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Weak compactness and the Eisenfeld-Lakshmikantham measure of nonconvexity

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Abstract

In this article, weakly compact subsets of real Banach spaces are characterized in terms of the Cantor property for the Eisenfeld-Lakshmikantham measure of nonconvexity. This characterization is applied to prove the existence of fixed points for condensing maps, nonexpansive maps, and isometries without convexity requirements on their domain.

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1. Introduction

Throughout this article, $(X, \|\cdot\|)$ will denote a real Banach space.

Definition 1.1. *The Eisenfeld-Lakshmikantham measure of nonconvexity (E-L measure of nonconvexity, for short) of a bounded subset A of X is defined by*

$$\mu(A) = \sup_{x\in\overline{\mathrm{co}}A} \inf_{a\in A} \|x-a\| = H(A, \overline{\mathrm{co}}A),$$

where $\overline{\mathrm{co}}A$ denotes the closed and convex hull of A and $H(C, D)$ is the Hausdorff-Pompeiu distance between the bounded subsets C and D of X.

The E-L measure of nonconvexity was introduced in [1]. The following properties of μ can be derived in a fairly straightforward manner from its definition. Here, $A, B \subset X$ are assumed to be bounded and $\overline{A}$ denotes the closure of A.

(i) $\mu(A) = 0$ if, and only if, $\overline{A}$ is convex.
(ii) $\mu(\lambda A) = |\lambda|\mu(A)$ $(\lambda \in \mathbb{R})$.
(iii) $\mu(A + B) \leq \mu(A)+\mu(B)$.
(iv) $|\mu(A) - \mu(B)| \leq \mu(A - B)$.
(v) $\mu(\overline{A}) = \mu(A)$.
(vi) $\mu(A) \leq \delta(A)$, where

$$\delta(A) = \sup_{x,y\in A} \|x-y\|$$

is the diameter of A.

(vii) $|\mu(A) - \mu(B)| \leq 2H(A, B)$.

The following result was obtained in [2].

Lemma 1.2 ([2, Lemma 2.4]). *Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty, closed, and bounded subsets of a Banach space X with*

$$\lim_{n\to\infty} \mu(A_n) = 0,$$

where μ is the E-L measure of nonconvexity of X, and let $A_\infty = \bigcap_{n=1}^{\infty} A_n$. Then $A_\infty = \bigcap_{n=1}^{\infty} \overline{\text{co}} A_n$.

Definition 1.3. *Let Y be a nonempty and closed subset of the Banach space X. The E-L measure of nonconvexity μ of X is said to have the Cantor property in Y if for every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty, closed, and bounded subsets of Y such that $\lim_{n\to\infty}\mu(A_n) = 0$, the closed and bounded (and, by Lemma 1.2, convex) set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty.*

Theorem 1.4 ([2, Theorem 2.5]). *For a Banach space X, the following statements are equivalent:*

(i) *X is reflexive.*
(ii) *The E-L measure of nonconvexity of X satisfies the Cantor property in X.*

In Section 2 below we prove a result (Theorem 2.1), more general than Theorem 1.4, which characterizes weak compactness also in terms of the Cantor property for the E-L measure of nonconvexity. As an application of this characterization, we show that the convexity requirements can be dropped from the hypotheses of a number of fixed point theorems in [3-5] for condensing maps (see Section 3.1), nonexpansive maps (see Section 3.2) and isometries (see Section 4).

2. A characterization of weak compactness

Theorem 2.1. *Let X be a Banach space with E-L measure of nonconvexity μ, and let C be a nonempty, weakly closed, and bounded subset of X. The following statements are equivalent:*

(i) *C is weakly compact.*
(ii) *The measure μ satisfies the Cantor property in $\overline{\text{co}}C$.*
(iii) *For every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty and closed subsets of $\overline{\text{co}}C$ such that $\lim_{n\to\infty}\mu(A_n) = 0$, the set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty.*

Proof. Part (iii) is just a rephrasement of part (ii).

Suppose (i) holds. By the Krein-Šmulian theorem [6, Theorem V.6.4], $\overline{\text{co}}C$ is weakly compact. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty and closed subsets of $\overline{\text{co}}C$ with $\lim_{n\to\infty} \mu(A_n) = 0$. By Lemma 1.2, $A_\infty = \bigcap_{n=1}^{\infty} \overline{\text{co}} A_n$, where $\{\overline{\text{co}} A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed, and convex subsets of the weakly compact and convex set $\overline{\text{co}}C$. The Šmulian theorem [6, Theorem V.6.2] then allows us to conclude that A_∞ is nonempty.

Conversely, assume (iii). If we take any decreasing sequence $\{C_n\}_{n=1}^{\infty}$ of nonempty, closed, and convex subsets of the bounded and convex set $\overline{\text{co}}C$, then $\mu(C_n) = 0$ ($n \in \mathbb{N}$), and therefore $C_\infty \neq \emptyset$. Appealing again to the Šmulian theorem [6, Theorem V.6.2] we find that the convex set $\overline{\text{co}}C$ is weakly compact. Finally, being a weakly closed subset of $\overline{\text{co}}C$, the set C itself is weakly compact.

Note that Theorem 1.4 can be easily derived from Theorem 2.1. For the sake of completeness, we give a proof of this fact.

Corollary 2.2. *For a Banach space X with E-L measure of nonconvexity μ, the following statements are equivalent:*

(i) *X is reflexive.*
(ii) *The closed unit ball B_X of X is weakly compact.*
(iii) *For every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty and closed subsets of B_X such that $\lim_{n\to\infty} \mu(A_n) = 0$, the set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and convex.*
(iv) *The measure μ satisfies the Cantor property in X.*

Proof. The equivalence of (i) and (ii) is well known [6, Theorem V.4.7]. To see that (ii) and (iii) are equivalent, take $C = B_X$ in Theorem 2.1, bearing in mind that $\overline{\text{co}}C = B_X$. For the proof that (iii) implies (iv), let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty, closed, and bounded subsets of X such that $\lim_{n\to\infty} \mu(A_n) = 0$. Since A_1 is bounded and $\{A_n\}_{n=1}^{\infty}$ is decreasing, there exists $\lambda > 0$ such that

$$B_n = \lambda A_n \subset B_X \quad (n \in \mathbb{N}).$$

Now $\{B_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty, closed, and bounded subsets of B_X with

$$\lim_{n\to\infty} \mu(B_n) = \lambda \lim_{n\to\infty} \mu(A_n) = 0.$$

Therefore $A_\infty = \lambda^{-1} B_\infty \neq \emptyset$, as asserted. Finally, it is apparent that (iv) implies (iii).

3. Fixed points for condensing and nonexpansive maps

Definition 3.1 ([2, Definition 4.3]). *Let Y be a nonempty, closed, and bounded subset of a Banach space X. A map $f: Y \to Y$ is said to have property (C) if $\lim_{n\to\infty} \mu(Y_n) = 0$, where μ is the E-L measure of nonconvexity in X and $\{Y_n\}_{n=1}^{\infty}$ is the decreasing sequence of nonempty, closed, and bounded subsets of X defined by*

$$Y_1 = \overline{f(Y)}, \quad Y_{n+1} = \overline{f(Y_n)} \quad (n \in \mathbb{N}).$$

Proposition 3.2. *Let Y be a nonempty and weakly compact subset of a Banach space X, and let $f: Y \to Y$ be a map with property (C). Then Y contains a nonempty, closed, and convex (hence, weakly compact) set K such that $f(K) \subset K$.*

Proof. Let $\{Y_n\}_{n=1}^{\infty}$ be as above. Since f has property (C), we have

$$\lim_{n\to\infty} \mu(Y_n) = 0.$$

Theorem 2.1 yields that $K = Y_\infty = \bigcap_{n=1}^{\infty} Y_n$ is nonempty, closed, and convex. Clearly, $f(K) \subset K$. Closed convex sets are weakly closed [6, Theorem V.3.18] and therefore K is weakly compact, as claimed.

As an application of Proposition 3.2, some fixed point theorems for condensing and nonexpansive maps will be proved.

3.1. Condensing maps

Definition 3.3. *Let Y be a nonempty and bounded subset of a Banach space X, and let γ denote some measure of noncompactness in X, in the sense of* [7, Definition 3.2]. *A map $f : Y \to Y$ is called γ-condensing provided that*

$$\gamma(f(B)) < \gamma(B)$$

for every $B \subset Y$ with $f(B) \subset B$ and $\gamma(B) > 0$.

The following result is an extension of [3, Theorem 4]. It can be also viewed as a version of Sadovskii's theorem [8].

Theorem 3.4. *Let γ be a measure of noncompactness in a Banach space X and let Y be a nonempty and closed subset of X such that $\overline{co}Y$ is weakly compact. Assume that the map $f:Y \to Y$ is continuous, γ-condensing and has property (C). Then f has at least one fixed point in Y.*

Proof. Arguing as in the proof of Proposition 3.2 we get a nonempty, closed, and convex set $K \subset Y$ such that $f(K) \subset K$. The required conclusion follows from [7, Corollary 3.5].

3.2. Nonexpansive maps

Definition 3.5. *Let $A \subset X$ be bounded. A point $x \in A$ is a diametral point of A provided that $\sup_{y\in A} \|x - y\| = \delta(A)$. The set A is said to have normal structure if for each convex subset B of A containing more than one point, there exists some $x \in B$ which is not a diametral point of B.*

The following is a version of Kirk's seminal theorem (cf. [4, Theorem 4.1]) which does not require the convexity of the domain.

Theorem 3.6. *Let Y be a nonempty and weakly compact subset of a Banach space X. Suppose Y has normal structure. If $f : Y \to Y$ has property (C) and is nonexpansive, that is, satisfies*

$$\|f(x) - f(y)\| \le \|x - y\| \quad (x, y \in Y),$$

then f has a fixed point.

Proof. The asserted conclusion can be derived from Proposition 3.2 and [4, Theorem 4.1].

4. Fixed points for isometries

Definition 4.1. *Let Y be a nonempty and weakly compact subset of a Banach space X. We say that Y has the fixed point property, FPP for short, if every isometry $f:Y \to Y$ has* *a fixed point. The set Y is said to have the hereditary FPP if every nonempty, closed, and convex subset of Y has the FPP.*

Definition 4.2. *Given a nonempty, closed, and bounded subset Y of a Banach space X, let*

$$r(x) = r(x, Y) = \sup_{y \in Y} \|x - y\| \quad (x \in X),$$

$$r(Y) = \inf_{x \in Y} r(x),$$

and

$$\widetilde{Y} = \{x \in Y : r(x) = r(Y)\}.$$

The number r(Y) and the members of $\tilde{Y}$ *are respectively called Chebyshev radius and Chebyshev centers of Y. Further, define*

$$\widetilde{Y}_n = \left\{x \in Y : r(Y) \le r(x) \le r(Y) + \frac{1}{n}\right\}$$
$$= \bigcap_{y \in Y} \left[y + \left(r(Y) + \frac{1}{n}\right) B_X\right] \cap Y \quad (n \in \mathbb{N}).$$

We say that Y has property (S) provided that $\lim_{n\to\infty}\mu(\tilde{Y}_n) = 0$, *where* μ *is the E-L measure of nonconvexity in X.*

Lemma 4.3. *Let Y be a nonempty and weakly compact subset of a Banach space X. If Y has property (S), then* $\tilde{Y}$ *is nonempty, closed, and convex.*

Proof. Note that $\left\{\tilde{Y}_n\right\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty and closed subsets of Y, with $\lim_{n\to\infty}\mu(\tilde{Y}_n) = 0$. From Theorem 2.1, the set of Chebyshev centers

$$\widetilde{Y} = \widetilde{Y}_\infty = \bigcap_{n=1}^{\infty} \widetilde{Y}_n$$

is nonempty closed, and convex.

Theorem 4.4. *Let Y be a nonempty and weakly compact subset of a Banach space X. Assume further that Y has both property (S) and the hereditary FPP. Then every isometry* $f{:}Y \to Y$ *such that* $f(\tilde{Y}) \subset \tilde{Y}$ *has a fixed point in* $\tilde{Y}$.

Proof. From Lemma 4.3, $\tilde{Y}$ is nonempty, closed, and convex. It suffices to invoke the hereditary FPP of Y.

Definition 4.5. *Let Y be a nonempty, closed, and bounded subset of a Banach space X. Given an isometry* $f{:}Y \to Y$, *let us consider*

$$R_{f,0}(x) = r(x, Y) = \sup_{z \in Y} \|x - z\| \quad (x \in X),$$

$$R_{f,m}(x) = r(x, Y_m) = \sup_{z \in Y_m} \|x - z\|$$
$$= r\left(x, f^m(Y)\right) = \sup_{y \in Y} \left\|x - f^m(y)\right\| \quad (x \in X, \quad m \in \mathbb{N}),$$

$$R_f(x) = \lim_{m\to\infty} R_{f,m}(x) = \inf_{m\in\mathbb{Z}_+} R_{f,m}(x) \quad (x \in X),$$
$$R_f(Y) = \inf_{x\in Y} R_f(x),$$

and

$$\widehat{Y}_f = \{x \in Y : R_f(x) = R_f(Y)\}.$$

The number $R_f(Y)$ *and the set* $\hat{Y}_f$ *are respectively called asymptotic Chebyshev radius and asymptotic Chebyshev center of* $\{Y_m\}_{m=0}^{\infty} = \{f^m(Y)\}_{m=0}^{\infty}$ *with respect to Y. Further, define*

$$\widehat{Y}_{f,n} = \left\{x \in Y : R_f(Y) \le R_f(x) \le R_f(Y) + \frac{1}{n}\right\}$$
$$= \bigcap_{m\in\mathbb{Z}_+} \bigcap_{z\in Y_m} \left[z + \left(R_f(Y) + \frac{1}{n}\right) B_X\right] \cap Y \quad (n \in \mathbb{N}).$$

We say that f has property (A) provided that $\lim_{n\to\infty}\mu(\hat{Y}_{f,n}) = 0$, *where* μ *is the E-L measure of nonconvexity in X.*

Lemma 4.6. *Let Y be a nonempty and weakly compact subset of a Banach space X, and let* $f{:}Y \to Y$ *be an isometry with property (A). Then* $\hat{Y}_f$ *is nonempty, closed, and convex.*

Proof. Note that $\{\widehat{Y}_{f,n}\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty and closed subsets of Y, with $\lim_{n\to\infty}\mu\left(\widehat{Y}_{f,n}\right) = 0$. From Theorem 2.1, the asymptotic Chebyshev center

$$\widehat{Y}_f = \widehat{Y}_{f,\infty} = \bigcap_{n=1}^{\infty} \widehat{Y}_{f,n}$$

is nonempty closed, and convex.

Lemma 4.7. *Let Y be a nonempty and weakly compact subset of a Banach space X, and let* $f : Y \to Y$ *be an isometry. Assume* $c \in \hat{Y}_f$ *is such that* $f(c) = c$. *then* $c \in \tilde{Y}$.

Proof. We argue as in the proof of [5, Theorem 2]. Since f is an isometry and $f(c) = c$, we have

$$R_{f,m}(c) = R_{f,m}\left(f(c)\right) = R_{f,m-1}(c) \quad (m \in \mathbb{N}),$$

whence

$$R_{f,m}(c) = R_{f,0}(c) \quad (m \in \mathbb{N}).$$

From Definition 4.5 and the hypothesis that $c \in \hat{Y}_f$, it follows that

$$r(c, Y) = R_{f,0}(c) = \lim_{m\to\infty} R_{f,m}(c) = R_f(c) = R_f(Y).$$

Now, for any $x \in Y$ we get

$$r(c, Y) = R_f(Y) \le \inf_{m\in\mathbb{Z}_+} R_{f,m}(x) \le R_{f,0}(x) = r(x, Y),$$

which proves that $c \in \tilde{Y}$.

Theorem 4.8. *Let Y be a nonempty and weakly compact subset of a Banach space X. Suppose Y has the hereditary FPP. Then every isometry* f:$Y \to Y$ *with property (A) has a fixed point in* $\tilde{Y}$.

Proof. Let $f : Y \to Y$ be an isometry with property (A). From Lemma 4.6, $\hat{Y}_f$ is nonempty, closed, and convex. Moreover, $f(\hat{Y}_f) \subset \hat{Y}_f$ (cf. [5, Proposition 3]). The hereditary FPP of Y then yields $c \in \hat{Y}_f$ such that $f(c) = c$, and Lemma 4.7 ensures that $c \in \tilde{Y}$.

Corollary 4.9 ([5, Theorem 2]). *Let Y be a nonempty, weakly compact, and convex subset of a Banach space X. Suppose Y has the hereditary FPP. Then every isometry* f:$Y \to Y$ *has a fixed point in* $\tilde{Y}$.

Proof. Since Y is convex, every isometry $f : Y \to Y$ has property (A). Theorem 4.8 completes the proof.

The following is an extension of Kirk's theorem [4, Theorem 4.1] for isometries.

Theorem 4.10. *Let Y be a nonempty and weakly compact subset of a Banach space X. Assume further that Y has normal structure. Then every isometry* f:$Y \to Y$ *with property (A) has a fixed point in* $\tilde{Y}$.

Proof. Let $f : Y \to Y$ be an isometry with property (A). From Lemma 4.6, $\hat{Y}_f$ is nonempty, closed, and convex. Moreover, $f(\hat{Y}_f) \subset \hat{Y}_f$ (cf. [5, Proposition 3]). Kirk's theorem [4, Theorem 4.1] along with Lemma 4.7 yield $c \in \tilde{Y}$ such that $f(c) = c$.

Corollary 4.11 ([5, Corollary 1]). *Let Y be a nonempty, weakly compact, and convex subset of a Banach space X. Assume further that Y has normal structure. Then every isometry* f:$Y \to Y$ *has a fixed point in* $\tilde{Y}$.

Proof. The convexity of Y guarantees that every isometry $f : Y \to Y$ satisfies property (A). The desired conclusion follows from Theorem 4.10.

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Competing interests
The author declares that she has no competing interests.

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