Güvenilir and Nizigiyimana *Advances in Difference Equations* 2014, 2014:45
http://www.advancesindifferenceequations.com/content/2014/1/45

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Oscillation criteria for second-order quasi-linear delay dynamic equations on time scales

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Abstract

This paper is concerned with oscillations of the second-order delay nonlinear dynamic equation $(a(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + q(t)x^{\beta}(\tau(t)) = 0$ on a time scale $\mathbb{T}$, where a and q are real-valued rd-continuous positive functions on $\mathbb{T}$, α and β are ratios of odd positive integers, $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t$, $\forall t \in \mathbb{T}$, and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions for this equation.

MSC: 34K11; 39A10; 39A99

Keywords: oscillation; delay quasi-linear dynamic equation; time scales

1 Introduction

The study of dynamic equations on time scales, which goes back to Hilger [1], provides a rapidly expanding body of literature where the main idea is to unify, extend, and generalise concepts from continuous, discrete and quantum calculus to arbitrary time scales analysis, where a time scale $\mathbb{T}$ is simply any nonempty closed subset of the reals.

For detailed information regarding calculus on time scales, we refer the reader to Bohner and Peterson [2, 3].

Now we consider the second-order delay nonlinear dynamic equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)x^{\beta}\left(\tau(t)\right) = 0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where a and q are real-valued rd-continuous positive functions defined on $\mathbb{T}$; α and $\beta > 0$ are ratios of odd positive integers; $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t$, $\forall t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$.

We shall also consider the case

$$\int_{t_0}^{\infty} \frac{\Delta s}{a^{1/\alpha}(s)} = \infty. \tag{1.2}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and we define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued function $x \in C^1_{\mathrm{rd}}[T_x, \infty)$, $T_x \ge t_0$ which has the property that $a(t)(x^{\Delta}(t))^{\alpha} \in C^1_{\mathrm{rd}}[T_x, \infty)$ and satisfies (1.1) on $[T_x, \infty)$; here C_{rd} is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$ and when $\alpha = \beta$ (1.1) becomes the half-linear delay differential equation

$$\big(a(t)\big(x'(t)\big)^{\alpha}\big)' + q(t)x^{\alpha}\big(\tau(t)\big) = 0. \tag{1.3}$$

When $a(t) = 1$ and $\alpha = 1$, (1.3) becomes

$$x''(t) + q(t)x\big(\tau(t)\big) = 0. \tag{1.4}$$

Ohriska [4] proved that every solution of (1.4) oscillates if

$$\limsup_{t\to\infty} t\int_t^{\infty} q(s)\left(\frac{\tau(s)}{s}\right) ds > 1 \tag{1.5}$$

holds. Agarwal *et al.* [5] considered (1.3) and extended the condition (1.5) and proved that if

$$\limsup_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\alpha} ds > 1, \tag{1.6}$$

then every solution of (1.3) oscillates.

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the quasi-linear difference equation

$$\Delta\big(a(t)\big(\Delta x(t)\big)^{\alpha}\big) + q(t)x^{\beta}\big(\tau(t)\big) = 0. \tag{1.7}$$

When $\alpha = \beta$, (1.1) becomes the half-linear delay dynamic equation which has been considered by some authors and some oscillation and nonoscillation results have been obtained [2, 3, 5–10]. As a special case of (1.1) Agarwal *et al.* [11] considered the second-order delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + q(t)x\big(\tau(t)\big) = 0 \tag{1.8}$$

and established some sufficient conditions for oscillations of (1.8) when

$$\int_{t_0}^{\infty} \tau(t)q(t)\Delta t = \infty. \tag{1.9}$$

Saker [12] studied (1.8) to extend the result of Lomtatidze [13]. Recently Higgins [14] proved oscillation results for (1.8). Saker [15] examined oscillations for the half-linear dynamic equation

$$\big(a(t)\big(x^{\Delta}(t)\big)^{\alpha}\big)^{\Delta} + q(t)x^{\alpha}(t) = 0 \tag{1.10}$$

on time scales, where $\alpha > 1$ is an odd positive integer and Agarwal *et al.* [6] and Grace *et al.* [10] studied oscillations for the same equation, (1.10), where $\alpha > 1$ is the quotient of odd positive integers which cannot be applied when $0 < \alpha \leq 1$. Han *et al.* [16] and Hassan [17] solved this problem and improved Agarwal's and Saker's results. Erbe *et al.* [7] considered the half-linear delay dynamic equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)x^{\alpha}\left(\tau(t)\right) = 0 \tag{1.11}$$

on time scales, where $\alpha > 1$ is the quotient of odd positive integers and

$$a^{\Delta}(t) \geq 0 \quad \text{and} \quad \int_{t_0}^{\infty} \tau^{\alpha}(t)q(t)\Delta t = \infty \tag{1.12}$$

and utilised a Riccati transformation technique and established some oscillation criteria for (1.11). Erbe *et al.* [8] considered the half-linear delay dynamic equation (1.11) on time scales, where $0 < \alpha \leq 1$ is the quotient of odd positive integers and established some sufficient conditions for oscillations when (1.12) holds. Han *et al.* [18] considered (1.11) and followed the proof that has been used in [15] and established some sufficient conditions for oscillations when $a^{\Delta}(t) \geq 0$. For oscillations of quasi-linear dynamic equations, Grace *et al.* [19] considered the equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)x^{\beta}(t) = 0; \tag{1.13}$$

and Saker and Grace [20] considered the delay equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)x^{\beta}\left(\tau(t)\right) = 0 \tag{1.14}$$

on an arbitrary time scale $\mathbb{T}$ where $\tau(t) \leq t$ or $\tau(t) \geq t$. The special case of (1.14) where $0 < \alpha = \beta \leq 1$ has been studied in [9] by Erbe *et al.*

In this paper, we establish some new sufficient conditions for oscillations of (1.1).

2 Preliminary result

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^{\Delta}(t)$ at $t \in \mathbb{T}$ is defined to be the number (if it exists) such that for all $\epsilon > 0$ there is a neighborhood U of t with

$$\left|f\left(\sigma(t)\right) - f(s) - f^{\Delta}\left(\sigma(t) - s\right)\right| \leq \epsilon\left|\sigma(t) - s\right| \tag{2.1}$$

for all $s \in U$. If the (delta) derivative $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}$. For two (delta) differentiable functions f and g, the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) are given as in [2, Theorem 1.20]

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \tag{2.2}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}} \tag{2.3}$$

as well as of the chain rule [2, Theorem 1.90] for the derivative of the composite function fog for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a (delta) differentiable function $g : \mathbb{T} \to \mathbb{R}$

$$(fog)^{\Delta} = \left[\int_0^1 f'\left(g + h\mu g^{\Delta}\right) dh\right] g^{\Delta}. \tag{2.4}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b) \tag{2.5}$$

and the improper integral is defined as

$$\int_b^{\infty} f(t)\Delta t = \lim_{c\to\infty} \int_b^c f(t)\Delta t. \tag{2.6}$$

Lemma 2.1 *If A and B are nonnegative real numbers and $\lambda > 1$, then*

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1)B^{\lambda} \geq 0,$$

where the equality holds if and only if $A = B$.

3 The main results

In this section, by employing a Riccati transformation technique, we establish oscillation criteria for (1.1). To prove our main result, we will use the formula

$$\left(x^{\alpha}(t)\right)^{\Delta} = \alpha\left\{\int_0^1 \left[hx^{\sigma}(t) + (1-h)x(t)\right]^{\alpha-1} dh\right\}x^{\Delta}(t), \tag{3.1}$$

which is a simple consequence of the Pötzsche chain rule [2].

In the following theorems, we assume that (1.2) holds.

Theorem 3.1 *Assume that there exists a positive nondecreasing delta differentiable function $\xi(t)$ such that, for all sufficiently large $T \geq t_0$, and for $\tau(t) > g(T)$, we have*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\theta^{\beta}\left(s, g(T)\right)\xi(s)q(s) - \frac{(\frac{\alpha}{\beta})^{\alpha} a(s)(\xi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\xi(s)\delta_1(s))^{\alpha}}\right]\Delta s = \infty, \tag{3.2}$$

where $(\xi^{\Delta}(s))_+ := \max\{0, \xi^{\Delta}(s)\}$,

$$\delta_1(t) = \begin{cases} c_1 \text{ is any positive constant}, & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ c_2(\eta^{\sigma}(t))^{\frac{\alpha-\beta}{\alpha}}, c_2 \text{ is any positive constant}, & \text{if } \beta < \alpha, \end{cases} \tag{3.3}$$

and

$$\theta\left(t, g(T)\right) = \frac{\int_{g(T)}^{\tau(t)} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}}{\int_{g(T)}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}}, \qquad \eta(t) = \left(\int_{t_1}^t \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}\right)^{-1}. \tag{3.4}$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Let x be a nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since $a(x^{\Delta})^{\alpha}$ is a strictly decreasing function, it is of one sign. We claim that $x^{\Delta} > 0$ on

$[t_1,\infty)_{\mathbb{T}}$. If not, then there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $x^\Delta(t_2) < 0$. Using the fact $a(x^\Delta)^\alpha$ is strictly decreasing, we get

$$x(t) \le x(t_2) + \left[a(t_2)\left(x^\Delta(t_2)\right)^\alpha\right]^{1/\alpha} \int_{t_2}^{t} \frac{\Delta s}{a^{1/\alpha}(s)} \to -\infty, \quad \text{as } t \to \infty,$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $x^\Delta > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = \frac{\xi(t)a(t)(x^\Delta(t))^\alpha}{x^\beta(t)}, \tag{3.5}$$

then $w(t) > 0$. By the product rule and then the quotient rule, we have

$$\begin{aligned} w^\Delta(t) &= \left(\frac{\xi(t)}{x^\beta(t)}\right)\left(a(t)\left(x^\Delta(t)\right)^\alpha\right)^\Delta + \left(a(t)\left(x^\Delta(t)\right)^\alpha\right)^\sigma \left(\frac{\xi(t)}{x^\beta(t)}\right)^\Delta \\ &= -q(t)\xi(t)\left(\frac{x(\tau(t))}{x(t)}\right)^\beta + \xi^\Delta(t)\frac{(a(t)(x^\Delta(t))^\alpha)^\sigma}{x^\beta(\sigma(t))} \\ &\quad - \xi(t)\frac{(a(t)(x^\Delta(t))^\alpha)^\sigma(x^\beta(t))^\Delta}{x^\beta(t)x^\beta(\sigma(t))}. \end{aligned} \tag{3.6}$$

Using the fact that $x(t)$ is increasing and $a(x^\Delta)^\alpha$ is decreasing, we have

$$\begin{aligned} x(t) - x\left(\tau(t)\right) &= \int_{\tau(t)}^{t} \frac{(a(s)(x^\Delta(s))^\alpha)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \Delta s \\ &\le \int_{\tau(t)}^{t} \frac{(a(\tau(t))(x^\Delta(\tau(t)))^\alpha)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \Delta s \\ &= a^{\frac{1}{\alpha}}\left(\tau(t)\right)x^\Delta\left(\tau(t)\right) \int_{\tau(t)}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)} \end{aligned}$$

and so

$$\frac{x(t)}{x(\tau(t))} \le 1 + \frac{a^{\frac{1}{\alpha}}(\tau(t))x^\Delta(\tau(t))}{x(\tau(t))} \int_{\tau(t)}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)} \tag{3.7}$$

for $t \ge t_1$. Also, we see that

$$\begin{aligned} x\left(\tau(t)\right) > x\left(\tau(t)\right) - x\left(g(T)\right) &= \int_{g(T)}^{\tau(t)} \frac{(a(s)(x^\Delta(s))^\alpha)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \Delta s \\ &\ge a^{\frac{1}{\alpha}}\left(\tau(t)\right)x^\Delta\left(\tau(t)\right) \int_{g(T)}^{\tau(t)} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}, \end{aligned}$$

$$\frac{a^{\frac{1}{\alpha}}(\tau(t))x^\Delta(\tau(t))}{x(\tau(t))} \le \left(\int_{g(T)}^{\tau(t)} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}\right)^{-1}. \tag{3.8}$$

Therefore, (3.7) and (3.8) imply that

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\int_{g(T)}^{\tau(t)} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}}{\int_{g(T)}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}} = \theta\bigl(t, g(T)\bigr). \tag{3.9}$$

In view of (3.1), we get

$$\begin{aligned}\bigl(x^{\beta}(t)\bigr)^{\Delta} &= \beta x^{\Delta}(t)\int_0^1 \bigl[\bigl(x(t)+h\mu(t)x^{\Delta}(t)\bigr)\bigr]^{\beta-1}\,dh \\ &\geq \begin{cases} \beta(x^{\sigma}(t))^{\beta-1}x^{\Delta}(t), & 0<\beta\leq 1, \\ \beta(x(t))^{\beta-1}x^{\Delta}(t), & \beta>1.\end{cases}\end{aligned} \tag{3.10}$$

From (3.9), (3.10), the fact that $x(t)$ is an increasing function and the definition of $w(t)$, if $0<\beta\leq 1$, we have

$$\begin{aligned} w^{\Delta}(t) &\leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)} - \beta\xi(t)\frac{(a(t)(x^{\Delta}(t))^{\alpha})^{\sigma}}{(x^{\sigma}(t))^{\beta+1}}x^{\Delta}(t) \\ &= -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)} - \beta\xi(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)},\end{aligned}$$

whereas, if $\beta>1$, we have

$$\begin{aligned} w^{\Delta}(t) &\leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)} - \beta\xi(t)\frac{(a(t)(x^{\Delta}(t))^{\alpha})^{\sigma}}{(x^{\sigma}(t))^{\beta+1}}x^{\Delta}(t) \\ &= -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)} - \beta\xi(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}.\end{aligned}$$

Using the fact that $a(t)(x^{\Delta}(t))^{\alpha}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we find

$$\frac{x^{\Delta}(t)}{x^{\sigma}(t)} \geq \frac{((a(t)(x^{\Delta}(t))^{\alpha})^{\sigma})^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(t)(x^{\sigma}(t))^{\frac{\beta}{\alpha}}}\bigl(x^{\sigma}(t)\bigr)^{\frac{\beta-\alpha}{\alpha}} = \left(\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)}\right)^{\frac{1}{\alpha}}\frac{(x^{\sigma}(t))^{\frac{\beta-\alpha}{\alpha}}}{a^{\frac{1}{\alpha}}(t)}.$$

Thus for $\beta>0$, we have

$$\begin{aligned} w^{\Delta}(t) &\leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) \\ &\quad + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)} - \frac{\beta\xi(t)}{a^{\frac{1}{\alpha}}(t)}\left(\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)}\right)^{\frac{\alpha+1}{\alpha}}\bigl(x^{\sigma}(t)\bigr)^{\frac{\beta-\alpha}{\alpha}}.\end{aligned} \tag{3.11}$$

We consider the following three cases:

Case (i). $\beta>\alpha$.

In this case, since $x^{\Delta}(t)>0$, there exists $t_2\geq t_1$ such that $x^{\sigma}(t)\geq x(t)\geq c>0$. This implies that $(x^{\sigma}(t))^{\frac{\beta-\alpha}{\alpha}}\geq c_1$, where $c_1=c^{\frac{\beta-\alpha}{\alpha}}$.

Case (ii). $\beta=\alpha$.

In this case, we see that $(x^{\sigma}(t))^{\frac{\beta-\alpha}{\alpha}}=1$.

Case (iii). $\beta<\alpha$.

Then there exists a positive constant $b := a(t_1)(x^\Delta(t_1))^\alpha$. Using the decreasing of $a(x^\Delta)^\alpha$, we have

$$x^\Delta(t) \le b^{\frac{1}{\alpha}} a^{\frac{-1}{\alpha}}(t) \quad \text{for } t \ge t_1.$$

Integrating this inequality from t_1 to t, we have

$$x(t) \le x(t_1) + b^{\frac{1}{\alpha}} \int_{t_1}^{t} a^{\frac{-1}{\alpha}}(s)\Delta s.$$

Thus, there exist a constant $b_1 > 0$ and $t_2 \ge t_1$ such that

$$x(t) \le b_1 \eta^{-1}(t) \quad \text{for } t \ge t_2$$

and hence

$$\left(x^\sigma(t)\right)^{\frac{\beta-\alpha}{\alpha}} \ge c_2 \left(\eta^\sigma\right)^{\frac{\alpha-\beta}{\alpha}}(t) \quad \text{for } t \ge t_2,$$

where $c_2 = b_1^{\frac{\beta-\alpha}{\alpha}}$.

By (3.3), we get

$$w^\Delta(t) \le -q(t)\xi(t)\theta^\beta\big(t, g(T)\big) + \xi^\Delta(t)\frac{w^\sigma(t)}{\xi^\sigma(t)} - \beta a^{-\frac{1}{\alpha}}(t)\xi(t)\left(\frac{w^\sigma(t)}{\xi^\sigma(t)}\right)^{\frac{\alpha+1}{\alpha}} \delta_1(t) \tag{3.12}$$

for $t \ge t_2$. Defining $A \ge 0$ and $B \ge 0$ by

$$A = \big(\beta\delta_1(t)\xi(t)\big)^{\frac{\alpha}{\alpha+1}} a^{\frac{-1}{\alpha+1}}(t)\frac{w^\sigma(t)}{\xi^\sigma(t)},$$

$$B = \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left(\frac{\xi^\Delta(t)}{\xi^\sigma(t)}\right)^\alpha \left[\big(\beta\delta_1(t)\xi(t)\big)^{\frac{-\alpha}{\alpha+1}} \xi^\sigma(t) a^{\frac{1}{\alpha+1}}(t)\right]^\alpha$$

and using Lemma 2.1 we get

$$\beta\delta_1(t)\xi(t)a^{\frac{-1}{\alpha}}(t)\left(\frac{w^\sigma(t)}{\xi^\sigma(t)}\right)^{\frac{\alpha+1}{\alpha}} - \xi^\Delta(t)\frac{w^\sigma(t)}{\xi^\sigma(t)} \ge -\frac{(\frac{\alpha}{\beta})^\alpha a(t)(\xi^\Delta(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\delta_1(t)\xi(t))^\alpha},$$

where $\lambda = \frac{\alpha+1}{\alpha}$. Therefore, by (3.12), we have

$$w^\Delta(t) \le -q(t)\xi(t)\theta^\beta\big(t, g(T)\big) + \frac{(\frac{\alpha}{\beta})^\alpha a(t)(\xi^\Delta(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\delta_1(t)\xi(t))^\alpha}. \tag{3.13}$$

Integrating (3.13) from t_2 to t, we get as $t \to \infty$

$$\begin{aligned} w(t) &\le w(t_2) - \int_{t_2}^{t} \left[q(s)\xi(s)\theta^\beta\big(s, g(T)\big) - \frac{(\frac{\alpha}{\beta})^\alpha a(s)(\xi^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\delta_1(s)\xi(s))^\alpha} \right]\Delta s \\ &= -\infty, \end{aligned}$$

which contradicts (3.5). □

Theorem 3.2 *Assume that there exists a positive nondecreasing delta differentiable function $\xi(t)$ such that, for all sufficiently large $T \geq t_0$, and for $\tau(t) > g(T)$, we have*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \bigl[q(s)\xi(s)\theta^{\beta}\bigl(s,g(T)\bigr) - \delta_2(s)\eta^{\alpha}(s)\xi^{\Delta}(s)\bigr]\Delta s = \infty, \tag{3.14}$$

where

$$\delta_2(t) = \begin{cases} c_1 \text{ is any positive constant}, & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ c_2\eta^{\beta-\alpha}(t), c_2 \text{ is any positive constant}, & \text{if } \beta < \alpha, \end{cases} \tag{3.15}$$

and θ is as in Theorem 3.1. *Then* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof Let x be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1,\infty)$. Proceeding as in the proof of Theorem 3.1, we obtain (3.11). Therefore

$$w^{\Delta}(t) \leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{w^{\sigma}(t)}{\xi^{\sigma}(t)}. \tag{3.16}$$

Using the definition of $w(t)$, it follows from (3.16) that

$$\begin{aligned} w^{\Delta}(t) &\leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{(a(t)(x^{\Delta}(t))^{\alpha})^{\sigma}}{x^{\beta}(\sigma(t))} \\ &\leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\frac{a(t)(x^{\Delta}(t))^{\alpha}}{x^{\beta}(t)} \\ &= -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)a(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} x^{\alpha-\beta}(t). \end{aligned} \tag{3.17}$$

Now, from

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} \frac{(a(s)(x^{\Delta}(s))^{\alpha})^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}\Delta s \\ &\geq a^{\frac{1}{\alpha}}(t)x^{\Delta}(t)\int_{t_1}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)} \end{aligned}$$

it follows that

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} \leq a^{-1}(t)\left(\int_{t_1}^{t} \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}\right)^{-\alpha} = \frac{\eta^{\alpha}(t)}{a(t)}. \tag{3.18}$$

Using (3.18) in (3.17), we have

$$w^{\Delta}(t) \leq -q(t)\xi(t)\theta^{\beta}\bigl(t,g(T)\bigr) + \xi^{\Delta}(t)\bigl(\eta(t)\bigr)^{\alpha}x^{\alpha-\beta}(t). \tag{3.19}$$

Next as in the proof of Theorem 3.1, we consider the Cases (i), (ii) and (iii).

Case (i). $\beta > \alpha$.

In this case, since $x^{\Delta}(t) > 0$, there exists $t_2 \geq t_1$ such that $x^{\sigma}(t) \geq x(t) \geq c > 0$. This implies that $x^{\alpha-\beta}(t) \leq c_1$, where $c_1 = c^{\alpha-\beta}$.

Case (ii). $\beta = \alpha$.

In this case, we see that $x^{\alpha-\beta}(t) = 1$.

Case (iii). $\beta < \alpha$.

Proceeding as in the proof of Theorem 3.1, there exist a constant $b_1 > 0$ and $t_2 \geq t_1$ such that

$$x(t) \leq b_1 \eta^{-1}(t) \quad \text{for } t \geq t_2$$

and hence

$$x^{\alpha-\beta}(t) \geq c_2 \eta^{\beta-\alpha}(t) \quad \text{for } t \geq t_2,$$

where $c_2 = b_1^{\alpha-\beta}$.

Using these three cases in (3.19) and the definition of $\delta_2(t)$, we get

$$w^{\Delta}(t) \leq -q(t)\xi(t)\theta^{\beta}\big(t, g(T)\big) + \delta_2(t)\big(\eta(t)\big)^{\alpha}\xi^{\Delta}(t)$$

for $t \geq t_2$. Integrating the above inequality from t_2 to t, we have

$$0 < w(t) \leq w(t_2) - \int_{t_2}^{t} \big[q(s)\xi(s)\theta^{\beta}\big(s, g(T)\big) - \delta_2(s)\eta^{\alpha}(s)\xi^{\Delta}(s)\big]\Delta s$$

which gives a contradiction using (3.14). □

We next state and prove a Philos-type oscillation criterion for (1.1).

Theorem 3.3 *Assume that there exist functions H and h such that for each fixed t, $H(t,s)$ and $h(t,s)$ are rd-continuous with respect to s on $D = \{(t,s), t \geq s \geq t_0\}$ such that*

$$H(t,t) = 0, \quad t \geq t_0, \qquad H(t,s) > 0, \quad t > s \geq t_0 \tag{3.20}$$

and H has a non-positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and that it satisfies

$$-H^{\Delta_s}(t,s) - H(t,s)\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} = \frac{h(t,s)}{\xi^{\sigma}(s)}\big(H(t,s)\big)^{\frac{\alpha}{\alpha+1}} \tag{3.21}$$

and for all sufficiently large $T \geq t_0$, and for $\forall t \geq T$, $\tau(t) > g(T)$, we have

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t,T)} \\ &\quad\times \int_{T}^{t} \left[q(s)\xi(s)\theta^{\beta}\big(s, g(T)\big)H(t,s) - \frac{(\frac{\alpha}{\beta})^{\alpha}(h_-(t,s))^{\alpha+1}a(s)}{(\alpha+1)^{\alpha+1}\xi^{\alpha}(s)(\delta_1(s))^{\alpha}}\right]\Delta s \\ &\quad= \infty,\end{aligned} \tag{3.22}$$

where $\xi(s)$ is a positive Δ-differentiable function and $h_-(t,s) := \max\{0, -h(t,s)\}$. Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Let x be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1,\infty)$. Again we define $w(t)$ function as in the proof of Theorem 3.1. Then, proceeding as in the proof of Theorem 3.1, we obtain (3.12). Multiplying both sides of inequality (3.12), with t replaced by s, by $H(t,s)$ and integrating with respect to s from t_2 to t, we get

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)q(s)\xi(s)\theta^{\beta}\bigl(s,g(T)\bigr)\Delta s \\
&\quad\leq -\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_2}^{t} H(t,s)\xi^{\Delta}(s)\frac{w^{\sigma}(s)}{\xi^{\sigma}(s)}\Delta s \\
&\qquad - \beta\int_{t_2}^{t} H(t,s)a^{-\frac{1}{\alpha}}(s)\xi(s)\biggl(\frac{w^{\sigma}(s)}{\xi^{\sigma}(s)}\biggr)^{\frac{\alpha+1}{\alpha}}\delta_1(s)\Delta s. \qquad (3.23)
\end{aligned}$$

Integrating (3.23) by parts and using (3.20) and (3.21), we obtain

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\theta^{\beta}\bigl(s,g(T)\bigr)q(s)\xi(s)\Delta s \\
&\quad\leq H(t,t_2)w(t_2) + \int_{t_2}^{t} \frac{h(t,s)}{\xi^{\sigma}(s)}\bigl(H(t,s)\bigr)^{\frac{\alpha}{\alpha+1}}w^{\sigma}(s)\Delta s \\
&\qquad - \beta\int_{t_2}^{t} H(t,s)a^{-\frac{1}{\alpha}}(s)\xi(s)\biggl(\frac{w^{\sigma}(s)}{\xi^{\sigma}(s)}\biggr)^{\frac{\alpha+1}{\alpha}}\delta_1(s)\Delta s. \qquad (3.24)
\end{aligned}$$

Again, defining $A \geq 0$ and $B \geq 0$ by

$$A = \biggl[\frac{\beta H(t,s)\delta_1(s)\xi(s)}{a^{1/\alpha}(s)}\biggr]^{\frac{\alpha}{\alpha+1}}\frac{w^{\sigma}(s)}{\xi^{\sigma}(s)}, \qquad B = \frac{(h_-(t,s))^{\alpha}a^{\frac{\alpha}{\alpha+1}}(s)}{(\frac{\alpha+1}{\alpha})^{\alpha}(\beta\delta_1(s)\xi(s))^{\frac{\alpha^2}{\alpha+1}}}$$

and using Lemma 2.1 where $\lambda = \frac{\alpha+1}{\alpha}$, we get

$$\begin{aligned}
&\frac{\beta H(t,s)\xi(s)\delta_1(s)(w^{\sigma}(s))^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)(\xi^{\sigma}(s))^{\frac{\alpha+1}{\alpha}}} - \frac{h_-(t,s)}{\xi^{\sigma}(s)}\bigl(H(t,s)\bigr)^{\frac{\alpha}{\alpha+1}}w^{\sigma}(s) \\
&\quad\geq \frac{\frac{1}{\alpha}(h_-(t,s))^{\alpha+1}a(s)}{\beta^{\alpha}(\frac{\alpha+1}{\alpha})^{\alpha+1}\xi^{\alpha}(s)(\delta_1(s))^{\alpha}}. \qquad (3.25)
\end{aligned}$$

From (3.24) and (3.25), we have

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\biggl[H(t,s)\theta^{\beta}\bigl(s,g(T)\bigr)q(s)\xi(s) - \frac{(\frac{\alpha}{\beta})^{\alpha}(h_-(t,s))^{\alpha+1}a(s)}{(\alpha+1)^{\alpha+1}\xi^{\alpha}(s)(\delta_1(s))^{\alpha}}\biggr]\Delta s \leq w(t_2),$$

which contradicts assumption (3.22).

Now we introduce the following notation for $T \in [t_1,\infty)_{\mathbb{T}}$. For all sufficiently large T such that $g(T) < \tau(T)$,

$$p_* := \liminf_{t\to\infty}\frac{t^{\alpha}}{a(t)}\int_{\sigma(t)}^{\infty}\theta^{\beta}\bigl(s,g(T)\bigr)q(s)\Delta s, \qquad (3.26)$$

$$q_* := \liminf_{t\to\infty}\frac{1}{t}\int_{T}^{t}\frac{s^{\alpha+1}}{a(s)}\theta^{\beta}\bigl(s,g(T)\bigr)q(s)\Delta s, \qquad (3.27)$$

$$r_* := \liminf_{t\to\infty} \frac{t^\alpha w^\sigma(t)}{a(t)}, \tag{3.28}$$

$$R_* := \limsup_{t\to\infty} \frac{t^\alpha w^\sigma(t)}{a(t)}. \tag{3.29}$$

Assume that $l = \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. Note that $0 \le l \le 1$. We assume that

$$\int_T^\infty \theta^\beta\big(s, g(T)\big) q(s)\Delta s < \infty, \quad T \in [t_1, \infty)_{\mathbb{T}}. \tag{3.30}$$

□

Theorem 3.4 *Let $\beta \ge \alpha$ and assume $a(t)$ is a delta differentiable function such that $a^\Delta(t) \ge 0$ and* (3.30) *holds. Furthermore, assume $l > 0$ and*

$$p_* > \frac{(\frac{\alpha^2}{\beta c_1})^\alpha}{(\alpha+1)^{\alpha+1} l^{\alpha^2}} \tag{3.31}$$

or

$$p_* + q_* > \left(\frac{\alpha}{\beta c_1}\right)^\alpha \frac{1}{l^{\alpha(\alpha+1)}} \tag{3.32}$$

for all large T. Then every solution of (1.1) *is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume that there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)$. Again we define $w(t)$ as in the proof of Theorem 3.1 by putting $\xi(t) = 1$ and $\delta(t) = c_1$. Proceeding as in the proof of Theorem 3.1, we obtain from (3.12)

$$w^\Delta(t) \le -q(t)\theta^\beta\big(t, g(T)\big) - \frac{\beta c_1 (w^\sigma(t))^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(t)}. \tag{3.33}$$

First, we assume that inequality (3.31) holds. It follows from (3.5) and $a(t)(x^\Delta(t))^\alpha$ being strictly decreasing that

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t \frac{(a(s)(x^\Delta(s))^\alpha)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \Delta s \\ &\ge x(t_0) + \int_{t_0}^t \frac{(a(t)(x^\Delta(t))^\alpha)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \Delta s \ge a^{\frac{1}{\alpha}}(t) x^\Delta(t) \int_{t_0}^t \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)} \end{aligned}$$

and it follows that

$$a(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\alpha \le \left(\int_{t_0}^t \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}\right)^{-\alpha}$$

and hence

$$w(t) = a(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\alpha x^{\alpha-\beta}(t) \le c_1 \left(\int_{t_0}^t \frac{\Delta s}{a^{\frac{1}{\alpha}}(s)}\right)^{-\alpha}. \tag{3.34}$$

Using (1.2), we have $\lim_{t\to\infty} w(t) = 0$. Integrating (3.33) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^\sigma(t) \geq \int_{\sigma(t)}^{\infty} q(s)\theta^\beta\bigl(s, g(T)\bigr)\Delta s + \beta c_1 \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}\Delta s. \tag{3.35}$$

Multiplying (3.35) by $\frac{t^\alpha}{a(t)}$ we get

$$\frac{t^\alpha}{a(t)} w^\sigma(t) \geq \frac{t^\alpha}{a(t)} \int_{\sigma(t)}^{\infty} q(s)\theta^\beta\bigl(s, g(T)\bigr)\Delta s + \beta c_1 \frac{t^\alpha}{a(t)} \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}\Delta s. \tag{3.36}$$

Let $0 < \epsilon < l$. Then by the definition of p_* and r_* we can pick $t_1 \in [T, \infty)_{\mathbb{T}}$ sufficiently large, so we have

$$\begin{aligned}
\frac{t^\alpha w^\sigma(t)}{a(t)} &\geq (p_* - \epsilon) + \beta c_1 \frac{t^\alpha}{a(t)} \int_{\sigma(t)}^{\infty} \frac{s(w^\sigma(s))^{\frac{1}{\alpha}} s^\alpha w^\sigma(s) a(s)}{a^{\frac{1}{\alpha}}(s) a(s) s^{\alpha+1}}\Delta s \\
&\geq (p_* - \epsilon) + \beta c_1 (r_* - \epsilon)^{\frac{\alpha+1}{\alpha}} \frac{t^\alpha}{a(t)} \int_{\sigma(t)}^{\infty} \frac{a(s)}{s^{\alpha+1}}\Delta s \\
&\geq (p_* - \epsilon) + \frac{\beta c_1}{\alpha} (r_* - \epsilon)^{\frac{\alpha+1}{\alpha}} t^\alpha \int_{\sigma(t)}^{\infty} \frac{\alpha \Delta s}{s^{\alpha+1}};
\end{aligned} \tag{3.37}$$

for $\sigma(t) \in [t_1, \infty)_{\mathbb{T}}$. Using the Pötzsche chain rule we get

$$\begin{aligned}
\left(\frac{-1}{s^\alpha}\right)^\Delta &= \alpha \int_0^1 \frac{1}{[s + \mu h]^{\alpha+1}}\,dh \\
&\leq \int_0^1 \left(\frac{\alpha}{s^{\alpha+1}}\right) dh = \frac{\alpha}{s^{\alpha+1}}.
\end{aligned} \tag{3.38}$$

Then from (3.37) and (3.38), we have $t \in [t_0, \infty)_{\mathbb{T}}$, and

$$\frac{t^\alpha w^\sigma(t)}{a(t)} \geq (p_* - \epsilon) + \frac{\beta c_1}{\alpha} (r_* - \epsilon)^{\frac{\alpha+1}{\alpha}} \left(\frac{t}{\sigma(t)}\right)^\alpha.$$

Taking the lim inf of both sides as $t \to \infty$ we get

$$r_* \geq p_* - \epsilon + \frac{\beta c_1}{\alpha} (r_* - \epsilon)^{\frac{\alpha+1}{\alpha}} l^\alpha.$$

Since $\epsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - \frac{\beta c_1}{\alpha} (r_*)^\lambda l^\alpha, \tag{3.39}$$

where $\lambda = \frac{\alpha+1}{\alpha}$. By using Lemma 2.1, with

$$A := \left(\frac{\beta c_1}{\alpha}\right)^{\frac{\alpha}{\alpha+1}} r_* l^{\frac{\alpha^2}{\alpha+1}} \quad \text{and} \quad B := \frac{1}{(\frac{\alpha+1}{\alpha})^\alpha (\frac{\beta c_1}{\alpha})^{\frac{\alpha^2}{\alpha+1}} l^{\frac{\alpha^3}{\alpha+1}}}$$

we get

$$\frac{\beta c_1}{\alpha}(r_*)^{\frac{\alpha+1}{\alpha}} l^\alpha - r_* \geq -\frac{(\frac{\alpha^2}{\beta c_1})^\alpha}{(\alpha+1)^{\alpha+1} l^{\alpha^2}}. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$p_* \leq \frac{(\frac{\alpha^2}{\beta c_1})^\alpha}{(\alpha+1)^{\alpha+1} l^{\alpha^2}},$$

which contradicts (3.31). Next, we assume that (3.32) holds. Multiplying both sides of (3.33) by $\frac{t^{\alpha+1}}{a(t)}$, and integrating from T to t ($t \geq T$) we get

$$\int_T^t \frac{s^{\alpha+1} w^\Delta(s)}{a(s)} \Delta s \leq -\int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s - \beta c_1 \int_T^t \frac{s^{\alpha+1}(w^\sigma(s))^{\frac{\alpha+1}{\alpha}}}{a^{\frac{\alpha+1}{\alpha}}(s)} \Delta s.$$

Using integration by parts, the quotient rule and applying the Pötzsche chain rule, we obtain

$$\begin{aligned}\frac{t^{\alpha+1} w(t)}{a(t)} &\leq \frac{T^{\alpha+1} w(T)}{a(T)} + (\alpha+1) \int_T^t \frac{(\sigma(s))^\alpha}{a(s)} w^\sigma(s) \Delta s \\ &\quad - \int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s \\ &\quad - \beta c_1 \int_T^t \left(\frac{s^\alpha w^\sigma(s)}{a(s)}\right)^{\frac{\alpha+1}{\alpha}} \Delta s.\end{aligned}$$

Let $0 < \epsilon < l$ be given. Then using the definition of l, we can assume without loss of generality that T is sufficiently large so that

$$\frac{s}{\sigma(s)} \geq l - \epsilon, \quad s \geq t.$$

It follows that

$$\sigma(s) \leq Ls, \qquad s \geq T, \quad \text{where } L := \frac{1}{l-\epsilon} > 0.$$

Therefore

$$\begin{aligned}\frac{t^{\alpha+1} w(t)}{a(t)} &\leq \frac{T^{\alpha+1} w(T)}{a(T)} - \int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s \\ &\quad + \int_T^t \left[(\alpha+1) \frac{L^\alpha s^\alpha w^\sigma(s)}{a(s)} - \beta c_1 \left(\frac{s^\alpha w^\sigma(s)}{a(s)}\right)^{\frac{\alpha+1}{\alpha}}\right] \Delta s. \end{aligned} \tag{3.41}$$

It follows that

$$\begin{aligned}\frac{t^{\alpha+1} w(t)}{a(t)} &\leq \frac{T^{\alpha+1} w(T)}{a(T)} - \int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s \\ &\quad + \int_T^t \left[(\alpha+1) L^\alpha u(s) - \beta c_1 u^\lambda(s)\right] \Delta s,\end{aligned}$$

where $u(s) = \frac{s^\alpha w^\sigma(s)}{a(s)}$ and $\lambda = \frac{\alpha+1}{\alpha}$. Using the inequality of Lemma 2.1 with

$$A^\lambda = \beta c_1 u^\lambda(s) \quad \text{and} \quad B^{\lambda-1} = \frac{\alpha}{(\beta c_1)^{\frac{1}{\lambda}}} L^\alpha$$

we get

$$\begin{aligned} \beta c_1 u^\lambda(s) - (\alpha+1)L^\alpha u(s) &\geq -(\lambda-1)\left(\frac{\alpha}{(\beta c_1)^{\frac{1}{\lambda}}} L^\alpha\right)^{\frac{\lambda}{\lambda-1}} \\ &= -\frac{1}{\alpha}\left(\frac{\alpha}{(\beta c_1)^{\frac{1}{\lambda}}} L^\alpha\right)^{\alpha+1} \\ &= -\left(\frac{\alpha}{\beta c_1}\right)^\alpha L^{\alpha(\alpha+1)}. \end{aligned} \tag{3.42}$$

Then we have

$$\frac{t^{\alpha+1} w(t)}{a(t)} \leq \frac{T^{\alpha+1} w(T)}{a(T)} - \int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s + \left(\frac{\alpha}{\beta c_1}\right)^\alpha L^{\alpha(\alpha+1)}(t-T).$$

Since $w^\Delta(t) \leq 0$ and dividing the last inequality by t we have

$$\frac{t^\alpha w^\sigma(t)}{a(t)} \leq \frac{T^{\alpha+1} w(T)}{ta(T)} - \frac{1}{t}\int_T^t \frac{s^{\alpha+1} q(s) \theta^\beta(s, g(T))}{a(s)} \Delta s + \left(\frac{\alpha}{\beta c_1}\right)^\alpha L^{\alpha(\alpha+1)}\left(1 - \frac{T}{t}\right).$$

Taking the lim sup of both sides as $t \to \infty$ we obtain

$$R_* \leq -q_* + \left(\frac{\alpha}{\beta c_1}\right)^\alpha L^{\alpha(\alpha+1)}.$$

Since $0 < \epsilon < l$ is arbitrary, we get

$$R_* \leq -q_* + \left(\frac{\alpha}{\beta c_1}\right)^\alpha \frac{1}{l^{\alpha(\alpha+1)}}.$$

Using (3.39), we have

$$p_* \leq r_* - \frac{\beta c_1}{\alpha}(r_*)^{\frac{\alpha+1}{\alpha}} l^\alpha \leq r_* \leq R_* \leq -q_* + \left(\frac{\alpha}{\beta c_1}\right)^\alpha \frac{1}{l^{\alpha(\alpha+1)}}.$$

Therefore

$$p_* + q_* \leq \left(\frac{\alpha}{\beta c_1}\right)^\alpha \frac{1}{l^{\alpha(\alpha+1)}},$$

which contradicts (3.32). □

4 Examples

In this section, we give some examples to illustrate our main results. To obtain the conditions for oscillations, we will use the following fact:

$$\int_{t_0}^\infty \frac{\Delta t}{t^p} = \infty, \quad 0 \leq p \leq 1, t \in [t_0, \infty)_{\mathbb{T}}.$$

Example 4.1 Consider the second-order delay half-linear dynamic equation on times scales

$$\left(\frac{1}{(t+\sigma(t))^3}\left(x^{\Delta}(t)\right)^3\right)^{\Delta}+\frac{(t+1)^3}{t^2}x^3(\sqrt{t})=0,\quad t\in[1,\infty)_{\mathbb{T}},\tag{4.1}$$

where $a(t)=\frac{1}{(t+\sigma(t))^3}$, $q(t)=\frac{(t+1)^3}{t^2}$, $\tau(t)=\sqrt{t}$ and $\alpha=\beta=3$. We see that

$$\int_{t_0}^{\infty}\frac{\Delta s}{a^{1/\alpha}(s)}=\int_{t_0}^{\infty}\left(t+\sigma(t)\right)\Delta t=\int_{t_0}^{\infty}\left(t^2\right)^{\Delta}\Delta t=\infty,\quad t\in[t_0,\infty)_{\mathbb{T}}.$$

Taking $T\geq t_0$ such that $g(T)=1$, we get $\forall t>t_0$, $\tau(t)>1$. Then

$$\theta\left(t,g(T)\right)=\frac{\int_1^{t^{1/2}}\frac{\Delta s}{a^{1/\alpha}(s)}}{\int_1^t\frac{\Delta s}{a^{1/\alpha}(s)}}=\frac{t-1}{t^2-1}=\frac{1}{t+1}.$$

Then, taking $\xi(t)=t$ by Theorem 3.1, we have

$$\begin{aligned}&\limsup_{t\to\infty}\int_T^t\left[\theta^{\alpha}(s,1)\xi(s)q(s)-\frac{a(s)(\xi^{\Delta}(s)_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\xi^{\alpha}(s)}\right]\Delta s\\&\quad=\limsup_{t\to\infty}\int_1^t\left[\frac{1}{(s+1)^3}\cdot s\cdot\frac{(s+1)^3}{s^2}-\frac{1}{4^4s^3(s+\sigma(s))^3}\right]\Delta s\\&\quad\geq\limsup_{t\to\infty}\int_1^t\left[\frac{1}{s}-\frac{1}{4^4s^3(2s)^3}\right]\Delta s=\infty,\quad t\in[1,\infty)_{\mathbb{T}}.\end{aligned}$$

Then (4.1) is oscillatory.

Example 4.2 Consider the second-order delay half-linear dynamic equation on times scales

$$\left(\frac{1}{(t+\sigma(t))^3}\left(x^{\Delta}(t)\right)^3\right)^{\Delta}+\frac{(T^2t^2-1)^3}{t^2(T^2t-1)^3}x^3(\sqrt{t})=0,\quad t\in[1,\infty)_{\mathbb{T}},\tag{4.2}$$

where T is sufficiently large, $a(t)=\frac{1}{(t+\sigma(t))^3}$, $q(t)=\frac{(T^2t^2-1)^3}{t^2(T^2t-1)^3}$, $\alpha=\beta=3$ and $\tau(t)=t^{1/2}$.

If we choose $g(T)=\frac{1}{T}$, we get

$$\theta\left(t,\frac{1}{T}\right)=\frac{\int_{\frac{1}{T}}^{t^{1/2}}\frac{\Delta s}{a^{1/\alpha}(s)}}{\int_{\frac{1}{T}}^{t}\frac{\Delta s}{a^{1/\alpha}(s)}}=\frac{t-\frac{1}{T^2}}{t^2-\frac{1}{T^2}}=\frac{T^2t-1}{T^2t^2-1}.$$

When $\xi(t)=t$ by Theorem 3.1, we have

$$\begin{aligned}&\limsup_{t\to\infty}\int_1^t\left[\left(\frac{T^2s-1}{T^2s^2-1}\right)^3\frac{(T^2s^2-1)^3}{s^2(T^2s-1)^3}s-\frac{1}{4^4s^3(s+\sigma(s))^3}\right]\Delta s\\&\quad\geq\limsup_{t\to\infty}\int_1^t\left[\frac{1}{s}-\frac{1}{4^4s^3(2s)^3}\right]\Delta s=\infty.\end{aligned}$$

Then (4.2) is oscillatory on $[1,\infty)_{\mathbb{T}}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AFG conceived of the study, and participated in its design and coordination. FN carried out the mathematical studies and participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

Received: 30 April 2013 Accepted: 10 January 2014 Published: 29 Jan 2014

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10.1186/1687-1847-2014-45
Cite this article as: Güvenilir and Nizigiyimana: **Oscillation criteria for second-order quasi-linear delay dynamic equations on time scales.** *Advances in Difference Equations* 2014, 2014:45