

RESEARCH

Open Access

Approximating common fixed points in hyperbolic spaces

Hafiz Fukhar-ud-din^{1,2} and Mohamed Amine Khamsi^{3*}

*Correspondence:

mohamed@math.utep.edu

³Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968, USA
Full list of author information is available at the end of the article**Abstract**

We establish strong convergence and Δ -convergence theorems of an iteration scheme associated to a pair of nonexpansive mappings on a nonlinear domain. In particular we prove that such a scheme converges to a common fixed point of both mappings. Our results are a generalization of well-known similar results in the linear setting. In particular, we avoid assumptions such as smoothness of the norm, necessary in the linear case.

MSC: Primary 47H09; secondary 46B20; 47H10; 47E10**Keywords:** Δ -convergence; fixed point; Ishikawa iterations; nonexpansive mapping; strong convergence; uniformly convex hyperbolic space

1 Introduction

Let C be a nonempty subset of a metric space (X, d) and $T : C \rightarrow C$ be a mapping. Denote the set of fixed points of T by $F(T)$. Then T is (i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x, y \in C$ (ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $d(Tx, y) \leq d(x, y)$ for $x \in C$ and $y \in F(T)$. For an initial value $x_1 \in C$, Das and Debata [1] studied the strong convergence of Ishikawa iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S(\beta_n Tx_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n \quad (1.1)$$

for two quasi-nonexpansive mappings S, T on a nonempty closed and convex subset of a strictly convex Banach space. Takahashi and Tamura [2] proved weak convergence of (1.1) to a common fixed point of two nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable and strong convergence in a strictly convex Banach space (see also [3, 4]). Mann and Ishikawa iterative procedures are well-defined in a vector space through its built-in convexity. In the literature, several mathematicians have introduced the notion of convexity in metric spaces; for example [5–8]. In this work, we follow the original metric convexity introduced by Menger [9] and used by many authors like Kirk [5, 6] and Takahashi [8]. Note that Mann iterative procedures were also investigated in hyperbolic metric spaces [10, 11].

In this paper we investigate the results published in [2] and generalize them to uniformly convex hyperbolic spaces. A particular example of such metric spaces is the class of CAT(0)-spaces (in the sense of Gromov) and \mathbb{R} -trees (in the sense of Tits). Heavy use of the linear structure of Banach spaces in [2] presents some difficulties when extending

these results to metric spaces. For example, a key assumption in many of their results is the smoothness of the norm which is hard to extend to metric spaces.

2 Menger convexity in metric spaces

Let (X, d) be a metric space. Assume that for any x and y in X , there exists a unique metric segment $[x, y]$, which is an isometric copy of the real line interval $[0, d(x, y)]$. Note by \mathcal{F} the family of the metric segments in X . For any $\beta \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - \beta)d(x, y) \quad \text{and} \quad d(z, y) = \beta d(x, y).$$

Throughout this paper we will denote such point by $\beta x \oplus (1 - \beta)y$. Such metric spaces are usually called *convex metric spaces* [9]. Moreover, if we have

$$d(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y) \leq \alpha d(p, q) + (1 - \alpha)d(x, y)$$

for all p, q, x, y in X and $\alpha \in [0, 1]$, then X is said to be a *hyperbolic metric space* (see [11–13]). For $q = y$, the hyperbolic inequality reduces to the convex structure inequality [8]. Throughout this paper, we will assume

$$\alpha x \oplus (1 - \alpha)y = (1 - \alpha)y \oplus \alpha x$$

for any $\alpha \in [0, 1]$ and any $x, y \in X$.

An example of hyperbolic spaces is the family of Banach vector spaces or any normed vector spaces. Hadamard manifolds [14], the Hilbert open unit ball equipped with the hyperbolic metric [15], and the CAT(0) spaces [6, 16–20] (see Example 2.1) are examples of nonlinear structures which play a major role in recent research in metric fixed point theory. A subset C of a hyperbolic space X is said to be convex if $[x, y] \subset C$, whenever $x, y \in C$ (see also [21]).

Definition 2.1 [22, 23] Let (X, d) be a hyperbolic metric space. For any $r > 0$ and $\varepsilon > 0$, set

$$\delta(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} d \left(\frac{1}{2}x \oplus \frac{1}{2}y, a \right); d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon \right\}$$

for any $a \in X$. X is said to be uniformly convex whenever $\delta(r, \varepsilon) > 0$, for any $r > 0$ and $\varepsilon > 0$.

Throughout this paper we assume that if X is a uniformly convex hyperbolic space, then for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0 \quad \text{for any } r > s.$$

Remark 2.1

- (i) We have $\delta(r, 0) = 0$. Moreover, $\delta(r, \varepsilon)$ is an increasing function of ε .
- (ii) For $r_1 \leq r_2$, we have

$$1 - \frac{r_2}{r_1} \left(1 - \delta \left(r_2, \varepsilon \frac{r_1}{r_2} \right) \right) \leq \delta(r_1, \varepsilon).$$

Next we give a very important example of uniformly convex hyperbolic metric space.

Example 2.1 [16] Let (X, d) be a metric space. A *geodesic* from x to y in X is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y . The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x, y]$, and called the segment joining x to y .

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space X consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [18]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a CAT(0) space, then the CAT(0) inequality implies

$$d^2\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

The above inequality is known as the (CN) inequality of Bruhat and Tits [24]. The (CN) inequality implies that CAT(0) spaces are uniformly convex with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

In a hyperbolic space X , (1.1) is written as

$$x_{n+1} = \alpha_n S(\beta_n T x_n \oplus (1 - \beta_n)x_n) \oplus (1 - \alpha_n)x_n, \tag{2.1}$$

where $\alpha_n, \beta_n \in [0, 1]$. If $S = T$ in (2.1), it reduces to the following Ishikawa iteration process of one mapping:

$$x_{n+1} = \alpha_n T(\beta_n T x_n \oplus (1 - \beta_n)x_n, n \geq 1) \oplus (1 - \alpha_n)x_n, \tag{2.2}$$

where $\alpha_n, \beta_n \in [0, 1]$. Let $\{x_n\}$ be a bounded sequence in a metric space X and C be a nonempty subset. Define $r(\cdot, \{x_n\}) : C \rightarrow [0, \infty)$, by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius ρ_C of $\{x_n\}$ with respect to C is given by

$$\rho_C = \inf\{r(x, \{x_n\}) : x \in C\}.$$

ρ will denote the asymptotic radius of $\{x_n\}$ with respect to X . A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to C if $r(\xi, \{x_n\}) = r(C, \{x_n\}) = \min\{r(x, \{x_n\}) : x \in C\}$. We denote with $A(C, \{x_n\})$, the set of asymptotic centers of $\{x_n\}$ with respect to C . When $C = X$, we call ξ an asymptotic center of $\{x_n\}$ and we use the notation $A(\{x_n\})$ instead of $A(X, \{x_n\})$. In general, the set $A(C, \{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or may even contain infinitely many points. Note that in the study of the geometry of Banach spaces, the function $r(\cdot, \{x_n\})$ is also known as a type. For more on types in metric spaces, we refer to [25].

The Δ -convergence, introduced independently several years ago by Kuczumow [26] and Lim [27], is shown in CAT(0) spaces to behave similarly as the weak convergence in Banach spaces.

Definition 2.2 A bounded sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \xrightarrow{\Delta} x$ ($\{x_n\}$ Δ -converges to x).

In this paper, we study the iteration schemes (2.1)-(2.2) for nonexpansive mappings. We study strong convergence of these iterates in strictly convex hyperbolic spaces and prove Δ -convergence results in uniformly convex hyperbolic spaces without requiring any condition similar to norm Fréchet differentiability.

In the sequel, the following results will be needed.

Lemma 2.1 [25, 28] *Let X be a hyperbolic metric spaces. Assume that X is uniformly convex. Let C be a nonempty, closed and convex subset of X . Then every bounded sequence $\{x_n\} \in X$ has a unique asymptotic center with respect to C .*

Lemma 2.2 [25, 28] *Let X be a hyperbolic metric spaces. Assume that X is uniformly convex. Let C be a nonempty, closed and convex subset of X . Let C be a nonempty closed and convex subset of X , and $\{x_n\}$ be a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is a sequence in C such that $\lim_{m \rightarrow \infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m \rightarrow \infty} y_m = y$.*

The following lemma [29] will be useful in studying the sequence generated by (2.1) in uniformly convex metric spaces. Here we give a proof based on the ideas developed in [25].

Lemma 2.3 *Let X be a uniformly convex hyperbolic space. Then for arbitrary positive numbers $\varepsilon > 0$ and $r > 0$, and $\alpha \in [0, 1]$, we have*

$$d(a, \alpha x \oplus (1 - \alpha)y) \leq r(1 - \delta(r, 2 \min\{\alpha, 1 - \alpha\}\varepsilon))$$

for all $a, x, y \in X$, such that $d(z, x) \leq r$, $d(z, y) \leq r$, and $d(x, y) \geq r\varepsilon$.

Proof Without loss of generality, we may assume $\alpha < \frac{1}{2}$. In this case, we have $\min\{\alpha, 1 - \alpha\} = \alpha$. Let $a \in X$ be fixed and $x, y \in X$. Set $\bar{x} = 2\alpha x \oplus (1 - 2\alpha)y$. Since

$$d\left(\frac{1}{2}\bar{x} \oplus \frac{1}{2}y, x\right) \leq (1 - \alpha)d(x, y) \quad \text{and} \quad d\left(\frac{1}{2}\bar{x} \oplus \frac{1}{2}y, y\right) = \alpha d(x, y),$$

the uniform convexity of X will imply $\frac{1}{2}\bar{x} \oplus \frac{1}{2}y = \alpha x \oplus (1 - \alpha)y$. Using the uniform convexity of X , we get

$$d\left(a, \frac{1}{2}\bar{x} \oplus \frac{1}{2}y\right) \leq r(1 - \delta(r, 2\alpha\varepsilon)).$$

Hence

$$d(a, \alpha x \oplus (1 - \alpha)y) \leq r(1 - \delta(r, 2 \min\{\alpha, 1 - \alpha\}\varepsilon)). \quad \square$$

Remark 2.2 If (X, d) is uniformly convex, then (X, d) is strictly convex, i.e., whenever

$$d(\alpha x \oplus (1 - \alpha)y, a) = d(x, a) = d(y, a)$$

for $\alpha \in (0, 1)$ and any $x, y, a \in X$, then we must have $x = y$.

The following result is very useful.

Lemma 2.4 [25] *Let (X, d) be a uniformly convex hyperbolic space. Let $R \in [0, +\infty)$ be such that*

$$\limsup_{n \rightarrow \infty} d(x_n, a) \leq R, \quad \limsup_{n \rightarrow \infty} d(y_n, a) \leq R \quad \text{and} \quad \lim_{n \rightarrow \infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = R.$$

Then we have

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

But since we use convex combinations other than the middle point, we will need the following generalization obtained by using Lemma 2.3.

Lemma 2.5 *Let (X, d) be a uniformly convex hyperbolic space. Let $R \in [0, +\infty)$ be such that $\limsup_{n \rightarrow \infty} d(x_n, a) \leq R$, $\limsup_{n \rightarrow \infty} d(y_n, a) \leq R$, and*

$$\lim_{n \rightarrow \infty} d(a, \alpha_n x_n \oplus (1 - \alpha_n)y_n) = R,$$

where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then we have

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

A subset C of a metric space X is Chebyshev if for every $x \in X$, there exists $c_0 \in C$ such that $d(c_0, x) < d(c, x)$ for all $c \in C$, $c \neq c_0$. In other words, for each point of the space, there is a well-defined nearest point of C . We can then define the nearest point projection $P : X \rightarrow C$ by sending x to c_0 . We have the following result.

Lemma 2.6 [25] *Let (X, d) be a complete uniformly convex hyperbolic space. Let C be nonempty, convex and closed subset of X . Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C , i.e., there exists a unique $c_0 \in C$ such that*

$$d(x, c_0) = d(x, C) = \inf\{d(x, c); c \in C\},$$

i.e., C is Chebyshev.

3 Convergence in strictly convex hyperbolic space

Let (X, d) be a hyperbolic metric space. Let C be a nonempty closed convex subset of X . Let $S, T : C \rightarrow C$ be two nonexpansive mappings. Throughout the paper, assume that $F = F(S) \cap F(T)$. Let $x_1 \in C$ and $p \in F$ (assuming F is not empty). Set $r = d(x_1, p)$. Then

$$C(x_1) = C \cap B(p, r) = \{x \in C; d(p, x) \leq r\}$$

is nonempty and invariant by both S and T . Therefore one may always assume that C is bounded provided that S and T have a common fixed point. Moreover, if $\{x_n\}$ is the sequence generated by (2.1), then we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n S y_n \oplus (1 - \alpha_n) x_n, p) \\ &\leq \alpha_n d(S y_n, p) + (1 - \alpha_n) d(x_n, p) \\ &\leq \alpha_n d(y_n, p) + (1 - \alpha_n) d(x_n, p) \\ &= \alpha_n d(\beta_n T x_n \oplus (1 - \beta_n) x_n, p) + (1 - \alpha_n) d(x_n, p) \\ &\leq \alpha_n [\beta_n d(T x_n, p) + (1 - \beta_n) d(x_n, p)] + (1 - \alpha_n) d(x_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

where $y_n = \beta_n T x_n \oplus (1 - \beta_n) x_n$. This proves that $\{d(x_n, p)\}$ is decreasing, which implies that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. Using the above inequalities, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} d(x_n, p) &= \lim_{n \rightarrow \infty} d(\alpha_n S y_n \oplus (1 - \alpha_n) x_n, p) \\ &= \lim_{n \rightarrow \infty} \alpha_n d(S y_n, p) + (1 - \alpha_n) d(x_n, p) \\ &= \lim_{n \rightarrow \infty} \alpha_n d(y_n, p) + (1 - \alpha_n) d(x_n, p) \\ &= \lim_{n \rightarrow \infty} \alpha_n d(\beta_n T x_n \oplus (1 - \beta_n) x_n, p) + (1 - \alpha_n) d(x_n, p) \\ &= \lim_{n \rightarrow \infty} \alpha_n [\beta_n d(T x_n, p) + (1 - \beta_n) d(x_n, p)] + (1 - \alpha_n) d(x_n, p). \end{aligned}$$

The first result of this work discusses the convergence behavior of the sequence generated by (2.1).

Theorem 3.1 *Let X be a strictly convex hyperbolic space. Let C be a nonempty bounded, closed and convex subset of X . Let $S, T : C \rightarrow C$ be two nonexpansive mappings. Assume that $F \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be given by (2.1). Then the following hold:*

- (i) *if $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$, with $0 < a \leq b < 1$, then $x_{n_i} \rightarrow y$ implies $y \in F(S)$;*
- (ii) *if $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_{n_i} \rightarrow y$ implies $y \in F(T)$;*
- (iii) *if $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_{n_i} \rightarrow y$ implies $y \in F$. In this case, we have $x_n \rightarrow y$.*

Proof Assume that $x_{n_i} \rightarrow y$. Let $p \in F$. Without loss of generality, we may assume $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha$ and $\lim_{n \rightarrow \infty} \beta_{n_i} = \beta$. Since $\{d(x_n, p)\}$ is decreasing, we get

$$\lim_{n_i \rightarrow \infty} d(x_n, p) = \lim_{n_i \rightarrow \infty} d(x_{n_i}, p) = d(y, p).$$

The above inequalities imply

$$\begin{aligned} d(y, p) &= d(\alpha S(\beta Ty \oplus (1 - \beta)y) \oplus (1 - \alpha)y, p) \\ &= \alpha d(S(\beta Ty \oplus (1 - \beta)y), p) + (1 - \alpha)d(y, p) \\ &= \alpha d(\beta Ty \oplus (1 - \beta)y, p) + (1 - \alpha)d(y, p) \\ &= \alpha[\beta d(Ty, p) + (1 - \beta)d(y, p)] + (1 - \alpha)d(y, p). \end{aligned}$$

Set $r = d(y, p)$. Without loss of generality we may assume $r > 0$ otherwise most of the conclusions in the theorem are trivial. Assume that $\liminf_{n \rightarrow \infty} \alpha_n > 0$. Then $\alpha \neq 0$. Hence

$$d(S(\beta Ty \oplus (1 - \beta)y), p) = d(\beta Ty \oplus (1 - \beta)y, p) = \beta d(Ty, p) + (1 - \beta)r = r,$$

which implies $\beta d(Ty, p) = \beta r$. If we assume that $\liminf_{n \rightarrow \infty} \beta_n > 0$, then $\beta \neq 0$, which implies $d(Ty, p) = r$.

(1) If $\alpha \in (0, 1)$ and $\beta > 0$, then

$$d(p, S(\beta Ty \oplus (1 - \beta)y)) = d(\alpha S(\beta Ty \oplus (1 - \beta)y) \oplus (1 - \alpha)y, p) = r.$$

The strict convexity of X will imply $S(\beta Ty \oplus (1 - \beta)y) = y$.

(2) If $\alpha \in (0, 1)$ and $\beta = 0$, then

$$d(p, y) = d(p, S(y)) = d(\alpha S(y) \oplus (1 - \alpha)y, p).$$

The strict convexity of X will imply $S(y) = y$.

(3) If $\beta \in (0, 1)$ and $\alpha > 0$, then

$$d(p, y) = d(p, T(y)) = d(p, \beta Ty \oplus (1 - \beta)y).$$

The strict convexity of X will imply $T(y) = y$.

(4) If $\alpha, \beta \in (0, 1)$, then $T(y) = y$ and $S(\beta Ty \oplus (1 - \beta)y) = y$. Hence $S(y) = y$.

Let us finish the proof of Theorem 3.1. Note that (i) implies $\alpha \in [a, b]$ and $\beta \in [0, b]$. If $\beta = 0$, then the conclusion (2) above implies $y \in F(S)$. Otherwise the conclusion (4) will imply $y \in F$. This proves (i).

For (ii), notice that $\alpha \in [a, 1]$ and $\beta \in [a, b]$. Hence the conclusion (3) will imply $y \in F(T)$ which proves (ii).

For (iii), notice that $\alpha, \beta \in [a, b]$. Hence the conclusion (4) will imply $y \in F(T)$. Since

$$\lim_{n \rightarrow \infty} d(y_n, y) = \lim_{n \rightarrow \infty} d(y_{n_i}, y) = 0,$$

we get $x_n \rightarrow y$, which completes the proof of (iii). □

If we assume compactness, Theorem 3.1 will imply the following result.

Theorem 3.2 *Let X be a strictly convex hyperbolic space. Let C be a nonempty bounded, closed and convex subset of X . Let $S, T : C \rightarrow C$ be two nonexpansive mappings. Assume*

that $F \neq \emptyset$. Fix $x_1 \in C$. Assume that $\overline{c\bar{o}}\{x_1\} \cup S(C) \cup T(C)$ is a compact subset of C . Define $\{x_n\}$ as in (2.1) where $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, and x_1 is the initial element of the sequence. Then $\{x_n\}$ converges strongly to a common fixed point of S and T .

Proof We have $x_n \in \overline{c\bar{o}}\{x_1\} \cup S(C) \cup T(C)$. Since $\overline{c\bar{o}}\{x_1\} \cup S(C) \cup T(C)$ is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$, i.e., $x_{n_i} \rightarrow z$. By Theorem 3.1, we have $z \in F$ and $x_n \rightarrow z$. \square

The existence of a common fixed point T and S is crucial. If one assumes that T and S commute, i.e., $S \circ T = T \circ S$, then a common fixed point exists under the assumptions of Theorem 3.2. Indeed, fix $x_0 \in C$ and define

$$T_n x = \frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) T x$$

for $x \in C$ and $n \geq 1$. Then

$$\begin{aligned} d(T_n x, T_n y) &= d\left(\frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) T x, \frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) T y\right) \\ &\leq \left(1 - \frac{1}{n}\right) d(T x, T y) \\ &\leq \left(1 - \frac{1}{n}\right) d(x, y). \end{aligned}$$

That is, T_n is a contraction. By the Banach contraction principle (BCP), T_n has a unique fixed point u_n in C . Since the closure of $T(C)$ is compact, there exists a subsequence $\{Tu_{n_i}\}$ of $\{Tu_n\}$ such that $Tu_{n_i} \rightarrow u$. Since $T(C)$ is bounded and

$$d(u_n, Tu_n) = d\left(\frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) Tu_n, Tu_n\right) \leq \frac{1}{n} d(x_0, Tu_n),$$

we have $d(u_n, Tu_n) \rightarrow 0$. In particular, we have $u_{n_i} \rightarrow u$. Continuity of T implies $Tu = u$. Since X is strictly convex, then $F(T)$ is a nonempty convex subset of X . Since T and S commute, we have $S(F(T)) \subset F(T)$. Moreover, since the closure of $T(C)$ is compact, we see that $F(T)$ is compact. The above proof shows that S has a fixed point in $F(T)$, i.e., $F = F(S) \cap F(T) \neq \emptyset$.

The case $S = T$ gives the following result.

Theorem 3.3 *Let C be a nonempty closed and convex subset of a complete strictly convex hyperbolic space X . Let $T : C \rightarrow C$ be a nonexpansive mapping such that $\overline{c\bar{o}}\{c_0\} \cup T(C)$ is a compact subset of C , where $c_0 \in C$. Define $\{x_n\}$ by (2.2), where $x_1 = c_0$, $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof We saw that in this case, we have $F(T) \neq \emptyset$. Since $x_n \in \overline{c\bar{o}}\{x_1\} \cup T(C)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow z \in C$. By Theorem 3.1, we have $Tz = z$ and $x_n \rightarrow z$. \square

4 Convergence in uniformly convex hyperbolic spaces

The following result is similar to the well-known demi-closedness principle discovered by Göhde in uniformly convex Banach spaces [30].

Lemma 4.1 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Let $\{x_n\} \in C$ be an approximate fixed point sequence of T , i.e., $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. If $x \in C$ is the asymptotic center of $\{x_n\}$ with respect to C , then x is a fixed point of T , i.e., $x \in F(T)$. In particular, if $\{x_n\} \in C$ is an approximate fixed point sequence of T , such that $x_n \xrightarrow{\Delta} x$, then $x \in F(T)$.*

Proof Let $\{x_n\}$ be an approximate fixed point sequence of T . Let $x \in C$ be the unique asymptotic center of $\{x_n\}$ with respect to C . Since

$$d(Tx, x_n) \leq d(Tx, Tx_n) + d(Tx_n, x_n) \leq d(x, x_n) + d(Tx_n, x_n),$$

we get

$$\begin{aligned} r(Tx, \{x_n\}) &= \limsup_{n \rightarrow \infty} d(Tx, x_n) \\ &\leq \limsup_{n \rightarrow \infty} [d(x, x_n) + d(Tx_n, x_n)] = r(x, \{x_n\}). \end{aligned}$$

By the uniqueness of the asymptotic center, we get $Tx = x$. □

The following theorem is necessary to discuss the behavior of the iterates in (2.1).

Theorem 4.1 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X . Let $S, T : C \rightarrow C$ be nonexpansive mappings such that $F \neq \emptyset$. Fix $x_1 \in C$ and generate $\{x_n\}$ by (2.1). Set*

$$y_n = \beta_n Tx_n \oplus (1 - \beta_n)x_n$$

for any $n \geq 1$.

(i) *If $\alpha_n \in [a, b]$, where $0 < a \leq b < 1$, then*

$$\lim_{n \rightarrow \infty} d(x_n, Sy_n) = 0.$$

(ii) *If $\liminf_{n \rightarrow \infty} \alpha_n > 0$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then*

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

(iii) *If $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, then*

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Proof Let $p \in F$. Then the sequence $\{d(x_n, p)\}$ is decreasing. Set $c = \lim_{n \rightarrow \infty} d(x_n, p)$. If $c = 0$, then all the conclusions are trivial. Therefore we will assume that $c > 0$. Note that

we have

$$d(x_{n+1}, p) \leq \alpha_n d(Sy_n, p) + (1 - \alpha_n) d(x_n, p) \tag{4.1}$$

and

$$d(Sy_n, p) \leq d(y_n, p) \leq \beta_n d(Tx_n, p) + (1 - \beta_n) d(x_n, p) \leq d(x_n, p) \tag{4.2}$$

for any $n \geq 1$. In order to prove (i), assume that $\alpha_n \in [a, b]$, where $0 < a \leq b < 1$. From the inequalities (4.1) and (4.2), we get

$$d(x_{n+1}, p) = d(\alpha_n Sy_n \oplus (1 - \alpha_n)x_n, p) \leq \alpha_n d(Sy_n, p) + (1 - \alpha_n) d(x_n, p) \leq d(x_n, p),$$

which implies $\lim_{n \rightarrow \infty} d(Sy_n, p) = c$. Indeed, let \mathcal{U} be an ultrafilter over \mathbb{N} . Then we have $\lim_{\mathcal{U}} \alpha_n = \alpha \in [a, b]$ and $\lim_{\mathcal{U}} d(x_n, p) = \lim_{\mathcal{U}} d(x_{n+1}, p) = c$. Hence

$$c \leq \alpha \lim_{\mathcal{U}} d(Sy_n, p) + (1 - \alpha)c \leq c.$$

Since $\alpha \neq 0$, we get $\lim_{\mathcal{U}} d(Sy_n, p) = c$. Since \mathcal{U} was an arbitrary ultrafilter, we get $\lim_{n \rightarrow \infty} d(Sy_n, p) = c$ as claimed. Therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, p) = \lim_{n \rightarrow \infty} d(Sy_n, p) = \lim_{n \rightarrow \infty} d(\alpha_n Sy_n \oplus (1 - \alpha_n)x_n, p) = c.$$

Using Lemma 2.5, we get $\lim_{n \rightarrow \infty} d(Sy_n, x_n) = 0$.

Next we prove (ii). Assume $\liminf_{n \rightarrow \infty} \alpha_n > 0$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. First note that from (4.1) and (4.2), we get

$$d(x_{n+1}, p) \leq \alpha_n d(y_n, p) + (1 - \alpha_n) d(x_n, p) \leq d(x_n, p),$$

which implies $\lim_{n \rightarrow \infty} \alpha_n d(y_n, p) + (1 - \alpha_n) d(x_n, p) = c$. Since $\liminf_{n \rightarrow \infty} \alpha_n > 0$, we conclude that $\lim_{n \rightarrow \infty} d(y_n, p) = c$. Since $\beta_n \geq a > 0$, we get in a similar fashion $\lim_{n \rightarrow \infty} d(Tx_n, p) = c$. Therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, p) = \lim_{n \rightarrow \infty} d(Tx_n, p) = \lim_{n \rightarrow \infty} d(\beta_n Tx_n \oplus (1 - \beta_n)x_n, p) = c.$$

Using Lemma 2.5, we get $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$.

Finally let us prove (iii). Assume that $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then from (i) and (ii), we know that

$$\lim_{n \rightarrow \infty} d(x_n, Sy_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Since

$$\begin{aligned} d(x_n, Sx_n) &\leq d(x_n, Sy_n) + d(Sy_n, Sx_n) \\ &\leq d(x_n, Sy_n) + d(y_n, x_n) \end{aligned}$$

$$\begin{aligned} &= d(x_n, Sy_n) + \beta_n d(Tx_n, x_n) \\ &\leq d(x_n, Sy_n) + d(Tx_n, x_n), \end{aligned}$$

we conclude that $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$. □

The conclusion of Theorem 4.1(iii) is amazing because the sequence generated by (2.1) gives an approximate fixed point sequence for both S and T without assuming that these mappings commute.

Remark 4.1 If we assume that $0 \leq \beta_n \leq b < 1$ and $\liminf_{n \rightarrow \infty} \alpha_n > 0$, then

$$\lim_{n \rightarrow \infty} \beta_n d(x_n, Tx_n) = 0.$$

Indeed, if we assume this not to be so, then there exists a subsequence $\{\beta_{n'}\}$ and $\delta > 0$ such that

$$\beta_{n'} d(x_{n'}, Tx_{n'}) \geq \delta$$

for any $n \geq 1$. In particular, it is clear, since $\{d(x_n, Tx_n)\}$ is bounded, that $\lim_{n \rightarrow \infty} \beta_{n'} \neq 0$. Without loss of generality, we may assume that $\beta_{n'} \geq a > 0$, for $n \geq 1$. The proof of (ii) will imply

$$\lim_{n \rightarrow \infty} d(x_{n'}, Tx_{n'}) = 0,$$

which is a contradiction since $\{\beta_n\}$ is a bounded sequence. Therefore we must have

$$\lim_{n \rightarrow \infty} \beta_n d(x_n, Tx_n) = 0.$$

In particular, if we assume $\alpha_n \in [a, b]$, then we get

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0.$$

As a direct consequence to Theorem 4.1 and Remark 4.1, we get the following result which discusses the Δ -convergence of the iterative sequence defined by (2.1).

Theorem 4.2 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X . Let $S, T : C \rightarrow C$ be two nonexpansive mappings such that $F \neq \emptyset$. Fix $x_1 \in C$ and generate $\{x_n\}$ by (2.1). Then*

- (i) *if $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$, with $0 < a \leq b < 1$, then $x_n \xrightarrow{\Delta} y$ and $y \in F(S)$;*
- (ii) *if $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_n \xrightarrow{\Delta} y$ and $y \in F(T)$;*
- (iii) *if $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_n \xrightarrow{\Delta} y$ and $y \in F$.*

Proof Let us prove (i). Assume $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$, with $0 < a \leq b < 1$. Theorem 4.1 and Remark 4.1 imply that $\{x_n\}$ is an approximate fixed point sequence of S , i.e.,

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0.$$

Let y be the unique asymptotic center of $\{x_n\}$. Then Lemma 4.1 implies that $y \in F(S)$. Let us prove that in fact $\{x_n\}$ Δ -converges to y . Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Let z be the unique asymptotic center of $\{x_{n_i}\}$. Again since $\{x_{n_i}\}$ is an approximate fixed point sequence of S , we get $z \in F(S)$. Hence

$$\limsup_{n_i \rightarrow \infty} d(x_{n_i}, z) \leq \limsup_{n_i \rightarrow \infty} d(x_{n_i}, y).$$

Since $y, z \in F(S)$, we get

$$\limsup_{n_i \rightarrow \infty} d(x_{n_i}, z) = \lim_{n \rightarrow \infty} d(x_n, z) \quad \text{and} \quad \limsup_{n_i \rightarrow \infty} d(x_{n_i}, y) = \lim_{n \rightarrow \infty} d(x_n, y).$$

Since y is the unique asymptotic center of $\{x_n\}$, we get $y = z$. This proves that $\{x_n\}$ Δ -converges to y .

Next we prove (ii). Assume $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then Theorem 4.1 implies that $\{x_n\}$ is an approximate fixed point sequence of T , i.e.,

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Following the same proof as given above for (i), we get $\{x_n\}$ Δ -converges to its unique asymptotic center which is a fixed point of T .

The conclusion (iii) follows easily from (i) and (ii). □

As a corollary to Theorem 4.2, we get the following result when $S = T$.

Corollary 4.1 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X . Let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point. Suppose that $\{x_n\}$ is given by (2.2), where $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then $x_n \xrightarrow{\Delta} p$, with $p \in F(T)$.*

Using the concept of near point projection, we establish the following amazing convergence result.

Theorem 4.3 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X . Let $S, T : C \rightarrow C$ be nonexpansive mappings such that $F \neq \emptyset$. Let P be the nearest point projection of C onto F . For an initial value $x_1 \in C$, define $\{x_n\}$ as given in (2.1), where $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then $\{Px_n\}$ converges strongly to the asymptotic center of $\{x_n\}$.*

Proof First, we claim that

$$d(Px_n, x_{n+m}) \leq d(Px_n, x_n) \quad \text{for } m \geq 1, n \geq 1. \tag{4.3}$$

We prove (4.3) by induction on $m \geq 1$. For $m = 1$, we have

$$\begin{aligned} d(Px_n, x_{n+1}) &= d(Px_n, \alpha_n Sy_n \oplus (1 - \alpha_n)x_n) \\ &\leq \alpha_n d(Px_n, Sy_n) + (1 - \alpha_n)d(Px_n, x_n) \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n d(Px_n, y_n) + (1 - \alpha_n) d(Px_n, x_n) \\
 &= \alpha_n d(Px_n, \beta_n Tx_n \oplus (1 - \beta_n)x_n) + (1 - \alpha_n) d(Px_n, x_n) \\
 &\leq \alpha_n [\beta_n d(Px_n, Tx_n) + (1 - \beta_n) d(Px_n, x_n)] + (1 - \alpha_n) d(Px_n, x_n) \\
 &\leq \alpha_n [\beta_n d(Px_n, x_n) + (1 - \beta_n) d(Px_n, x_n)] + (1 - \alpha_n) d(Px_n, x_n) \\
 &= d(Px_n, x_n).
 \end{aligned}$$

That is,

$$d(Px_n, x_{n+1}) \leq d(Px_n, x_n)$$

for $n \geq 1$. Assume that (4.3) is true for $m = k$. That is,

$$d(Px_n, x_{n+k}) \leq d(Px_n, x_n)$$

for $n \geq 1$. Hence

$$\begin{aligned}
 d(Px_n, x_{n+k+1}) &= d(Px_n, \alpha_{n+k} Sy_{n+k} \oplus (1 - \alpha_{n+k})x_{n+k}) \\
 &\leq \alpha_{n+k} d(Px_n, Sy_{n+k}) + (1 - \alpha_{n+k}) d(Px_n, x_{n+k}) \\
 &\leq \alpha_{n+k} d(Px_n, y_{n+k}) + (1 - \alpha_{n+k}) d(Px_n, x_{n+k}) \\
 &= \alpha_{n+k} d(Px_n, \beta_{n+k} Tx_{n+k} \oplus (1 - \beta_{n+k})x_{n+k}) \\
 &\quad + (1 - \alpha_{n+k}) d(Px_n, x_{n+k}) \\
 &\leq \alpha_{n+k} [\beta_{n+k} d(Px_n, Tx_{n+k}) + (1 - \beta_{n+k}) d(Px_n, x_{n+k})] \\
 &\quad + (1 - \alpha_{n+k}) d(Px_n, x_{n+k}) \\
 &\leq \alpha_{n+k} [\beta_{n+k} d(Px_n, x_{n+k}) + (1 - \beta_{n+k}) d(Px_n, x_{n+k})] \\
 &\quad + (1 - \alpha_{n+k}) d(Px_n, x_{n+k}) \\
 &= d(Px_n, x_{n+k}) \\
 &\leq d(Px_n, x_n).
 \end{aligned}$$

This completes the proof of (4.3). Let us complete the proof of Theorem 4.3. We know from Theorem 4.2(iii) that $\{x_n\}$ Δ -converges to its unique asymptotic center y , which is in F . Let us prove that $\{Px_n\}$ converges strongly to y . Assume not, *i.e.*, there exist $\varepsilon > 0$ and a subsequence $\{Px_{n_i}\}$ such that $d(Px_{n_i}, y) \geq \varepsilon$, for any $n_i \geq 1$. It is clear that we must have $R = d(x_1, y) > 0$, otherwise $\{x_n\}$ is a constant sequence. Since

$$\begin{cases}
 d(x_{n_i}, y) \leq d(x_{n_i}, y), \\
 d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y), \\
 d(Px_{n_i}, y) \geq \varepsilon = d(x_{n_i}, y) \frac{\varepsilon}{d(x_{n_i}, y)} \geq d(x_{n_i}, y) \frac{\varepsilon}{R}
 \end{cases}$$

we get

$$d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y) \left(1 - \delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right)\right)$$

for any $n_i \geq 1$. Using the properties of the modulus of uniform convexity, there exists $\eta > 0$ such that

$$\delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right) \geq \eta$$

for any $n_i \geq 1$. Hence

$$d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y)(1 - \eta)$$

for any $n_i \geq 1$. Using the definition of the nearest point projection P , we get

$$d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y)(1 - \eta)$$

for any $n_i \geq 1$. Using the inequality (4.3) above, we get

$$d(x_{n_i+m}, Px_{n_i}) \leq d(x_{n_i}, y)(1 - \eta)$$

for any $n_i \geq 1$ and $m \geq 1$. Since $Px_{n_i} \in F$, we know that $\{d(x_n, Px_{n_i})\}$ is decreasing (in n and fixed n_i). Hence

$$\limsup_{m \rightarrow \infty} d(x_{n_i+m}, Px_{n_i}) = \lim_{n \rightarrow \infty} d(x_n, Px_{n_i}) \leq d(x_{n_i}, y)(1 - \eta)$$

for any $n_i \geq 1$. Since y is the asymptotic center of $\{x_n\}$, we get

$$\lim_{n \rightarrow \infty} d(x_n, y) \leq \lim_{n \rightarrow \infty} d(x_n, Px_{n_i}) \leq d(x_{n_i}, y)(1 - \eta)$$

for any $n_i \geq 1$. Finally since $y \in F$, if we let $n_i \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x_n, y) \leq \lim_{n \rightarrow \infty} d(x_n, y)(1 - \eta).$$

Since $\varepsilon \leq d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y)$, we conclude that $\varepsilon \leq \lim_{n \rightarrow \infty} d(x_n, y)$, which implies $1 \leq 1 - \eta$ which is our desired contradiction. Therefore $\{Px_n\}$ converges strongly to y . \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia. ²Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan. ³Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968, USA.

Acknowledgements

The authors are grateful to King Fahd University of Petroleum and Minerals for supporting research project IN121055.

Received: 14 February 2014 Accepted: 16 April 2014 Published: 08 May 2014

References

1. Das, G, Debata, P: Fixed points of quasi-nonexpansive mappings. *Indian J. Pure Appl. Math.* **17**, 1263-1269 (1986)
2. Takahashi, W, Tamura, T: Convergence theorems for a pair of nonexpansive mappings. *J. Convex Anal.* **5**, 45-56 (1998)
3. Khan, SH, Fukhar-ud-din, H: Weak and strong convergence of a scheme with errors for two nonexpansive mappings. *Nonlinear Anal.* **61**, 1295-1301 (2005)
4. Takahashi, W, Kim, GE: Approximating fixed points of nonexpansive mappings in Banach spaces. *Math. Jpn.* **48**, 1-9 (1998)
5. Kirk, WA: An abstract fixed point theorem for nonexpansive mappings. *Proc. Am. Math. Soc.* **82**, 640-642 (1981)
6. Kirk, WA: Fixed point theory for nonexpansive mappings II. *Contemp. Math.* **18**, 121-140 (1983)
7. Penot, JP: Fixed point theorems without convexity. *Bull. Soc. Math. Fr.* **60**, 129-152 (1979)
8. Takahashi, W: A convexity in metric spaces and nonexpansive mappings. *Kodai Math. Semin. Rep.* **22**, 142-149 (1970)
9. Menger, K: Untersuchungen über allgemeine Metrik. *Math. Ann.* **100**, 75-163 (1928)
10. Borwein, J, Reich, S, Shafrir, I: Krasnoselski-Mann iterations in normed spaces. *Can. Math. Bull.* **35**, 21-28 (1992)
11. Reich, S, Shafrir, I: Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal.* **15**, 537-558 (1990)
12. Khan, AR, Fukhar-ud-din, H, Khan, MAA: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. *Fixed Point Theory Appl.* **2012**, 54 (2012)
13. Kohlenbach, U: Some logical metatheorems with applications in functional analysis. *Trans. Am. Math. Soc.* **357**, 89-128 (2005)
14. Busemann, H: Spaces with non-positive curvature. *Acta Math.* **80**, 259-310 (1948)
15. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Dekker, New York (1984)
16. Kirk, WA: A fixed point theorem in CAT(0) spaces and \mathbb{R} -trees. *Fixed Point Theory Appl.* **2004**(4), 309-316 (2004)
17. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0)-spaces. *J. Math. Anal. Appl.* **325**, 386-399 (2007)
18. Bridson, M, Haefliger, A: Metric Spaces of Non-Positive Curvature. Springer, Berlin (1999)
19. Dhompongsa, S, Panyanak, B: On Δ -convergence theorems in CAT(0) spaces. *Comput. Math. Appl.* **56**, 2572-2579 (2008)
20. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0)-spaces. *Nonlinear Anal.* **74**, 783-791 (2011)
21. Shimizu, T, Takahashi, W: Fixed points of multivalued mappings in certain convex metric spaces. *Topol. Methods Nonlinear Anal.* **8**, 197-203 (1996)
22. Ibn Dehaish, BA, Khamsi, MA, Khan, AR: Mann iteration process for asymptotic pointwise nonexpansive mappings in metric spaces. *J. Math. Anal. Appl.* **397**, 861-868 (2013)
23. Fukhar-ud-din, H, Khan, AR, Akhtar, Z: Fixed point results for a generalized nonexpansive map in uniformly convex metric spaces. *Nonlinear Anal.* **75**, 4747-4760 (2012)
24. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. I. Données radicielles valuées. *Publ. Math. IHES* **41**, 5-251 (1972)
25. Khamsi, MA, Khan, AR: Inequalities in metric spaces with applications. *Nonlinear Anal.* **74**, 4036-4045 (2011)
26. Kuczumow, T: An almost convergence and its applications. *Ann. Univ. Mariae Curie-Skłodowska, Sect. A* **32**, 79-88 (1978)
27. Lim, TC: Remarks on some fixed point theorems. *Proc. Am. Math. Soc.* **60**, 179-182 (1976)
28. Fukhar-ud-din, H: Existence and approximation of fixed points in convex metric spaces. *Carpath. J. Math.* (to appear)
29. Shimizu, T: A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces. In: Proceedings of the International Conference on Nonlinear and Convex Analysis, pp. 575-585 (2005)
30. Göhde, D: Zum Prinzip der kontraktiven Abbildung. *Math. Nachr.* **30**, 251-258 (1965)

10.1186/1687-1812-2014-113

Cite this article as: Fukhar-ud-din and Khamsi: Approximating common fixed points in hyperbolic spaces. *Fixed Point Theory and Applications* 2014, **2014**:113

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com