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An order-type existence theorem and applications to periodic problems

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Abstract

Based on the fixed point index and partial order method, one new order-type existence theorem concerning cone expansion and compression is established. As applications, we present sufficient existence conditions for the first- and second-order periodic problems.

MSC: 34B15

Keywords: fixed point index; order-type existence theorem; cone expansion and compression; positive solutions; periodic boundary value problems

1 Introduction and preliminaries

Let X, Y be real Banach spaces. Consider a linear mapping $L : \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N : X \to Y$. Here we assume that L is a Fredholm operator of index zero, that is, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. Then the solvability of the operator equation

$$Lx = Nx$$

has been studied by many researchers in the literature; see [1–8] and the references therein. In [1], Cremins established a fixed point index for A-proper semilinear operators defined on cones which includes and improves the results in [5, 8, 9]. Using the fixed point index and the concept of a quasi-normal cone introduced in [10], Cremins established a norm-type existence theorem concerning cone expansion and compression in [11], which generalizes some corresponding results contained in [12].

In this paper, we will use the properties of the fixed point index in [1] and partial order to present a new order-type existence theorem concerning cone expansion and compression which extends the corresponding results in [12]. We recall that a partial order in X induced by a cone $K \subset X$ is defined by

$$x \leq y \quad \Longleftrightarrow \quad y - x \in K.$$

As applications, we study the first- and second-order periodic boundary problems and obtain new existence results. During the last few decades, periodic boundary value problems have been studied by many researchers in the literature; see, for example, [13–19] and the references therein. Our new results improve those contained in [13, 18].

Next we recall some notations and results which will be needed in this paper. Let X and Y be Banach spaces, D be a linear subspace of X, $\{X_n\} \subset D$ and $\{Y_n\} \subset Y$ be the sequences of oriented finite dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$, where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y. A map $T : D \subset X \to Y$ is called approximation-proper (abbreviated A-proper) at a point $y \in Y$ with respect to an admissible scheme Γ if $T_n \equiv Q_n T|_{D \cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \to y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x \in D$ and $Tx = y$. T is simply called A-proper if it is A-proper at all points of Y. $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero if $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. As a consequence of this property, X and Y may be expressed as direct sums; $X = X_0 \bigoplus X_1$, $Y = Y_0 \oplus Y_1$ with continuous linear projections $P : X \to \operatorname{Ker} L = X_0$ and $Q : Y \to Y_0$. The restriction of L to $\operatorname{dom} L \cap X_1$, denoted L_1, is a bijection onto $\operatorname{Im} L = Y_1$ with continuous inverse $L_1^{-1} : Y_1 \to \operatorname{dom} L \cap X_1$. Since X_0 and Y_0 have the same finite dimension, there exists a continuous bijection $J : Y_0 \to X_0$. Let $H = L + J^{-1}P$, then $H : \operatorname{dom} L \subset X \to Y$ is a linear bijection with bounded inverse. Let K be a cone in a Banach space X. Then $K_1 = H(K \cap \operatorname{dom} L)$ is a cone in Y. In [20], Petryshyn has shown that an admissible scheme Γ_L can be constructed such that L is A-proper with respect to Γ_L. The following properties of the fixed point index ind_K and two lemmas can be found in [1].

Proposition 1.1 *Let $\Omega \subset X$ be open and bounded and $\partial\Omega_K = \partial\Omega \cap K$. Assume that $Q_n K_1 \subset K_1$, $P + JQN + L_1^{-1}(I - Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$.*

- (P_1) (*Existence property*) *If* $\operatorname{ind}_K([L, N], \Omega) \neq \{0\}$, *then there exists* $x \in \Omega_K$ *such that* $Lx = Nx$.
- (P_2) (*Normality*) *If* $x_0 \in \Omega_K$, *then* $\operatorname{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \{1\}$, *where* $\hat{y}_0 = Hx_0$ *and* $\hat{y}_0(y) = y_0$ *for every* $y \in H\Omega_K$.
- (P_3) (*Additivity*) *If* $Lx \neq Nx$ *for* $x \in \overline{\Omega}_K \backslash (\Omega_1 \cup \Omega_2)$, *where* Ω_1 *and* Ω_2 *are disjoint relatively open subsets of* Ω_K, *then*

  $$\operatorname{ind}_K\big([L, N], \Omega\big) \subseteq \operatorname{ind}_K\big([L, N], \Omega_1\big) + \operatorname{ind}_K\big([L, N], \Omega_2\big)$$

  with equality if either of indices on the right is a singleton.
- (P_4) (*Homotopy invariance*) *If* $L - N(\lambda, x)$ *is an A-proper homotopy on* Ω_K *for* $\lambda \in [0, 1]$ *and* $(N(\lambda, x) + J^{-1}P)H^{-1} : K_1 \to K_1$ *and* $\theta \notin (L - N(\lambda, x))(\operatorname{dom} L \cap \partial\Omega_K)$ *for* $\lambda \in [0, 1]$, *then* $\operatorname{ind}_K([L, N(\lambda, x)], \Omega) = \operatorname{ind}_{K_1}(T_\lambda, U)$ *is independent of* $\lambda \in [0, 1]$, *where* $T_\lambda = (N(\lambda, x) + J^{-1}P)H^{-1}$.

Lemma 1.1 *If $L : \operatorname{dom} L \to Y$ is Fredholm of index zero, Ω is an open bounded set and $\Omega_K \cap \operatorname{dom} L \neq \emptyset$, $\theta \in \Omega \subset X$. Let $L - \lambda N$ be A-proper for $\lambda \in [0, 1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I - Q)N$ maps K to K. If $Lx \neq \mu Nx - (1 - \mu)J^{-1}Px$ on $\partial\Omega_K$ for $\mu \in [0, 1]$, then*

$$\operatorname{ind}_K\big([L, N], \Omega\big) = \{1\}.$$

Lemma 1.2 *If $L : \operatorname{dom} L \to Y$ is Fredholm of index zero, Ω is an open bounded set and $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Let $L - \lambda N$ be A-proper for $\lambda \in [0, 1]$. Assume that N is bounded and*

$P + JQN + L_1^{-1}(I - Q)N$ *maps* K *to* K. *If there exists* $e \in K_1 \backslash \{\theta\}$ *such that*

$$Lx - Nx \neq \mu e,$$

for every $x \in \partial\Omega_K$ *and all* $\mu \geq 0$, *then*

$$\operatorname{ind}_K\big([L,N],\Omega\big) = \{0\}.$$

2 An abstract result

We will establish an abstract existence theorem concerning cone expansion and compression of order type, which reads as follows.

Theorem 2.1 *If* $L : \operatorname{dom} L \to Y$ *is Fredholm of index zero, let* $L - \lambda N$ *be A-proper for* $\lambda \in [0,1]$. *Assume that* N *is bounded and* $P + JQN + L_1^{-1}(I - Q)N$ *maps* K *to* K. *Suppose further that* Ω_1 *and* Ω_2 *are two bounded open sets in* X *such that* $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, $\Omega_1 \cap K \cap \operatorname{dom} L \neq \emptyset$ *and* $\Omega_2 \cap K \cap \operatorname{dom} L \neq \emptyset$. *If one of the following two conditions is satisfied*:

- (C_1) $(P + JQN)x + L_1^{-1}(I - Q)Nx \ngeq x$ *for all* $x \in \partial\Omega_1 \cap K$ *and* $(P + JQN)x + L_1^{-1}(I - Q)Nx \nleq x$ *for all* $x \in \partial\Omega_2 \cap K$;
- (C_2) $(P + JQN)x + L_1^{-1}(I - Q)Nx \nleq x$ *for all* $x \in \partial\Omega_1 \cap K$ *and* $(P + JQN)x + L_1^{-1}(I - Q)Nx \ngeq x$ *for all* $x \in \partial\Omega_2 \cap K$.

Then there exists $x \in (\overline{\Omega}_2 \backslash \Omega_1) \cap K$ *such that* $Lx = Nx$.

Proof We assume that (C_1) is satisfied. First we show that

$$Lx \neq \mu Nx - (1 - \mu)J^{-1}Px, \quad \text{for any } x \in \partial\Omega_1 \cap K, \mu \in [0,1]. \tag{2.1}$$

In fact, otherwise, there exist $x_1 \in \partial\Omega_1 \cap K$ and $\mu_1 \in [0,1]$ such that

$$Lx_1 = \mu_1 Nx_1 - (1 - \mu_1)J^{-1}Px_1,$$

then we obtain

$$\big(L + J^{-1}P\big)x_1 = \mu_1\big(N + J^{-1}P\big)x_1.$$

Therefore,

$$\begin{aligned} x_1 &= \mu_1\big(L + J^{-1}P\big)^{-1}\big(N + J^{-1}P\big)x_1 \\ &= \mu_1\big[(P + JQN)x_1 + L_1^{-1}(I - Q)Nx_1\big] \\ &\leq (P + JQN)x_1 + L_1^{-1}(I - Q)Nx_1, \end{aligned}$$

which contradicts condition (C_1). From (2.1) and Lemma 1.1, we have

$$\operatorname{ind}_K\big([L,N],\Omega_1\big) = \{1\}. \tag{2.2}$$

Choosing an arbitrary $e \in K_1 \backslash \{\theta\}$, next we prove that

$$Lx - Nx \neq \mu e. \tag{2.3}$$

In fact, otherwise, there exist $x_2 \in \partial\Omega_2 \cap K$ and $\mu_2 \geq 0$ such that

$$Lx_2 - Nx_2 = \mu_2 e,$$

then we obtain

$$\left(L + J^{-1}P\right)x_2 = \left(N + J^{-1}P\right)x_2 + \mu_2 e \geq_1 \left(N + J^{-1}P\right)x_2,$$

in which the partial order is induced by the cone K_1 in Y. So,

$$x_2 \geq \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right)x_2 = (P + JQN)x_2 + L_1^{-1}(I - Q)Nx_2,$$

which is a contradiction to condition (C_1). Hence (2.3) holds, and then by Lemma 1.2, we have

$$\operatorname{ind}_K\left([L, N], \Omega_2\right) = \{0\}. \tag{2.4}$$

It follows therefore from (2.2), (2.4) and the additivity property (P_3) of Proposition 1.1 that

$$\begin{aligned} \operatorname{ind}_K\left([L, N], \Omega_2 \backslash \Omega_1\right) &= \operatorname{ind}_K\left([L, N], \Omega_2\right) - \operatorname{ind}_K\left([L, N], \Omega_1\right) \\ &= \{0\} - \{1\} \\ &= \{-1\}. \end{aligned} \tag{2.5}$$

Since the index is nonzero, the existence property (P_1) of Proposition 1.1 implies that there exists $x \in (\overline{\Omega}_2 \backslash \Omega_1) \cap K$ such that $Lx = Nx$.

Similarly, when (C_2) is satisfied, instead of (2.2), (2.4) and (2.5), we have

$$\operatorname{ind}_K\left([L, N], \Omega_1\right) = \{0\}, \qquad \operatorname{ind}_K\left([L, N], \Omega_2\right) = \{1\},$$

and therefore

$$\operatorname{ind}_K\left([L, N], \Omega_2 \backslash \Omega_1\right) = \{1\}.$$

Also, we can assert that there exists $x \in (\overline{\Omega}_2 \backslash \Omega_1) \cap K$ such that $Lx = Nx$. □

3 Applications

3.1 First-order periodic boundary value problems

We consider the following first-order periodic boundary value problem:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1), \end{cases} \tag{3.1}$$

where $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ is continuous and $f(0, x) = f(1, x)$ for all $x \in \mathbb{R}$.

Consider the Banach spaces $X = Y = C[0,1]$ endowed with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Define the cone K in X by

$$K = \{x \in X : x(t) \geq 0, t \in [0,1]\}.$$

Let L be the linear operator from $\operatorname{dom} L \subset X$ to Y with

$$\operatorname{dom} L = \{x \in X : x' \in C[0,1], x(0) = x(1)\},$$

and

$$Lx(t) = x'(t), \quad x \in \operatorname{dom} L, t \in [0,1].$$

Let us define $N : X \to Y$ by

$$Nx(t) = f(t, x(t)), \quad t \in [0,1].$$

Then (3.1) is equivalent to the equation

$$Lx = Nx.$$

It is obvious that L is a Fredholm operator of index zero with

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) \equiv c \text{ on } [0,1], c \in \mathbb{R}\},$$
$$\operatorname{Im} L = \left\{y \in Y : \int_0^1 y(s)\,ds = 0\right\},$$
$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1.$$

Next we define the projections $P : X \to X$, $Q : Y \to Y$ by

$$Px = \int_0^1 x(s)\,ds,$$
$$Qy = \int_0^1 y(s)\,ds,$$

and the isomorphism $J : \operatorname{Im} Q \to \operatorname{Im} P$ as $Jy = y$. Note that for $y \in \operatorname{Im} L$, the inverse operator

$$L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$$

of

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is given by

$$(L_1^{-1} y)(t) = \int_0^1 K(t,s) y(s)\,ds,$$

where

$$K(t,s)=\begin{cases} s+1, & 0\le s<t\le 1,\\ s, & 0\le t\le s\le 1.\end{cases}$$

Set

$$G(t,s)=1+K(t,s)-\int_0^1 K(t,s)\,ds.$$

We can verify that

$$G(t,s)=\begin{cases} \frac{3}{2}-(t-s), & 0\le s<t\le 1,\\ \frac{1}{2}+(s-t), & 0\le t\le s\le 1,\end{cases}$$

and

$$\frac{1}{2}\le G(t,s)\le\frac{3}{2},\quad t,s\in[0,1].$$

To state the existence result, we introduce two conditions:

(H_1) $f(t,b)<0$ for all $t\in[0,1]$,
(H_2) $f(t,x)>0$ for all $(t,x)\in[0,1]\times[0,a]$.

Theorem 3.1 *Assume that there exist two positive numbers* $0<a<b$ *such that* (H_1), (H_2) *and*

(H_3) $f(t,x)\ge-\frac{2}{3}x$ *for all* $(t,x)\in[0,1]\times[0,b]$

hold. Then (3.1) *has at least one positive periodic solution* $x^*\in K$ *with* $a\le\|x^*\|\le b$.

Proof First, we note that L, as defined, is Fredholm of index zero, L_1^{-1} is compact by the Arzela-Ascoli theorem and thus $L-\lambda N$ is A-proper for $\lambda\in[0,1]$ by [20, Lemma 2(a)].

For each $x\in K$, then by condition (H_3),

$$\begin{aligned} &Px+JQNx+L_1^{-1}(I-Q)Nx\\ &\quad=\int_0^1 x(s)\,ds+\int_0^1 f\bigl(s,x(s)\bigr)\,ds\\ &\qquad+\int_0^1 K(t,s)\left(f\bigl(s,x(s)\bigr)-\int_0^1 f\bigl(s,x(s)\bigr)\,ds\right)ds\\ &\quad=\int_0^1 x(s)\,ds+\int_0^1 G(t,s)f\bigl(s,x(s)\bigr)\,ds\\ &\quad\ge\int_0^1\left(1-\frac{2}{3}G(t,s)\right)x(s)\,ds\ge 0.\end{aligned}$$

Thus $(P+JQN+L_1^{-1}(I-Q)N)(K)\subset K$.

Let

$$\Omega_1=\bigl\{x\in X:\|x\|<a\bigr\},\qquad\Omega_2=\bigl\{x\in X:\|x\|<b\bigr\}.$$

Clearly, Ω_1 and Ω_2 are bounded open sets and

$$\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2.$$

We now show that

$$(P + JQN)x + L_1^{-1}(I - Q)Nx \not\geq x \quad \text{for any } x \in \partial\Omega_2 \cap K. \tag{3.2}$$

In fact, if there exists $x_3 \in \partial\Omega_2 \cap K$ such that

$$(P + JQN)x_3 + L_1^{-1}(I - Q)Nx_3 \geq x_3.$$

Then

$$x_3'(t) \leq f\bigl(t, x_3(t)\bigr), \quad t \in [0,1].$$

Let $t_1 \in [0,1]$ be such that $x_3(t_1) = b$. Clearly, the function x_3^2 attains a maximum on $[0,1]$ at $t = t_1$. Therefore $2x_3(t_1)x_3'(t_1) = 0$. As a consequence,

$$0 = 2bx_3'(t_1) \leq 2bf\bigl(t_1, x_3(t_1)\bigr) = 2bf(t_1, b),$$

which is a contradiction to (H_1). Therefore (3.2) holds.

On the other hand, we claim that

$$(P + JQN)x + L_1^{-1}(I - Q)Nx \not\leq x \quad \text{for any } x \in \partial\Omega_1 \cap K. \tag{3.3}$$

In fact, if not, there exists $x_4 \in \partial\Omega_1 \cap K$ such that

$$(P + JQN)x_4 + L_1^{-1}(I - Q)Nx_4 \leq x_4.$$

For any $x_4 \in \partial\Omega_1 \cap K$, we have $\|x_4\| = a$, then $0 \leq x_4(t) \leq a$ for $t \in [0,1]$. By condition (H_2), we have

$$\begin{aligned} x_4(t) &\geq (P + JQN)x_4(t) + L_1^{-1}(I - Q)Nx_4(t) \\ &= \int_0^1 x_4(s)\,ds + \int_0^1 G(t,s)f\bigl(s, x_4(s)\bigr)\,ds \\ &> \int_0^1 x_4(s)\,ds, \quad \text{for any } t \in [0,1], \end{aligned}$$

which is a contradiction. As a result, (3.3) is verified.

It follows from (3.2), (3.3) and Theorem 2.1 that there exists $x^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $Lx^* = Nx^*$ with $a \leq \|x^*\| \leq b$. □

Remark 3.1 In [18], the following condition is required instead of (H_2):

(H^*) there exist $a \in (0,b)$, $t_0 \in [0,1]$, $r \in (0,1]$, and continuous functions $g : [0,1] \to [0,\infty)$, $h : (0,a] \to [0,\infty)$ such that $f(t,x) \geq g(t)h(x)$ for all $t \in [0,1]$ and $x \in (0,a]$, $h(x)/x^r$ is nonincreasing on $(0,a]$ with

$$\frac{h(a)}{2^{r-1}} \int_0^1 G(t_0,s)g(s)\,ds \geq a.$$

Obviously, our condition (H_2) is much weaker and less strict compared with (H^*). Moreover, (H_2) is easier to check than (H^*). So, our result generalizes and improves [18, Theorem 5].

Remark 3.2 From the proof of Theorem 3.1, we can see that condition (H_2) can be replaced by one of the following two relatively weaker conditions:

(H_2^*) $f(t,x) \geq 0$ for all $(t,x) \in [0,1] \times [0,a]$ and $f(t,\cdot)$ is positive for almost everywhere on $[0,a]$.

(H_2^{**}) $\lim_{x\to 0^+} \min_{t\in[0,1]} f(t,x) > 0$.

Remark 3.3 Finally in this section, we note that conditions (H_1) and (H_2) can be replaced by the following asymptotic conditions:

(H_1') $\lim_{x\to+\infty} \frac{f(t,x)}{x} < 0$ uniformly for t;

(H_2') $\lim_{x\to 0^+} \frac{f(t,x)}{x} > 0$ uniformly for t.

Example 3.1 Let the nonlinearity in (3.1) be

$$f(t,x) = c(t)x^\alpha + \mu d(t)x^\beta - kx,$$

where $0 < \alpha < 1 < \beta$, $c(t), d(t) \in C[0,1]$ are positive 1-periodic functions, $k \in (0, 2/3)$ and $\mu > 0$ is a positive parameter. Then (3.1) has at least one positive 1-periodic solution for each $0 < \mu < \mu^*$, here μ^* is some positive constant.

Proof We will apply Theorem 3.1 with $f(t,x) = c(t)x^\alpha + \mu d(t)x^\beta - kx$. Since $k \in (0, 2/3)$, it is easy to see that (H_3) holds. Set

$$T(x) = \frac{kx - c^* x^\alpha}{d^* x^\beta},$$

where

$$c^* = \max_t c(t), \qquad d^* = \max_t d(t).$$

Since $0 < \alpha < 1 < \beta$, we have

$$T(0^+) = -\infty, \qquad T(+\infty) = 0.$$

One may easily see that there exists $b > 0$ such that

$$T(b) = \frac{kb - c^* b^\alpha}{d^* b^\beta} = \sup_{x>0} T(x) > 0.$$

Let

$$\mu^* = \frac{kb - c^* b^\alpha}{d^* b^\beta}.$$

Then, for each $\mu \in (0, \mu^*)$, we have

$$\begin{aligned} f(t,b) &= c(t)b^\alpha + \mu d(t)b^\beta - kb \\ &< c^* b^\alpha + \mu^* d^* b^\beta - kb \\ &= 0, \end{aligned}$$

which implies that (H_1) holds.

On the other hand, we have

$$\lim_{x\to 0^+} \frac{f(t,x)}{x} = \lim_{x\to 0^+} \left(\frac{c(t)}{x^{1-\alpha}} + \mu d(t) x^{\beta-1} \right) - k > 0,$$

which implies that (H_2') holds. Now we have the desired result. □

3.2 Second-order periodic boundary value problems

Let $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ be continuous and $f(0,x) = f(1,x)$ for all $x \in \mathbb{R}$. We will discuss the existence of positive solutions of the second-order periodic boundary value problem

$$\begin{cases} -x''(t) = f(t,x), & t \in (0,1), \\ x(0) = x(1), & x'(0) = x'(1). \end{cases} \tag{3.4}$$

Since some parts of the proof are in the same line as that of Theorem 3.1, we will outline the proof with the emphasis on the difference.

Let X, Y be Banach spaces and the cone K be as in Section 3.1. In this case, we may define

$$\operatorname{dom} L = \{x \in X : x'' \in C[0,1], x(0) = x(1), x'(0) = x'(1)\},$$

and let the linear operator $L : \operatorname{dom} L \to Y$ be defined by

$$Lx = -x'', \quad \text{for } x \in \operatorname{dom} L.$$

Then L is Fredholm of index zero,

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) \equiv \text{constants}\},$$

and

$$\operatorname{Im} L = \left\{ y \in Y : \int_0^1 y(s)\,ds = 0 \right\}.$$

Define $N : X \to Y$ by

$$Nx(t) = f(t, x(t)).$$

Thus it is clear that (3.4) is equivalent to

$$Lx = Nx.$$

We use the same projections P, Q as in Section 3.1 and define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Im} P$ as

$$Jy = \beta y,$$

where $\beta = \frac{1}{6}$. It is easy to verify that the inverse operator $L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is

$$\left(L_1^{-1} y\right)(t) = \int_0^1 \Lambda(t,s) y(s)\, ds,$$

where

$$\Lambda(t,s) = \begin{cases} \frac{s}{2}(1 - 2t + s), & 0 \le s < t \le 1, \\ \frac{1}{2}(1 - s)(2t - s), & 0 \le t \le s \le 1. \end{cases}$$

Set

$$H(t,s) = \frac{1}{6} + \Lambda(t,s) - \int_0^1 \Lambda(t,s)\, ds.$$

We can verify that

$$H(t,s) = \begin{cases} \frac{1}{4} + \frac{s}{2}(1 - 2t + s) + \frac{t^2}{2} - \frac{t}{2}, & 0 \le s < t \le 1, \\ \frac{1}{4} + \frac{1}{2}(1 - s)(2t - s) + \frac{t^2}{2} + \frac{t}{2}, & 0 \le t \le s \le 1, \end{cases}$$

and

$$\frac{1}{8} \le H(t,s) \le \frac{1}{4}, \quad t,s \in [0,1].$$

Theorem 3.2 *Assume that there exist two positive numbers* $0 < a < b$ *such that* (H_1), (H_2) *and*

(H_4) $f(t,x) \ge -4x$ *for all* $(t,x) \in [0,1] \times [0,b]$

hold. Then (3.4) *has at least one positive periodic solution* $x^* \in K$ *with* $a \le \|x^*\| \le b$.

Proof It is again easy to show that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ by [20, Lemma 2(a)].

For each $x \in K$, then by condition (H_4),

$$\begin{aligned} &Px + JQNx + L_1^{-1}(I - Q)Nx \\ &\quad = \int_0^1 x(s)\, ds + \frac{1}{6} \int_0^1 f\big(s, x(s)\big)\, ds \\ &\qquad + \int_0^1 \Lambda(t,s) \left(f\big(s, x(s)\big) - \int_0^1 f\big(s, x(s)\big)\, ds \right) ds \end{aligned}$$

$$= \int_0^1 x(s)\,ds + \int_0^1 H(t,s)f\bigl(s,x(s)\bigr)\,ds$$
$$\geq \int_0^1 \bigl(1 - 4H(t,s)\bigr)x(s)\,ds \geq 0.$$

Thus $(P + JQN + L_1^{-1}(I - Q)N)(K) \subset K$.

Let

$$\Omega_3 = \bigl\{x \in X : \|x\| < a\bigr\}, \qquad \Omega_4 = \bigl\{x \in X : \|x\| < b\bigr\}.$$

Clearly, Ω_3 and Ω_4 are bounded and open sets and

$$\theta \in \Omega_3 \subset \overline{\Omega}_3 \subset \Omega_4.$$

Next, we show that

$$(P + JQN)x + L_1^{-1}(I - Q)Nx \ngeq x, \quad \text{for any } x \in \partial\Omega_4 \cap K. \tag{3.5}$$

On the contrary, suppose that there exists $x_5 \in \partial\Omega_4 \cap K$ such that

$$(P + JQN)x_5 + L_1^{-1}(I - Q)Nx_5 \geq x_5.$$

Then

$$-x_5''(t) \leq f\bigl(t, x_5(t)\bigr), \quad t \in [0,1].$$

Let $t_2 \in [0,1]$ such that $x_5(t_2) = \max_{t\in[0,1]} x_5(t) = b$. Using the boundary conditions, we have $t_2 \in (0,1)$. In this case, $x_5'(t_2) = 0$, $x_5''(t_2) \leq 0$. This gives

$$0 \leq -x_5''(t_2) \leq f\bigl(t_2, x_5(t_2)\bigr) = f(t_2, b),$$

which is a contradiction to condition (H_1). Therefore (3.5) holds.

Finally, similar to the proof of (3.3), it follows from condition (H_2) that

$$(P + JQN)x + L_1^{-1}(I - Q)Nx \nleq x, \quad \text{for any } x \in \partial\Omega_3 \cap K.$$

Consequently all conditions of Theorem 2.1 are satisfied. Therefore, there exists $x^* \in K \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $Lx^* = Nx^*$ with $x^* \in K$ and $a \leq \|x^*\| \leq b$ and the assertion follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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