24] *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integers i, j with $i < j$,*

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.2}$$

Lemma 2.4 *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{b_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} b_n = \infty$, $\{u_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ be a real numbers with $\limsup_{n\to\infty} t_n \leq 0$. If*

$$a_{n+1} \leq (1 - b_n)a_n + b_n t_n + u_n, \quad \textit{for each } n \geq 1,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 [25] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

Lemma 2.6 [22] *Let H be a real Hilbert space, $B : H \to 2^H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_β^B be the resolvent mapping of B.*

(i) *For each $\beta > 0$, J_β^B is a single-valued and firmly nonexpansive mapping*;
(ii) *$D(J_\beta^B) = H$ and $F(J_\beta^B) = B^{-1}(0) := \{x \in D(B) : 0 \in Bx\}$*;
(iii) *$(I - J_\beta^B)$ is a firmly nonexpansive mapping for each $\beta > 0$*;
(iv) *suppose that $B^{-1}(0) \neq \emptyset$, then for each $x \in H$, each $x^* \in B^{-1}(0)$ and each $\beta > 0$*

$$\|x - J_\beta^B x\|^2 + \|J_\beta^B x - x^*\| \leq \|x - x^*\|^2;$$

(v) *suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$ and each $w \in B^{-1}(0)$, and each $\beta > 0$.*

Lemma 2.7 *Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear bounded operator and A^* be the adjoint of A. Let $B : H_2 \to 2_2^H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_β^B be the resolvent mapping of B, then*

(i) $\|(I - J_\beta^B)Ax - (I - J_\beta^B)Ay\|^2 \leq \langle (I - J_\beta^B)Ax - (I - J_\beta^B)Ay, Ax - Ay\rangle$;
(ii) $\|A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay\|^2 \leq \|A\|^2\langle (I - J_\beta^B)Ax - (I - J_\beta^B)Ay, Ax - Ay\rangle$;
(iii) *if $\rho \in (0, \frac{2}{\|A\|^2})$, then $(I - \rho A^*(I - J_\beta^B)A)$ is a nonexpansive mapping.*

Proof By Lemma 2.6(iii), the mapping $(I - J_\beta^B)$ is firmly nonexpansive, hence the conclusions (i) and (ii) are obvious.

Now we prove the conclusion (iii).

In fact, for any $x, y \in H_1$, it follows from the conclusions (i) and (ii) that

$$\begin{aligned}
&\big\|\big(I - \rho A^*\big(I - J_\beta^B\big)A\big)x - \big(I - \rho A^*\big(I - J_\beta^B\big)A\big)y\big\|^2 \\
&\quad = \|x - y\|^2 - 2\rho\big\langle x - y, A^*\big(I - J_\beta^B\big)Ax - A^*\big(I - J_\beta^B\big)Ay\big\rangle \\
&\qquad + \rho^2\big\|A^*\big(I - J_\beta^B\big)Ax - A^*\big(I - J_\beta^B\big)Ay\big\|^2 \\
&\quad \leq \|x - y\|^2 - 2\rho\big\langle Ax - Ay, \big(I - J_\beta^B\big)Ax - \big(I - J_\beta^B\big)Ay\big\rangle \\
&\qquad + \rho^2\|A\|^2\big\|\big(I - J_\beta^B\big)Ax - \big(I - J_\beta^B\big)Ay\big\|^2 \\
&\quad \leq \|x - y\|^2 - \rho\big(2 - \rho\|A\|^2\big)\big\|\big(I - J_\beta^B\big)Ax - \big(I - J_\beta^B\big)Ay\big\|^2 \\
&\quad \leq \|x - y\|^2 \quad \big(\text{since } \rho\big(2 - \rho\|A\|^2\big) \geq 0\big).
\end{aligned}$$

This completes the proof of Lemma 2.7. □

3 Main results

The following lemma will be used in proving our main results.

Lemma 3.1 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator, and A^* be the adjoint of A. Let $B_i : H_1 \to 2^{H_1}$, and $K_i : H_2 \to 2^{H_2}$, $i = 1, 2, \dots$, be two families of set-valued maximal monotone mappings, and let $\beta > 0$ and $\gamma > 0$. If $\Omega \neq \emptyset$ (the solution set of (GSVIP) (1.4)), then $x^* \in H_1$ is a solution of (GSVIP) (1.4) if and only if for each $i \geq 1$, for each $\gamma > 0$ and for each $\beta > 0$*

$$x^* = J_\beta^{B_i}\big(x^* - \gamma A^*\big(I - J_\beta^{K_i}\big)Ax^*\big). \tag{3.1}$$

Proof Indeed, if x^* is a solution of (GSVIP) (1.4), then for each $i \geq 1$, $\gamma > 0$ and $\beta > 0$,

$$x^* \in B_i^{-1}(0) \quad \text{and} \quad Ax^* \in K_i^{-1}(0), \quad \textit{i.e.}, x^* = J_\beta^{B_i}x^* \text{ and } Ax^* = J_\beta^{K_i}Ax^*.$$

This implies that $x^* = J_\beta^{B_i}(x^* - \gamma Ax^*(I - J_\beta^{K_i})Ax^*)$.

Conversely, if x^* solves (3.1), by Lemma 2.6(v), we have

$$\big\langle x^* - \big(x^* - \gamma A^*\big(I - J_\beta^{K_i}\big)Ax^*\big), y - x^*\big\rangle \geq 0, \quad \forall y \in B_i^{-1}(0).$$

Hence we have

$$\big\langle \big(I - J_\beta^{K_i}\big)Ax^*, Ay - Ax^*\big\rangle \geq 0, \quad \forall y \in B_i^{-1}(0). \tag{3.2}$$

On the other hand, by Lemma 2.6(v) again

$$\big\langle (Ax^* - J_\beta^{K_i}Ax^*, J_\beta^{K_i}Ax^* - v\big\rangle \geq 0, \quad \forall v \in K_i^{-1}(0). \tag{3.3}$$

Adding up (3.2) and (3.3), we have

$$\big\langle Ax^* - J_\beta^{K_i}Ax^*, J_\beta^{K_i}Ax^* + Ay - Ax^* - v\big\rangle \geq 0, \quad \forall y \in B_i^{-1}(0), \text{ and } v \in K_i^{-1}(0).$$

Simplifying it, we have

$$\big\|Ax^* - J_\beta^{K_i}Ax^*\big\|^2 \leq \big\langle Ax^* - J_\beta^{K_i}Ax^*, Ay - v\big\rangle \geq 0, \quad \forall y \in B_i^{-1}(0), \text{ and } v \in K_i^{-1}(0). \tag{3.4}$$

By the assumption that $\Omega \neq \emptyset$. Taking $w \in \Omega$, hence for each $i \geq 1$ $w \in B_i^{-1}(0)$ and $Aw \in K_i^{-1}(0)$. In (3.4), taking $y = w$ and $v = Aw$, then we have

$$\big\|Ax^* - J_\beta^{K_i}Ax^*\big\|^2 = 0.$$

This implies that $Ax^* = J_\beta^{K_i}Ax^*$, and so $Ax^* \in K_i^{-1}(0)$ for each $i \geq 1$. Hence from (3.1), $x^* = J_\beta^{B_i}x^*$, *i.e.*, $x^* \in B_i^{-1}(0)$. Hence x^* is a solution of (GSVIP)(1.4).

This completes the proof of Lemma 3.1. □

We are now in a position to prove the following main result.

Theorem 3.2 *Let* $H_1, H_2, A, A^*, \{B_i\}, \{K_i\}, \Omega$ *be the same as in Lemma* 3.1. *Let* $f : H_1 \to H_1$ *be a contractive mapping with contractive constant* $k \in (0,1)$. *Let* $\{\alpha_n\}$, $\{\xi_n\}$, $\{\gamma_{n,i}\}$ *be the sequences in* $(0,1)$ *with* $\alpha_n + \xi_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$, *for each* $n \geq 0$. *Let* $\{\beta_i\}$ *be a sequence in* $(0,\infty)$, *and* $\{\lambda_{n,i}\}$ *be a sequence in* $(0, \frac{2}{\|A\|^2})$. *Let* $\{x_n\}$ *be the sequence defined by* (1.11). *If* $\Omega \neq \emptyset$ *and the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \xi_n = 0$, *and* $\sum_{n=0}^{\infty} \xi_n = \infty$;

(ii) $\liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$ *for each* $i \geq 1$;

(iii) $0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \frac{2}{\|A\|^2}$,

then $x_n \to x^* \in \Omega$ *where* $x^* = P_\Omega f(x^*)$, *where* P_Ω *is the metric projection from* H_1 *onto* Ω.

Proof (I) First we prove that $\{x_n\}$ is bounded.

In fact, letting $z \in \Omega$, by Lemma 3.1, for each $i \geq 1$,

$$z = J_{\beta_i}^{B_i}\big[z - \lambda_{n,i} A^*\big(I - J_{\beta_i}^{K_i}\big)Az\big].$$

Hence it follows from Lemma 2.7(iii) that for each $i \geq 1$ and each $n \geq 1$ we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\| \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\beta_i}^{B_i}\big[x_n - \lambda_{n,i} A^*\big(I - J_{\beta_i}^{K_i}\big)Ax_n\big] - z \right\| \\
&\leq \alpha_n \|x_n - z\| + \xi_n \big\|f(x_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \big\|J_{\beta_i}^{B_i}\big[x_n - \lambda_{n,i} A^*\big(I - J_{\beta_i}^{K_i}\big)Ax_n\big] - z\big\| \\
&\leq \alpha_n \|x_n - z\| + \xi_n \big\|f(x_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \big\|J_{\beta_i}^{B_i}\big[x_n - \lambda_{n,i} A^*\big(I - J_{\beta_i}^{K_i}\big)Ax_n\big] - z\big\| \\
&\leq \alpha_n \|x_n - z\| + \xi_n \big\|f(x_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|x_n - z\| \\
&= (1 - \xi_n)\|x_n - z\| + \xi_n \big\|f(x_n) - z\big\| \\
&\leq (1 - \xi_n)\|x_n - z\| + \xi_n \big\|f(x_n) - f(z)\big\| + \xi_n \big\|f(z) - z\big\| \\
&\leq \big(1 - \xi_n(1-k)\big)\|x_n - z\| + \frac{\xi_n(1-k)}{1-k} \big\|f(z) - z\big\| \\
&\leq \max\left\{ \|x_n - z\|, \frac{1}{1-k} \big\|f(z) - z\big\| \right\}.
\end{aligned}$$

By induction, we can prove that

$$\|x_n - z\| \leq \max\left\{ \|x_0 - z\|, \frac{1}{1-k} \big\|f(z) - z\big\| \right\}, \quad \forall n \geq 0. \tag{3.5}$$

This implies that $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

(II) Now we prove that for each $j \geq 1$

$$\begin{aligned}
&\alpha_n \gamma_{n,j} \big\|x_n - J_{\beta_i}^{B_i}\big[x_n - \lambda_{n,i} A^*\big(I - J_{\beta_i}^{K_i}\big)Ax_n\big]\big\|^2 \\
&\quad \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \xi_n \big\|f(x_n) - z\big\|^2, \quad \text{for each } i \geq 1.
\end{aligned} \tag{3.6}$$

Indeed, it follows from Lemma 2.3 that for any positive $j \geq 1$

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - z \right\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + \xi_n \left\|f(x_n) - z\right\|^2 \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} \left\|J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - z\right\|^2 \\
&\quad - \alpha_n \gamma_{n,j} \left\|x_n - J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right]\right\|^2 \\
&\leq (1 - \xi_n)\|x_n - z\|^2 + \xi_n \left\|f(x_n) - z\right\|^2 \\
&\quad - \alpha_n \gamma_{n,j} \left\|x_n - J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right]\right\|^2.
\end{aligned}$$

Simplifying it, (3.6) is proved.

By the assumption that $\Omega \neq \emptyset$, and it is easy to prove that Ω is closed and convex. This implies that P_Ω is well defined. Again since $P_\Omega f : H_1 \to \Omega$ is a contraction mapping with contractive constant $k \in (0,1)$, there exists a unique $x^* \in \Omega$ such that $x^* = P_\Omega f x^*$. Since $x^* \in \Omega$, it solves (GSVIP) (1.4). By Lemma 3.1,

$$x^* = J_{\beta_j}^{B_j}\left(x^* - \lambda_{n,j} A^*\left(I - J_{\beta_j}^{K_j}\right)Ax^*\right), \quad \forall j \geq 1, n \geq 0. \tag{3.7}$$

(III) Now we prove that $x_n \to x^*$.

In order to prove that $x_n \to x^*$ (as $n \to \infty$), we consider two cases.

Case 1. Assume that $\{\|x_n - x^*\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nondecreasing or non-increasing. Since $\{\|x_n - x^*\|\}$ is bounded, $\{\|x_n - x^*\|\}$ is convergence. Again since $\lim_{n\to\infty} \xi_n = 0$, and $\{f(x_n)\}$ is bounded, from (3.6) we get

$$\lim_{n\to\infty} \alpha_n \gamma_{n,j} \left\|x_n - J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right]\right\|^2 = 0.$$

By condition (ii), we obtain

$$\lim_{n\to\infty} \left\|x_n - J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right]\right\| = 0. \tag{3.8}$$

Now we prove that

$$\limsup_{n\to\infty} \left\langle f\left(x^*\right) - x^*, x_n - x^* \right\rangle \leq 0. \tag{3.9}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w$, $\lambda_{n_k,i} \to \lambda_i \in (0, \frac{2}{\|A\|^2})$ for each $i \geq 1$, and

$$\limsup_{n\to\infty} \left\langle f\left(x^*\right) - x^*, x_n - x^* \right\rangle = \lim_{n_k\to\infty} \left\langle f\left(x^*\right) - x^*, x_{n_k} - x^* \right\rangle. \tag{3.10}$$

It follows from (3.8) that

$$\begin{aligned}
&\left\| J_{\beta_i}^{B_i}\left[x_n - \lambda_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - x_n \right\| \\
&\quad \le \left\| J_{\beta_i}^{B_i}\left[x_n - \lambda_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] \right\| \\
&\qquad + \left\| J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - x_n \right\| \\
&\quad \le \left\| \left[x_n - \lambda_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - \left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] \right\| \\
&\qquad + \left\| J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - x_n \right\| \\
&\quad \le |\lambda_i - \lambda_{n,i}| \left\| A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n \right\| \\
&\qquad + \left\| J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - x_n \right\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

For each $i \ge 1$, $J_{\beta_i}^{B_i}[I - \lambda_i A^*(I - J_{\beta_i}^{K_i})A]$ is a nonexpansive mapping. Thus from Lemma 2.1, $w = J_{\beta_i}^{B_i}[I - \lambda_i A^*(I - J_{\beta_i}^{K_i})A]w$. By Lemma 3.1 $w \in \Omega$, *i.e.*, w is a solution of (GSVIP) (1.4). Consequently we have

$$\begin{aligned}
\limsup_{n\to\infty} \left\langle f\left(x^*\right) - x^*, x_n - x^* \right\rangle &= \lim_{n_k\to\infty} \left\langle f\left(x^*\right) - x^*, x_{n_k} - x^* \right\rangle \\
&= \left\langle f\left(x^*\right) - x^*, w - x^* \right\rangle \le 0.
\end{aligned}$$

(IV) Finally, we prove that $x_n \to P_\Omega f(x^*)$.

In fact, from Lemma 2.2 we have

$$\begin{aligned}
&\left\| x_{n+1} - x^* \right\|^2 \\
&\quad \le \left\| \alpha_n\left(x_n - x^*\right) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\beta_i}^{B_i}\left[x_n - \lambda_{n,i} A^*\left(I - J_{\beta_i}^{K_i}\right)Ax_n\right] - x^* \right\|^2 \\
&\qquad + 2\xi_n \left\langle f(x_n) - x^*, x_{n+1} - x^* \right\rangle \\
&\quad \le (1-\xi_n)^2 \left\| x_n - x^* \right\|^2 + 2\xi_n \left\langle f(x_n) - f\left(x^*\right), x_{n+1} - x^* \right\rangle + 2\xi_n \left\langle f\left(x^*\right) - x^*, x_{n+1} - x^* \right\rangle \\
&\quad \le (1-\xi_n)^2 \left\| x_n - x^* \right\|^2 + 2\xi_n k \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\| + 2\xi_n \left\langle f\left(x^*\right) - x^*, x_{n+1} - x^* \right\rangle \\
&\quad \le (1-\xi_n)^2 \left\| x_n - x^* \right\|^2 + \xi_n k \left\{ \left\| x_{n+1} - x^* \right\|^2 + \left\| x_n - x^* \right\|^2 \right\} \\
&\qquad + 2\xi_n \left\langle f\left(x^*\right) - x^*, x_{n+1} - x^* \right\rangle.
\end{aligned}$$

Simplifying it, we have

$$\begin{aligned}
\left\| x_{n+1} - x^* \right\|^2 &\le \frac{(1-\xi_n)^2 + \xi_n k}{1 - \xi_n k} \left\| x_n - x^* \right\|^2 + \frac{2\xi_n}{1 - \xi_n k} \left\langle f\left(x^*\right) - x^*, x_{n+1} - x^* \right\rangle \\
&\le \frac{1 - 2\xi_n + \xi_n k}{1 - \xi_n k} \left\| x_n - x^* \right\|^2 + \frac{\xi_n^2}{1 - \xi_n k} \left\| x_n - x^* \right\|^2 \\
&\quad + \frac{2\xi_n}{1 - \xi_n k} \left\langle f\left(x^*\right) - x^*, x_{n+1} - x^* \right\rangle \\
&\le (1 - \eta_n) \left\| x_n - x^* \right\|^2 + \eta_n \delta_n, \quad \forall n \ge 0,
\end{aligned}$$

where $\delta_n = \frac{\xi_n M}{2(1-k)} + \frac{1}{1-k}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$, $M = \sup_{n\geq 0} \|x_n - x^*\|^2$, and $\eta_n = \frac{2(1-k)\xi_n}{1-\xi_n k}$. It is easy to see that $\eta_n \to 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence by Lemma 2.4, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then, by Lemma 2.3, we can define a sequence of positive integers: $\{\tau(n)\}$, $n \geq n_0$ (where n_0 large enough) by

$$\tau(n) = \max\{k \leq n : \|x_k - x^*\| \leq \|x_{k+1} - x^*\|\}. \tag{3.11}$$

Clearly $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq n_0$

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.12}$$

Therefore $\{\|x_{\tau(n)} - x^*\|\}$ is a nondecreasing sequence. According to Case (1), $\lim_{n\to\infty} \|x_{\tau(n)} - x^*\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - x^*\| = 0$. Hence we have

$$0 \leq \|x_n - x^*\| \leq \max\{\|x_n - x^*\|, \|x_{\tau(n)} - x^*\|\} \leq \|x_{\tau(n)+1} - x^*\| \to 0, \quad \text{as } n \to \infty.$$

This implies that $x_n \to x^*$ and $x^* = P_\Omega f(x^*)$ is a solution of (GSVIP) (1.4).

This completes the proof of Theorem 3.2. □

In Theorem 3.2, if $B_i = B$ and $K_i = K$, for each $i \geq 1$, where $B : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ are two set-valued maximal monotone mappings, then from Theorem 3.2 we have the following.

Theorem 3.3 *Let H_1, H_2, A, A^*, B, K, Ω, f be the same as in Theorem* 3.2. *Let $\{\alpha_n\}$, $\{\xi_n\}$, $\{\gamma_n\}$ be the sequence in* $(0,1)$ *with $\alpha_n + \xi_n + \gamma_n = 1$ for each $n \geq 0$. Let $\beta > 0$ be any given positive number, and $\{\lambda_n\}$ be a sequence in $(0, \frac{2}{\|A\|^2})$. Let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n J_\beta^B[x_n - \lambda_n A^*(I - J_\beta^K)Ax_n], \quad \forall n \geq 0. \tag{3.13}$$

If $\Omega \neq \emptyset$ and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \xi_n = 0$, *and* $\sum_{n=0}^{\infty} \xi_n = \infty$;
(ii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$,

then $x_n \to x^* \in \Omega$ where $x^* = P_\Omega f(x^*)$.

4 Applications

In this section we shall utilize the results presented in Theorem 3.2 and Theorem 3.3 to study some problems.

4.1 Application to split optimization problem

Let H_1 and H_2 be two real Hilbert spaces. Let $h : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ be two proper, convex and lower semicontinuous functions, and $A : H_1 \to H_2$ be a linear and bounded operators. The so-called *split optimization problem* (SOP) is

$$\text{to find } x^* \in H_1 \text{ such that } h(x^*) = \min_{y\in H_1} h(y) \text{ and } g(Ax^*) = \min_{z\in H_2} g(z). \tag{4.1}$$

Denote by $\partial h = B$ and $\partial g = K$. It is know that $B : H_1 \to 2^{H_1}$ (resp. $K : H_2 \to 2^{H_2}$) is a maximal monotone mapping, so we can define the resolvent $J_\beta^B = (I + \beta B)^{-1}$ and $J_\beta^K = (I + \beta K)^{-1}$, where $\beta > 0$. Since x^* and Ax^* is a minimum of h on H_1 and g on H_2, respectively, for any given $\beta > 0$, we have

$$x^* \in B^{-1}(0) = F\bigl(J_\beta^B\bigr), \quad \text{and} \quad Ax^* \in K^{-1}(0) = F\bigl(J_\beta^K\bigr). \tag{4.2}$$

This implies that the (SOP) (4.1) is equivalent to the split variational inclusion problem (SVIP) (4.2). From Theorem 3.3 we have the following.

Theorem 4.1 *Let H_1, H_2, A, B, K, h, g be the same as above. Let f, $\{\alpha_n\}$, $\{\xi_n\}$, $\{\gamma_n\}$ be the same as in Theorem* 3.3. *Let $\beta > 0$ be any given positive number, and $\{\lambda_n\}$ be a sequence in $(0, \frac{2}{\|A\|^2})$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H_1$*

$$\begin{cases} y_n = \operatorname{argmin}_{z\in H_2}\{g(z) + \frac{1}{2\beta}\|z - Ax_n\|^2\}, \\ z_n = x_n - \lambda_n A^*(Ax_n - y_n), \\ w_n = \operatorname{argmin}_{y\in H_1}\{h(y) + \frac{1}{2\beta}\|y - z_n\|^2\}, \\ x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n w_n, \quad n \geq 0. \end{cases} \tag{4.3}$$

If $\Omega_1 \neq \emptyset$, the solution set of the split optimization problem (4.1), *and the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \xi_n = 0$, *and* $\sum_{n=0}^\infty \xi_n = \infty$;
(ii) $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$,

then $x_n \to x^* \in \Omega_1$ where $x^* = P_{\Omega_1} f(x^*)$.

Proof Since $\partial h = B$, $\partial g := K$, and $y_n = \operatorname{argmin}_{z\in H_2}\{g(z) + \frac{1}{2\beta}\|z - Ax_n\|^2\}$, we have

$$0 \in \left[K(z) + \frac{1}{\beta}(z - Ax_n)\right]_{z=y_n}, \quad \text{i.e.,}\ Ax_n \in (\beta K + I)(y_n).$$

This implies that

$$y_n = J_\beta^K(Ax_n). \tag{4.4}$$

Similarly, from (4.3), we have

$$w_n = J_\beta^B(z_n). \tag{4.5}$$

From (4.3)-(4.5), we have

$$w_n = J_\beta^B\bigl(x_n - \lambda_n A^*\bigl(I - J_\beta^K\bigr)Ax_n\bigr). \tag{4.6}$$

Therefore (4.3) can be rewritten as

$$x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n J_\beta^B\bigl(x_n - \lambda_n A^*\bigl(I - J_\beta^K\bigr)Ax_n\bigr), \quad n \geq 0. \tag{4.7}$$

The conclusion of Theorem 4.1 can be obtained from Theorem 3.3 immediately. □

4.2 Application to split feasibility problem

Let $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex subsets and $A : H_1 \to H_2$ be a bounded linear operator. Now we consider the following *split feasibility problem*, *i.e.*: to find

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{4.8}$$

Let i_C and i_Q be the indicator functions of C and Q defined by (1.9). Let $N_C(u)$ be the *normal cone at* $u \in H_1$ defined by

$$N_C(u) = \{z \in H_1 : \langle z, v - u\rangle \le 0, \forall v \in C\}.$$

Since i_C and i_Q both are proper convex and lower semicontinuous functions on H_1 and H_2, respectively, and the subdifferential ∂i_C of i_C (resp. ∂i_Q of i_Q) is a maximal monotone operator, we can define the resolvents $J_\beta^{\partial i_C}$ of ∂i_C and $J_\beta^{\partial i_Q}$ of ∂i_Q by

$$J_\beta^{\partial i_C}(x) = (I + \beta \partial i_C)^{-1}(x), \quad \forall x \in H_1,$$
$$J_\beta^{\partial i_Q}(x) = (I + \beta \partial i_Q)^{-1}(x), \quad \forall x \in H_2,$$

where $\beta > 0$. By definition, we know that

$$\begin{aligned}\partial i_C(x) &= \{z \in H_1 : i_C(x) + \langle z, y - x\rangle \le i_C(y), \forall y \in H_1\} \\ &= \{z \in H_1 : \langle z, y - x\rangle \le 0, \forall y \in C\} = N_C(x), \quad x \in C.\end{aligned}$$

Hence, for each $\beta > 0$, we have

$$\begin{aligned}u = J_\beta^{\partial i_C}(x) \quad &\Leftrightarrow \quad x - u \in \beta N_C(u) \\ \Leftrightarrow \quad \langle x - u, y - u\rangle \le 0, \quad \forall y \in C \quad &\Leftrightarrow \quad u = P_C(x).\end{aligned}$$

This implies that $J_\beta^{\partial i_C} = P_C$. Similarly $J_\beta^{\partial i_Q} = P_Q$. Taking $h(x) = i_C(x)$ and $g(x) = i_Q(x)$ in (4.1), then the (SFP) (4.8) is equivalent to the following split optimization problem:

$$\text{to find } x^* \in H_1 \text{ such that } i_C(x^*) = \min_{y \in H_1} i_C(y) \text{ and } i_Q(Ax^*) = \min_{z \in H_2} i_Q(z). \tag{4.9}$$

Hence, the following result can be obtained from Theorem 4.1 immediately.

Theorem 4.2 *Let* H_1, H_2, A, A^*, i_C, i_Q *be the same as above. Let* f, $\{\alpha_n\}$, $\{\xi_n\}$, $\{\gamma_n\}$ *be the same as in Theorem* 4.1. *Let* $\{\lambda_n\}$ *be a sequence in* $(0, \frac{2}{\|A\|^2})$. *Let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n P_C[x_n - \lambda_n A^*(I - P_Q)Ax_n], \quad \forall n \ge 0. \tag{4.10}$$

If the solution set of the split optimization problem (4.4) $\Omega_2 \neq \emptyset$, *and the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \xi_n = 0$, *and* $\sum_{n=0}^{\infty} \xi_n = \infty$;
(ii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$,

then $x_n \to x^* \in \Omega_2$ *where* $x^* = P_{\Omega_2} f(x^*)$.

Remark 4.3 Theorem 4.2 extends and improves the main results in Censor and Elfving [1] and Byrne [2].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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