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Analytical study of time-fractional Navier-Stokes equation by using transform methods

Kangle Wang* and Sanyang Liu

*Correspondence:
kangle83718@163.com
School of Mathematics and Statistics, XiDian University, Xi'an, 710118, China

Abstract
In this paper, we establish a modified reduced differential transform method and a new iterative Elzaki transform method, which are successfully applied to obtain the analytical solutions of the time-fractional Navier-Stokes equations. The obtained results show that the proposed techniques are simple, efficient, and easy to implement for fractional differential equations.

Keywords: Elzaki transform; fractional Navier-Stokes equation; reduced differential transform method

1 Introduction

In recent years, the fractional differential equations have been used in various fields such as colored noise, electromagnetic waves, boundary layer effects in ducts, viscoelastic mechanics, diffusion processes, and so on [1–5]. However, most fractional differential equations are very difficult to exactly solve, so numerical and approximation techniques have to be used. Recently, many powerful methods have been used to approximate linear and nonlinear fractional differential equations. These methods include the Adomain decomposition method (ADM) [6, 7], the homotopy perturbation method (HPM) [8–12], the variational iteration method (VIM) [13, 14], and so on.

The time-fractional Navier-Stokes equation can be written in operator form as [15, 16]

$$\begin{cases} D_t^\alpha u + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u, \\ \nabla\cdot u = 0, \quad 0 < \alpha \le 1, \end{cases} \tag{1.1}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order α, p is the pressure, ρ is the density, u is the velocity, ν is the kinematic viscosity, and t is the time. When $\alpha = 1$, equation (1.1) is the classical Navier-Stokes equation, the form given by

$$\begin{cases} u_t + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u, \\ \nabla\cdot u = 0. \end{cases} \tag{1.2}$$

In this paper, we consider the unsteady flow of a viscous fluid in a tube, the velocity field is a function of only one space coordinate, the time is a dependent variable. This kind of time-fractional Navier-Stokes equation has been studied by Momani and Odibat [15], Kumar *et al.* [16, 17], and Khan [18] by using the Adomian decomposition method (ADM), the homotopy perturbation transform method (HPTM), the modified Laplace decomposition method (MLDM), the variational iteration method (VIM), and the homotopy perturbation method (HPM), respectively.

In 2006, Daftardar-Gejji and Jafari [19] were first to propose the Gejji-Jafari iteration method for solving a linear and nonlinear fractional differential equation. The Gejji-Jafari iteration method is easy to implement and obtains a highly accurate result. The reduced differential transform method (RDTM) was first proposed by Keskin and Oturanc [20, 21]. The RDTM was also applied by many researchers to handle nonlinear equations arising in science and engineering. In recent years, Kumar *et al.* [22–28] used various methods to study the solutions of linear and nonlinear fractional differential equation combined with a Laplace transform.

Based on the Gejji-Jafari iteration method and RDTM, we established the new iterative Elzaki transform method (NIETM) and the modified reduced differential transform method (MRDTM) with the help of the Elzaki transform [29, 30] and we successfully applied this to time-fractional Navier-Stokes equations. The results show that our proposed methods are efficient and easy to implement with less computation for fractional differential equations.

2 Basic definitions

In this section, we set up notation and review some basic definitions from fractional calculus and Elzaki transforms.

Definition 2.1 A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number p ($p > \mu$), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined as [5]

$$I^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, & \alpha > 0, x > 0, \\ I^0 f(x) = f(x), & \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 2.3 The fractional derivative of $f(x)$ in the Caputo sense is defined as [5]

$$D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \tag{2.2}$$

where $n-1 < \alpha \leq n$, $n \in N$, $x > 0$, $f \in C_{-1}^n$.

The following are the basic properties of the operator D^α:

(1) $D^\alpha I^\alpha f(x) = f(x)$,

(2) $I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x}{k!}, \quad x > 0.$

Definition 2.4 The Elzaki transform is defined over the set of functions $A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k}}, t \in (-1)^j, X \in [0,\infty)\}$ by the following formula [29, 30]:

$$T(s) = E\big[f(t)\big] = s\int_s^{\infty} e^{-\frac{t}{s}} f(t)\,dt, \quad s \in [-k_1, k_2].$$

Lemma 2.1 *The Elzaki transform of the Riemann-Liouville fractional integral is defined as follows* [29, 30]:

$$E\big[I^{\alpha} f(t)\big] = s^{\alpha+1} T(s). \tag{2.3}$$

Lemma 2.2 *The Elzaki transform of the Caputo fractional derivative is given as follows* [29, 30]:

$$E\big[D_x^{n\alpha} u(x,t)\big] = \frac{T(s)}{s^{n\alpha}} - \sum_{k=0}^{n-1} s^{2-n\alpha+k} u^{(k)}(0,t), \quad n-1 < n\alpha \le n. \tag{2.4}$$

3 New iterative Elzaki transform method (NIETM)

Consider an unsteady, one-dimensional motion of a viscous fluid in a tube. The equations of motions which govern the flow field in the tube are the Navier-Stokes equations in cylindrical coordinates and they are given by [15, 16]

$$\begin{cases} u_t = -\frac{1}{\rho} p_z + \nu(u_{rr} + \frac{1}{r} u_r), \\ u(r,0) = f(r). \end{cases} \tag{3.1}$$

If the fractional derivative model is used to present the time derivative term, the equation of motion (3.1) assumes the form

$$\begin{cases} u_t^{\alpha} = P + \nu(u_{rr} + \frac{1}{r} u_r), \quad 0 < \alpha \le 1, \\ u(r,0) = f(r), \end{cases} \tag{3.2}$$

where $P = -\frac{1}{\rho} p_z$.

Applying the Elzaki transform on both sides of equation (3.2), we have

$$E\big[u_t^{\alpha}\big] = E\left[P + \nu\left(u_{rr} + \frac{1}{r} u_r\right)\right]. \tag{3.3}$$

Using the property of the Elzaki transform and the initial condition, we get

$$E\big[u(r,t)\big] = s^2 f(r) + s^{\alpha} E\left[P + \nu\left(u_{rr} + \frac{1}{r} u_r\right)\right]. \tag{3.4}$$

Applying the inverse Elzaki operator on both sides of (3.4), we obtain

$$u(r,t) = E^{-1}\big[s^2 f(r)\big] + E^{-1}\left[s^{\alpha} E\left[P + \nu\left(u_{rr} + \frac{1}{r} u_r\right)\right]\right]. \tag{3.5}$$

Assume

$$\begin{cases} g(r,t) = E^{-1}[s^2 f(r)], \\ N(u(r,t)) = E^{-1}[s^{\alpha} E[P + \nu(u_{rr} + \frac{1}{r} u_r)]]. \end{cases} \tag{3.6}$$

We can obtain

$$u(r,t) = g(r,t) + N\big(u(r,t)\big). \tag{3.7}$$

The solution of equation (3.7) has the series form

$$u(r,t) = \sum_{i=0}^{\infty} u_i(r,t). \tag{3.8}$$

The operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{3.9}$$

According to equations (3.8) and (3.9), equation (3.7) is equivalent to

$$\sum_{i=0}^{\infty} u_i = g + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{3.10}$$

We define the recurrence relation

$$\begin{cases} u_0 = g, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + u_1 + \cdots + u_m) - N(u_0 + u_1 + \cdots + u_{m-1}), \quad m = 1,2,3,\ldots. \end{cases} \tag{3.11}$$

Then

$$(u_1 + u_2 + \cdots + u_{m+1}) = N(u_0 + u_1 + \cdots + u_m), \quad m = 1,2,3,\ldots, \tag{3.12}$$

and

$$\sum_{i=0}^{\infty} u_i = g + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{3.13}$$

The k-term approximate solution of equation (3.7) is given by $u = u_0 + u_1 + u_2 + \cdots + u_{k-1}$.

Similarly, for the proof of the convergence of the NIETM, see [19].

4 Modified reduced differential transform method (MRDTM)

In this section, the basic definition of the modified reduced differential transform method is introduced as in [31, 32].

Definition 4.1 The modified reduced differential transform of $u(x,t)$ at $t = t_0$ is represented as

$$U_k(x) = \frac{1}{\Gamma(k\alpha + 1)}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=t_0}, \tag{4.1}$$

where α is a parameter which describes the order of the time-fractional derivative.

Definition 4.2 The differential inverse transform of $U_k(x,t)$ is represented as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0} t^{k\alpha}. \tag{4.2}$$

According to (4.1) and (4.2), the following theorems can be obtained.

Theorem 4.1 *If* $w(x,t) = u(x,t) \pm v(x,t)$, *then* $MRDT[w(x,t)] = U_k(x) \pm V_k(x)$.

Theorem 4.2 *If* $w(x,t) = \lambda u(x,t)$, *then* $MRDT[w(x,t)] = \lambda U_k(x)$.

Theorem 4.3 *If* $w(x,t) = x^m t^n$, *then* $MRDT[w(x,t)] = x^m \delta(k-n)$, *where*

$$\delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Theorem 4.4 *If* $w(x,t) = u(x,t)v(x,t)$, *then* $MRDT[w(x,t)] = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$.

Theorem 4.5 *If* $w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$, *then* $MRDT[w(x,t)] = \frac{(k+r)!}{k!}\frac{\partial^r}{\partial t^r}U_{k+r}(x)$.

Theorem 4.6 *If* $w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$, *then* $MRDT[w(x,t)] = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x)$.

Theorem 4.7 *If* $w(x,t) = [u(x,t)]^k$, *then*

$$MRDT\big[w(x,t\big] = W_k(x) = \begin{cases} U_0(x), & k = 0, \\ \sum_{n=1}^{k} \frac{(m+1)n-k}{kU_0(x)} U_n(x)W_{k-n}(x), & k \geq 1. \end{cases}$$

Applying the modified reduced differential transform on both sides of equation (3.2), we have

$$MRDT\big[u_t^{\alpha}\big] = MRDT[P] + MRDT\big[v(u_{rr})\big] + MRDT\left[v\left[\left(\frac{1}{r}u_r\right)\right]\right]. \tag{4.3}$$

Using the property of MRDT, we can obtain

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1} = v\left[(U_k)_{rr} + \frac{1}{r}(U_k)_r\right] + P\delta(k-0). \tag{4.4}$$

According to equation (4.4), we have the following result:

$$U_1(x) = \varphi_1(x), \qquad U_2(x) = \varphi_2(x), \qquad U_3(x) = \varphi_3(x),$$
$$U_4(x) = \varphi_4(x), \qquad \ldots.$$

So, we get the solution of equation (3.2) as follows:

$$u(x,t) = U_0(x)t^0 + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \cdots + U_n(x)t^{n\alpha} + \cdots. \tag{4.5}$$

5 Illustrative examples

Example 1 Consider the following time-fractional Navier-Stokes equation:

$$\begin{cases} u_t^{\alpha}(r,t) = P + u_{rr}(r,t) + \frac{1}{r}u_r(r,t), \\ u(r,0) = 1 - r^2. \end{cases} \tag{5.1}$$

5.1 Applying the NIETM

Applying the Elzaki transform on both sides of equation (5.1), we have

$$E\big[u(r,t)\big] = s^2 f(r) + s^{\alpha} E\left[P + u_{rr} + \frac{1}{r}u_r\right]. \tag{5.2}$$

Using the inverse Elzaki transform and the initial condition on both sides of equation (5.2), we get

$$u(r,t) = E^{-1}\big[s^2\big(1-r^2\big)\big] + E^{-1}\left[s^{\alpha} E\left[P + u_{rr} + \frac{1}{r}u_r\right]\right]. \tag{5.3}$$

Assume

$$\begin{cases} g(r,t) = E^{-1}[s^2(1-r^2)], \\ N(u(r,t)) = E^{-1}[s^{\alpha}E[P + u_{rr} + \frac{1}{r}u_r]]. \end{cases} \tag{5.4}$$

According to (3.11), we get the following results:

$$\begin{aligned} u_0 &= 1 - r^2, \\ u_1 &= \frac{(P-4)t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2 &= 0, \\ u_3 &= 0, \\ &\cdots \\ u_n &= 0. \end{aligned}$$

Therefore, the solution of equation (5.1) is

$$\begin{aligned} u(r,t) &= \sum_{i=0}^{\infty} u_i \\ &= 1 - r^2 + \frac{(P-4)t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{5.5}$$

Remark 5.1 The result is the same as ADM, HPTM, HPM, and VIM by Momani and Odibat [15], Kumar *et al.* [16, 17], and Khan [18].

Remark 5.2 When $\alpha = 1$, equation (5.5) is the exact solution of the classical Navier-Stokes equation as follows:

$$u(r,t) = \big(1 - r^2\big) + (P-4)t.$$

5.2 Applying the MRDTM

Applying the MRDT on both sides of equation (5.1), we get

$$MRDT\big[u_t^{\alpha}(r,t)\big] = MRDT[P] + MRDT\big[u_{tt}(r,t)\big] + MRDT\left[\frac{1}{r}u_r(r,t)\right]. \tag{5.6}$$

Using the property of MRDT, we have

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1} = (U_k)_{rr} + \frac{1}{r}(U_k)_r + P\delta(k-0). \tag{5.7}$$

By iteration with $U_0 = 1 - r^2$, we get the result

$$\begin{aligned}
U_0 &= 1 - r^2, \\
U_1 &= \frac{(P-4)}{\Gamma(\alpha+1)}, \\
U_2 &= 0, \\
U_3 &= 0, \\
&\cdots \\
U_n &= 0.
\end{aligned}$$

So, the solution of equation (5.1) is given as

$$\begin{aligned}
u(r,t) &= \sum_{n=0}^{\infty} U_n t^{n\alpha} \\
&= 1 - r^2 + \frac{(P-4)t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{5.8}$$

Example 2 We consider the following time-fractional Navier-Stokes equation:

$$\begin{cases} u_t^{\alpha}(r,t) = u_{rr}(r,t) + \frac{1}{r}u_r(r,t), & 0 < \alpha \leq 1, \\ u(r,0) = r. \end{cases} \tag{5.9}$$

5.3 Applying the NIETM

Applying the Elzaki transform on both sides of equation (5.9), we have

$$E\big[u(r,t)\big] - s^2 u(r,0) - s^{\alpha} E\left[u_{rr} + \frac{1}{r}u_r\right] = 0. \tag{5.10}$$

According to the initial condition, we can obtain

$$E\big[u(r,t)\big] - s^2 r - s^{\alpha} E\left[u_{rr} + \frac{1}{r}u_r\right] = 0. \tag{5.11}$$

Using the inverse Elzaki operator on both sides of equation (5.11), we have

$$u(r,t) = E^{-1}\big[s^2 r\big] + E^{-1}\left[s^{\alpha} E\left[u_{rr} + \frac{1}{r}u_r\right]\right]. \tag{5.12}$$

Assume

$$\begin{cases} g(r,t) = E^{-1}[s^2 r], \\ N(u(r,t)) = E^{-1}[s^\alpha E[u_{rr} + \frac{1}{r}u_r]]. \end{cases} \tag{5.13}$$

According to (3.11), we have the result

$$\begin{aligned} u_0 &= r, \\ u_1 &= \frac{t^\alpha}{r\Gamma(\alpha+1)}, \\ u_2 &= \frac{t^{2\alpha}}{r^3\Gamma(2\alpha+1)}, \\ u_3 &= \frac{9t^{3\alpha}}{r^5\Gamma(3\alpha+1)}, \\ u_4 &= \frac{225t^{4\alpha}}{r^7\Gamma(4\alpha+1)}, \\ &\ldots \\ u_n &= \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned}$$

Therefore, the solution of equation (5.9) is given as

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{5.14}$$

Remark 5.3 The result is the same as ADM, HPTM, HPM, and VIM by Momani and Odibat [15], Kumar *et al.* [16, 17], and Khan [18].

Remark 5.4 When $\alpha = 1$, equation (5.14) is the same as the exact solution of the classical Navier-Stokes equation [15],

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}} \frac{t^n}{n!}.$$

5.4 Applying the MRDTM

Applying the MRDT on both sides of equation (5.9), we get

$$MRDT\big[u_t^\alpha(r,t)\big] = MRDT\big[u_{rr}(r,t)\big] + MRDT\left[\frac{1}{r}u_r(r,t)\right]. \tag{5.15}$$

Applying the property of MRDT, we get

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1} = (U_k)_{rr} + \frac{1}{r}(U_k)_r. \tag{5.16}$$

Using $U_0 = r$, we get

$$u_0 = r,$$

Figure 1 **The solution of equation (5.1) for $\alpha = 0.5$, $P = 5$.**

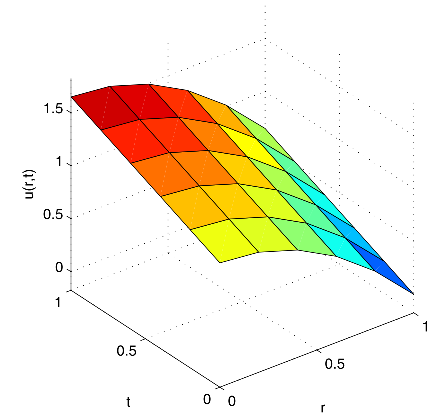
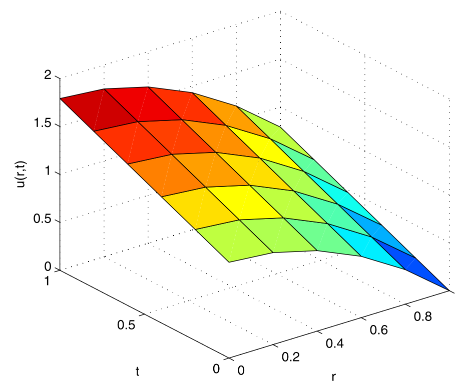


Figure 2 **The solution of equation (5.1) for $\alpha = 0.7$, $P = 5$.**



$$
\begin{aligned}
u_1 &= \frac{1}{r\Gamma(\alpha+1)},\\
u_2 &= \frac{1}{r^3\Gamma(2\alpha+1)},\\
u_3 &= \frac{9}{r^5\Gamma(3\alpha+1)},\\
u_4 &= \frac{225}{r^7\Gamma(4\alpha+1)},\\
&\cdots\\
u_n &= \frac{1^2\times 3^2\times\cdots\times(2n-3)^2}{r^{2n-1}}\frac{1}{\Gamma(n\alpha+1)}.
\end{aligned}
$$

So, the solution of equation (5.9) is given as

$$
u(r,t) = \sum_{n=0}^{\infty} U_n t^{n\alpha} = r + \sum_{n=1}^{\infty} \frac{1^2\times 3^2\times\cdots\times(2n-3)^2}{r^{2n-1}} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{5.17}
$$

Remark 5.5 We apply the NIETM and MRDTM to solve the time-fractional Navier-Stokes equations, and we get complete agreement with HPM, HPTM, ADM, and VIM. By comparing, NIETM and MRDTM are more easy to understand and implement than other methods with less computation.

Figure 3 The solution of equation (5.1) for $\alpha = 1$, $P = 5$.

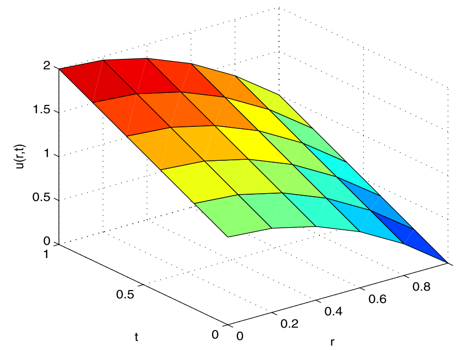


Figure 4 The exact solution of equation (5.1) for $\alpha = 1$, $P = 5$.

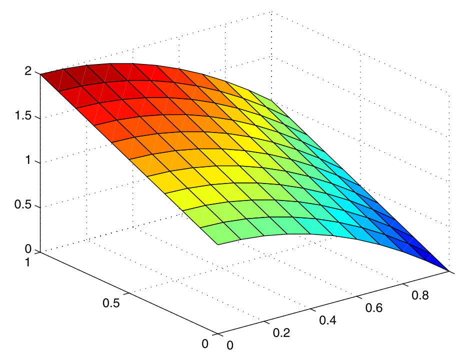
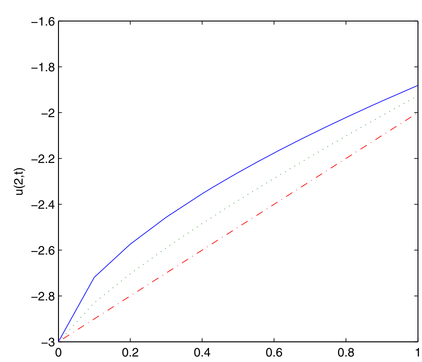


Figure 5 The different solutions of equation (5.1) for $\alpha = 0.6, 0.8, 1$ at $P = 5$, $r = 2$. Solid line: $\alpha = 0.6$, dotted line: $\alpha = 0.8$, dashed line: $\alpha = 1$.

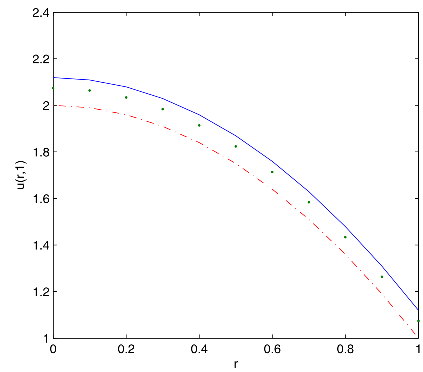


Remark 5.6 Figures 1-3, respectively, show the approximate solution of equation (5.1) for $\alpha = 0.5, 0.7, 1$ at $P = 5$. Figure 4 shows the exact solution of equation (5.1) for $\alpha = 1$. Figure 5 shows the different solution of equation (5.1) for $\alpha = 0.6, 0.8, 1$ at $P = 5$, $r = 2$. Figure 6 shows the different solution of equation (5.1) for $\alpha = 0.6, 0.8, 1$ at $P = 5$, $t = 1$. By comparison, it is easy for us to find that the solution continuously depends on the values of the time-fractional derivative.

6 Conclusion

In this paper, we apply the modified reduced differential transform method and new iterative Elzaki transform method for solving the time-fractional Navier-Stokes equation. The numerical results show that the MRDTM and NIETM are very powerful and efficient techniques for fractional differential equations.

Figure 6 The different solutions of equation (5.1) for $\alpha = 0.6, 0.8, 1$ at $P = 5$, $t = 1$. Solid line: $\alpha = 0.6$, dotted line: $\alpha = 0.8$, dashed line: $\alpha = 1$.



Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors, KW and SL, contributed substantially to this paper, participated in drafting and checking the manuscript, and have approved the version to be published.

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