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Hybrid iterative scheme for a generalized equilibrium problems, variational inequality problems and fixed point problem of a finite family of κ_i-strictly pseudocontractive mappings

Atid Kangtunyakarn

Correspondence:
beawrock@hotmail.com
Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this article, by using the *S*-mapping and hybrid method we prove a strong convergence theorem for finding a common element of the set of fixed point problems of a finite family of κ_i-strictly pseudocontractive mappings and the set of generalized equilibrium defined by Ceng et al., which is a solution of two sets of variational inequality problems. Moreover, by using our main result we have a strong convergence theorem for finding a common element of the set of fixed point problems of a finite family of κ_i-strictly pseudocontractive mappings and the set of solution of a finite family of generalized equilibrium defined by Ceng et al., which is a solution of a finite family of variational inequality problems.

Keywords: κ-strict pseudo contraction mapping, α-inverse strongly monotone, generalized equilibrium problem, variational inequality, the *S*-mapping

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$) Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

Recall the mapping T is said to be *κ-strict pseudo-contration* if there exist $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \forall x, y \in D(T). \tag{1.1}$$

Note that the class of κ-strict pseudo-contractions strictly includes the class of nonexpansive mappings, that is T is nonexpansive if and only if T is 0-strict pseudo-contractive. If $\kappa = 1$, T is said to be *pseudo-contraction mapping. T* is *strong pseudo-contraction* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contraction. In a real Hilbert space H (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2 \ \forall x, y \in D(T). \tag{1.2}$$

T is pseudo-contraction if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \quad \forall x, y \in D(T).$$

T is strong pseudo-contraction if there exists a positive constant $\lambda \in (0, 1)$

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2 \quad \forall x, y \in D(T)$$

The class of κ-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings and class of strong pseudo-contraction mappings is independent of the class of κ-strict pseudo-contraction.

A mapping A of C into H is called *inverse-strongly monotone*, see [2] if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$.

The equilibrium problem for G is to determine its equilibrium points, i.e., the set

$$EP(G) = \{x \in G : G(x, y) \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

Given a mapping $T : C \to H$, let $G(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0 \tag{1.4}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$.

In 2005, Combettes and Hirstoaga [3] introduced some iterative schemes of finding the best approximation to the initial data when EP(G) is nonempty and proved strong convergence theorem.

Also in [3] Combettes and Hiratoaga, following [4] define $S_r : H \to C$ by

$$S_r(x) = \{z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \forall y \in C\}. \tag{1.5}$$

hey proved that under suitable hypotheses G, S_r is single-valued and firmly nonexpansive with $F(S_r) = EP(G)$.

Numerous problems in physics, optimization, and economics reduce to find a element of $EP(G)$ (see, e.g., [5-16])

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(.,.)$ be the Hausdorff metric on CB(H) defined as

$$\mathcal{H}(U, V) = \max\left\{\sup_{u\in U} d(u, V), \sup_{v\in V} d(U, v)\right\}, \quad \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v\in V} d(u, v)$, $d(U, v) = \inf_{u\in U} d(u, v)$, and $d(u, v) = \|u - v\|$.

Let C be a nonempty closed convex subset of H. Let $\phi : C \to \mathbb{R}$ be a real-valued function, $T : C \to CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \to \mathbb{R}$ an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued map $T : C \to CB(H)$.

(H1) For each fixed $v \in C$, $(\omega, u) \mapsto \Phi(\omega, u, v)$ is an upper semicontinuous function from $H \times C$ to $\mathbb{R}$, that is, for $(\omega, u) \in H \times C$, whenever $\omega_n \to \omega$ and $u_n \to u$ as $n \to \infty$,

$$\limsup_{n\to\infty} \Phi(\omega_n, u_n, v) \leq \Phi(\omega, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

In 2009, Ceng et al. [17] introduced the following generalized equilibrium problem (GEP) as follows:

$$(\text{GEP}) \begin{cases} \text{Find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \ \forall v \in C. \end{cases} \tag{1.6}$$

The set of such solutions $u \in C$ of (GEP) is denote by $(GEP)_s(\Phi, \phi)$.

In the case of $\phi \equiv 0$ and $\Phi(w, u, v) \equiv G(u, v)$, then $(GEP)_s(\Phi, \phi)$ is denoted by $EP(G)$. By using Nadler's theorem they introduced the following algorithm:

Let $x_1 \in C$ and $w_1 \in T(x_1)$, there exists sequences $\{w_n\} \subseteq H$ and $\{x_n\}$, $\{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi\,(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \geq 0, \ \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad n = 1, 2, ... \end{cases} \tag{1.7}$$

They proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.7) as follows:

Theorem 1.1. *(See* [17]*) Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H and let $\phi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$ -Lipschitz continuous with constant μ, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3) and S be a nonexpansive mapping of C into itself such that $F(S) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}$, $\{w_n\}$, and $\{u_n\}$ be sequences generated by (1.7), where $\{\alpha_n\} \subseteq [0,1]$ and $\{r_n\} \subseteq (0, \infty)$ satisfy*

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If there exists a constant $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda \left\| T_{r_1}(x_1) - T_{r_2}(x_2) \right\|^2 \tag{1.8}$$

for all $(r_1, r_2) \in \Xi \times \Xi$,$(x_1, x_2) \in C \times C$ and $w_i \in T(x_i)$, $i = 1, 2$, where $\Xi = \{r_n : n \geq 1\}$, then for $\widehat{x} = P_{F(S)\cap(GEP)_s(\Phi,\varphi)} f(\widehat{x})$, there exists $\widehat{w} \in T(\widehat{x})$ such that $(\widehat{x}, \widehat{w})$ is a solution of (GEP) and

$$x_n \to \widehat{x}, \ w_n \to \widehat{w} \text{ and } u_n \to \widehat{x} \text{ as } n \to \infty.$$

In 2011, Kangtunyakarn [18] proved the following theorem for strict pseudocontractive mapping in Hilbert space by using hybrid method as follows:

Theorem 1.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F and G be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A_1)-(A_4), respectively. Let $A : C \to H$ be a α-inverse strongly monotone mapping and let $B : C \to H$ be a β-inverse strongly monotone mapping. Let $T : C \to C$ be a κ-strict pseudo-contraction mapping with $\mathbb{F} = F(T) \cap EP(F, A) \cap EP(G, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C = C_1$ and*

$$\begin{cases} F(u_n, u) + (Ax_n, u - u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ G(v_n, v) + (Bx_n, v - v_n) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ z_n = \delta_n u_n + (1 - \delta_n)v_n \\ y_n = \alpha_n z_n + (1 - \alpha_n)Tz_n \\ C_{n+1} = \left\{z \in C_n : \left\|y_n - z\right\| \leq \|x_n - z\|\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in $[0,1]$, $r_n \in [a, b] \subset (0, 2\alpha)$ and $s_n \subset [c, d] \subset (0, 2\beta)$ satisfy the following condition:

$(i)\ \lim_{n\to\infty} \delta_n = \delta \in (0, 1)$

$(ii)\ 0 \leq \kappa \leq \alpha_n < 1, \quad \forall n \geq 1$

Then x_n converges strongly to $P_{\mathbb{F}}x_1$.

From motivation of (1.7) and (1.9), we define the following algorithm as follows:

Algorithm 1.3. *Let T_i, i = 1,2,...,N, be κ_i-pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1,2,..., N\}$ and let S_n be the S-mappings generated by T_1, T_2, ..., T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = \left(\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}\right) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \ldots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all j = 1,2,...,N. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{w_n^1\}, \{w_n^2\} \in H$ and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$ such that*

$$\begin{cases} w_n^1 \in T(x_n), \quad \left\|w_n^1 - w_{n+1}^1\right\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \ \left\|w_n^2 - w_{n+1}^2\right\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1\left(w_n^1, u_n, u\right) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \quad \forall u \in C, \\ \Phi_2\left(w_n^2, v_n, v\right) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \quad \forall v \in C, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n)P_C(I - \eta B)v_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \left\{z \in C_n : \left\|y_n - z\right\| \leq \|x_n - z\|\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1. \end{cases} \tag{1.10}$$

where D, $T : C \to CB(H)$ are $\mathcal{H}$ -Lipschitz continuous with constant μ_1, μ_2, respectively, Φ_1, $\Phi_2 : H \times C \times C \to \mathbb{R}$ are equilibrium-like functions satisfying $(H1)$-$(H3)$, A :

$C \to H$ *is* a α-inverse strongly monotone mapping and $B : C \to H$ is a β-inverse strongly monotone mapping.

In this article, we prove under some control conditions on $\{\delta_n\}$, $\{\alpha_n\}$, $\{s_n\}$, and $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $P_{\mathbb{F}}x_1$ where $\mathbb{F} = \cap_{i=1}^{N} F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, $G_1, G_2 : C \to C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ and $P_{\mathbb{F}}x_1$ is solution of the following system of variational inequality:

$$\begin{cases} \langle Ax^*,\ x - x^* \rangle \geq 0, \\ \langle Bx^*,\ x - x^* \rangle \geq 0. \end{cases}$$

2 Preliminaries

In this section, we need the following lemmas and definition to prove our main result.

Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

The following lemma is a property of P_C.

Lemma 2.1. *(See* [19]*.) Given $x \in H$ and $y \in C$. Then $P_Cx = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2. *(See* [20]*) Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \cap_{n=1}^{\infty} F(T_n)$ hold.

The following lemma is well known.

Lemma 2.3. *Let H be Hilbert space, C be a nonempty closed convex subset of H. Let $T : C \to C$ be a κ-strictly pseudo-contractive, then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

In 2009, Kangtunyakarn and Suantai [21] introduced the S-mapping generated by a finite family of κ-strictly pseudo contractive mappings and real numbers as follows:

Definition 2.1. *Let C be a nonempty convex subset of real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strict pseudo-contractions of C into itself. For each j =* 1,2,..., *N, let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ *where* $I \in [0,1]$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. *We define the mapping* $S : C \to C$ *as follows:*

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\ &\ \ \vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \tag{2.1}$$

This mapping is called *S-mapping* generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 2.4. *(See* [21]*) Let C be a nonempty closed convex subset of real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ*-strict pseudo contraction mapping of C into C with* $\cap_{i=1}^N F(T_i) \neq \emptyset$ *and* $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *j =* 1,2,3,...,*N, where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ *for all j =* 1,2,...,*N* - 1 *and* $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ *for all j =* 1,2,..., *N. Let S be the mapping generated by* $T_1, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$. *Then* $F(S) = \cap_{i=1}^N F(T_i)$ *and S is a nonexpansive mapping.*

Lemma 2.5. *(See* [22]*) Let C be a nonempty closed convex subset of a real Hilbert space H and* $S : C \to C$ *be a self-mapping of C. If S is a* κ*-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \le \frac{1+\kappa}{1-\kappa} \|x - y\|, \quad \forall x, \quad y \in C.$$

We prove the following lemma by using the concept of the *S*-mapping as follows:

Lemma 2.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* T_i, *i =* 1,2,...,*N be* κ_i *strictly pseudo-contraction mappings of C into itself and* $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ *and let* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ *such that* $\alpha_i^{n,j} \to \alpha_i^j \in [0, 1]$ *as* $n \to \infty$ *for i =* 1, 3 *and j =* 1,2,3,..., *N. For every* $n \in \mathbb{N}$, *let S and* S_n *be the S-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$ *and* $T_1, T_2, ..., T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, *respectively. Then* $\lim_{n\to\infty} \|S_n x_n - S x_n\| = 0$ *for every bounded sequence* $\{x_n\}$ *in C.*

Proof. Let $\{x_n\}$ be bounded sequence in *C*, U_k and $U_{n,k}$ be generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$ and $T_1, T_2, ..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\left\|U_{n,1}x_n - U_1x_n\right\| &= \left\|\alpha_1^{n,1}T_1x_n + (1-\alpha_1^{n,1})x_n - \alpha_1^1T_1x_n - (1-\alpha_1^1)x_n\right\| \\ &= \left\|\alpha_1^{n,1}T_1x_n - \alpha_1^{n,1}x_n - \alpha_1^1T_1x_n + \alpha_1^1x_n\right\| \\ &= \left\|(\alpha_1^{n,1}-\alpha_1^1)T_1x_n - (\alpha_1^{n,1}-\alpha_1^1)x_n\right\| \\ &= \left|a_1^{n,1}-\alpha_1^1\right| \|T_1x_n - x_n\|\end{aligned} \tag{2.2}$$

and for $k \in \{2, 3,..., N\}$, by using Lemma 2.5, we obtain

$$\begin{aligned}\left\|U_{n,k}x_n - U_kx_n\right\| &= \left\|\alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_2^{n,k}U_{n,k-1}x_n + \alpha_3^{n,k}x_n\right. \\ &\quad \left. -\alpha_1^kT_kU_{k-1}x_n - \alpha_2^kU_{k-1}x_n - \alpha_3^kx_n\right\| \\ &= \left\|\alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_3^{n,k}x_n - \alpha_1^kT_kU_{k-1}x_n - \alpha_3^kx_n\right. \\ &\quad \left. +\alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n\right\| \\ &= \left\|\alpha_1^{n,k}T_kU_{n,k-1}x_n - \alpha_1^{n,k}T_kU_{k-1}x_n + \alpha_1^{n,k}T_kU_{k-1}x_n\right. \\ &\quad \left. -\alpha_1^kT_kU_{k-1}x_n + (\alpha_3^{n,k}-\alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n\right\| \\ &= \left\|\alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k}-\alpha_1^k)T_kU_{k-1}x_n\right. \\ &\quad \left. +(\alpha_3^{n,k}-\alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n\right\| \\ &= \left\|\alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k}-\alpha_1^k)T_kU_{k-1}x_n\right. \\ &\quad + (\alpha_3^{n,k}-\alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^{n,k}U_{k-1}x_n \\ &\quad \left. +\alpha_2^{n,k}U_{k-1}x_n - \alpha_2^kU_{k-1}x_n\right\| \\ &= \left\|\alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k}-\alpha_1^k)T_kU_{k-1}x_n\right. \\ &\quad + (\alpha_3^{n,k}-\alpha_3^k)x_n + \alpha_2^{n,k}(U_{n,k-1}x_n - U_{k-1}x_n) \\ &\quad \left. +(\alpha_2^{n,k}-\alpha_2^k)U_{k-1}x_n\right\| \\ &\le \alpha_1^{n,k}\left\|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\right\| + \left|\alpha_1^{n,k}-\alpha_1^k\right| \|T_kU_{k-1}x_n\| \\ &\quad + \left|\alpha_3^{n,k}-\alpha_3^k\right| \|x_n\| + \alpha_2^{n,k}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| \\ &\quad + \left|\alpha_2^{n,k}-\alpha_2^k\right| \|U_{k-1}x_n\| \\ &= \alpha_1^{n,k}\left\|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\right\| + \left|\alpha_1^{n,k}-\alpha_1^k\right| \|T_kU_{k-1}x_n\| \\ &\quad + \alpha_2^{n,k}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left|1-\alpha_1^{n,k}-\alpha_3^{n,k}-1\right. \\ &\quad \left.+\alpha_1^k+\alpha_3^k\right| \|U_{k-1}x_n\| + \left|\alpha_3^{n,k}-\alpha_3^k\right| \|x_n\| \\ &\le \alpha_1^{n,k}\frac{1+\kappa}{1-\kappa}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left|\alpha_1^{n,k}\right. \\ &\quad \left.-\alpha_1^k\right| \|T_kU_{k-1}x_n\| + \alpha_2^{n,k}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left(\left|\alpha_1^k-\alpha_1^{n,k}\right|\right. \\ &\quad \left.+\left|\alpha_3^{n,k}-\alpha_3^k\right|\right) \|U_{k-1}x_n\| + \left|\alpha_3^{n,k}-\alpha_3^k\right| \|x_n\| \\ &\le \frac{1+\kappa}{1-\kappa}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left|\alpha_1^{n,k}-\alpha_1^k\right| \|T_kU_{k-1}x_n\| \\ &\quad + \frac{1-\kappa}{1-\kappa}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left(\left|\alpha_1^k-\alpha_1^{n,k}\right|\right. \\ &\quad \left.+\left|\alpha_3^{n,k}-\alpha_3^k\right|\right) \|U_{k-1}x_n\| + \left|\alpha_3^{n,k}-\alpha_3^k\right| \|x_n\| \\ &\le \frac{2}{1-\kappa}\left\|U_{n,k-1}x_n - U_{k-1}x_n\right\| + \left|\alpha_1^{n,k}-\alpha_1^k\right| (\|T_kU_{k-1}x_n\| + \|U_{k-1}x_n\|) \\ &\quad + \left|\alpha_3^{n,k}-\alpha_3^k\right| (\|U_{k-1}x_n\| + \|x_n\|).\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
\|S_nx_n - Sx_n\| &= \left\|U_{n,N}x_n - U_Nx_n\right\| \\
&\le \frac{2}{1-\kappa}\left\|U_{n,N-1}x_n - U_{N-1}x_n\right\| + \left|\alpha_1^{n,N} - \alpha_1^N\right| (\|T_NU_{N-1}x_n\| \\
&\quad + \|U_{N-1}x_n\|) + \left|\alpha_3^{n,N} - \alpha_3^N\right| (\|U_{N-1}x_n\| + \|x_n\|) \\
&\le \frac{2}{1-\kappa}\left(\frac{2}{1-\kappa}\left\|U_{n,N-2}x_n - U_{N-2}x_n\right\| + \left|\alpha_1^{n,N-1}\right.\right. \\
&\quad \left.-\alpha_1^{N-1}\right| (\|T_{N-1}U_{N-2}x_n\| \\
&\quad \left.+ \|U_{N-2}x_n\|) + \left|\alpha_3^{n,N-1} - \alpha_3^{N-1}\right| (\|U_{N-2}x_n\| + \|x_n\|)\right) \\
&\quad + \left|\alpha_1^{n,N} - \alpha_1^N\right| (\|T_NU_{N-1}x_n\| + \|U_{N-1}x_n\|) \\
&\quad + \left|\alpha_3^{n,N} - \alpha_3^N\right| (\|U_{N-1}x_n\| + \|x_n\|) \\
&= \left(\frac{2}{1-\kappa}\right)^2 \left\|U_{n,N-2}x_n - U_{N-2}x_n\right\| + \sum_{j=N-1}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_1^{n,j}\right. \\
&\quad \left.-\alpha_1^j\right| \left(\left\|T_jU_{j-1}x_n\right\| + \left\|U_{j-1}x_n\right\|\right) \\
&\quad + \sum_{j=N-1}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^j\right| \left(\left\|U_{j-1}x_n\right\| + \|x_n\|\right) \\
&\;\;\vdots \\
&\le \left(\frac{2}{1-\kappa}\right)^{N-1}\left\|U_{n,1}x_n - U_1x_n\right\| + \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_1^{n,j}\right. \\
&\quad \left.-\alpha_1^j\right| \left(\left\|T_jU_{j-1}x_n\right\| + \left\|U_{j-1}x_n\right\|\right) + \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^j\right| \left(\left\|U_{j-1}x_n\right\| + \|x_n\|\right) \\
&= \left(\frac{2}{1-\kappa}\right)^{N-1}\left|\alpha_1^{n,1} - \alpha_1^1\right| \|T_1x_n - x_n\| + \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_1^{n,j}\right. \\
&\quad \left.-\alpha_1^j\right| \left(\left\|T_jU_{j-1}x_n\right\| + \left\|U_{j-1}x_n\right\|\right) \\
&\quad + \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^j\right| \left(\left\|U_{j-1}x_n\right\| + \|x_n\|\right).
\end{aligned}$$

This together with the assumption $\alpha_i^{n,j} \to \alpha_i^j$ *as* $n \to \infty$ (i = 1, 3, j = 1,2,..., N), we can conclude that

$$\lim_{n\to\infty} \|S_nx_n - Sx_n\| = 0.$$

Lemma 2.7. *(See* [23]*) Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and* $S : C \to C$ *be a nonexpansive mapping. Then I - S is demi-closed at zero.*

Lemma 2.8. *(See* [24]*) Let C be a closed convex subset of H. Let* $\{x_n\}$ *be a sequence in H and* $u \in H$*. Let* $q = P_Cu$*, if* $\{x_n\}$ *is such the* $\omega(x_n) \subset C$ *and satisfy the condition*

$$\|x_n - u\| \le \left\|u - q\right\|, \quad \forall n \in \mathbb{N}.$$

Then $x_n \to q$*, as* $n \to \infty$.

Definition 2.2. *A multivalued map* $T : C \to CB(H)$ *is say to be* $\mathcal{H}$ *-Lipschitz continuous if there exists a constant* $\mu > 0$ *such that*

$$\mathcal{H}(T(u) - T(v)) \le \mu \|u - v\|, \quad \forall u, v \in C,$$

where $\mathcal{H}(.,.)$ *is the Hausdorff metric on* $CB(H)$.

Lemma 2.9. *(Nadler's theorem, see* [25]*) Let* $(X, \|\cdot\|)$ *be a normed vector space and* $\mathcal{H}(.,.)$ *is the Hausdorff metric on* $CB(H)$. *If* $U, V \in CB(X)$, *then for any given* $\epsilon > 0$ *and* $u \in U$, *there exists* $v \in V$ *such that*

$$\|u - v\| \le (1 + \varepsilon)\mathcal{H}(U, V).$$

Let C be a nonempty closed convex subset of a real Hilbert space H. *Let* $\phi: C \to H$ *be a* real-valued function, $T: C \to CB(H)$ be a multivalued map and $\Phi: H \times C \times C \to \mathbb{R}$ be an equilibrium-like function.

To solve the GEP, let us assume that the equilibrium-like function $\Phi: H \times C \times C \to \mathbb{R}$ satisfies the following conditions with respect to the multivalued map $T: C \to CB(H)$.

$(H1)$ For each fixed $v \in C$, $(\omega, u) \mapsto \Phi(\omega, u, v)$ is an upper semicontinuous function from $H \times C$ *to* $\mathbb{R}$, that is, for $(\omega, u) \in H \times C$, whenever $\omega_n \to \omega$ and $u_n \to u$ as $n \to \infty$,

$$\limsup_{n\to\infty} \Phi(\omega_n, u_n, v) \le \Phi(\omega, u, v);$$

$(H2)$ For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

$(H3)$ For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

Theorem 2.10. *(See* [17]*) Let* C *be a nonempty, bounded, closed, and convex subset of a real Hilbert space* H, *and let* $\phi: C \to \mathbb{R}$ *be a lower semicontinuous and convex functional. Let* $T: C \to CB(H)$ *be* $\mathcal{H}$ *-Lipschitz continuous with constant* μ, *and* $\Phi: H \times C \times C \to \mathbb{R}$ *be an equilibrium-like function satisfying* $(H1)$-$(H3)$. *Let* $r > 0$ *be a constant. For each* $x \in C$, *take* $w_x \in T(x)$ *arbitrarily and define a mapping* $T_r: C \to C$ *as follows:*

$$T_r(x) = \left\{ u \in C : \Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \ge 0, \quad \forall v \in C \right\}.$$

Then, there hold the following:

(a) T_r *is single-valued;*

(b) T_r *is firmly nonexpansive (that is, for any* $u, v \in C$, $\|T_r u - T_r v\|^2 \le \langle T_r u - T_r v, u - v\rangle$) *if*

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \le 0,$$

for all $(x_1, x_2) \in C \times C$ *and all* $w_i \in T(x_i)$, $i = 1,2$;

(c) $F(T_r) = (GEP)_s(\Phi, \phi)$

(d) $(GEP)_s(\Phi, \phi)$ *is closed and convex.*

Lemma 2.11. *(See* [26]*) Let* C *be a nonempty closed convex subset of a Hilbert space* H *and let* $G: C \to C$ *be defined by*

$$G(x) = P_C(x - \lambda Ax), \quad \forall x \in C,$$

with $\forall \lambda > 0$. *Then* $x^* \in VI(C, A)$ *if and only if* $x^* \in F(G)$.

3 Main results

In this section, we prove a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.10) to $P_{\mathbb{F}}x_1$.

Theorem 3.1. *Let C be a nonempty bounded, closed, and convex subset of Hilbert space H and let ϕ_1, ϕ_2 : be a lower semicontinuous and convex function. Let D, T : $C \to CB(H)$ be $\mathcal{H}$ -Lipschitz continuous with constant μ_1, μ_2, respectively, Φ_1,Φ_2: $H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying (H1) - (H3). Let A: $C \to H$ be a α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping, let T_i, i = 1,2,...,N, be κ_i-pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1,2,..., N\}$ with $\mathbb{F} = \cap_{i=1}^{N} F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, where G_1, G_2 : $C \to C$ are defined by $G_1(x)$ $P_C(x-\lambda Ax)$, $G_2(x) = P_C(x-\eta Bx)$, $\forall x \in C$. Let S_n be the S-mappings generated by $T_1, T_2, ..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_N^{(n)}$ where $\alpha_j^{(n)} = \left(\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}\right) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \le \alpha_1^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1, 2, ..., N-1, \kappa < c \le \alpha_1^{n,N} \le 1, \kappa \le \alpha_3^{n,N} \le d < 1, \kappa \le \alpha_2^{n,j} \le e < 1$ for all j = 1,2,...,N and let $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n^1\}$, and $\{w_n^2\}$ be sequences generated by (1.10), where $\{\alpha_n\}$ is a sequence in [0,1], r_n, $\lambda \in [a, b] \subset (0, 2\alpha)$ and s_n, $\eta \in [c, d] \subset (0, 2\beta)$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$,

(ii) $0 \le \kappa \le \alpha_n < 1$, $\forall n \ge 1$,

(iii) $\sum_{n=1}^{\infty} \left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right| < \infty, \sum_{n=1}^{\infty} \left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right| < \infty$, *for all j ∈ {1,2,3,...,N}.*

(iv) *There exists λ_1, λ_2 such that*

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \le -\lambda_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2 \text{and} \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \le -\lambda_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2 \end{cases} \tag{3.1}$$

for all $(r_1, r_2) \in \Theta \times \Theta$, $(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for i = 1,2 where $\Theta = \{r_n : n \ge 1\}$ and $\Xi = \{s_n : n \ge 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$ which is a solution of (3.2):

$$\begin{cases} \langle Ax^*, \ x - x^* \rangle \ge 0, \\ \langle Bx^*, \ x - x^* \rangle \ge 0. \end{cases} \tag{3.2}$$

Proof. From (3.1) for every $r \in \Theta$, we have

$$\Phi_1(w_1^1, T_r(x_1), T_r(x_2)) + \Phi_1(w_2^1, T_r(x_2), T_r(x_1)) \le -\lambda_1 \|T_r(x_1) - T_r(x_2)\|^2 \le 0, \tag{3.3}$$

for all $(x_1, x_2) \in C \times C$ and $w_i^1 \in T(x_i)$, $i = 1, 2$.

Similarly, for every $s \in \Xi$, we have

$$\Phi_2(w_1^2, T_s(x_1), T_s(x_2)) + \Phi_2(w_2^2, T_s(x_2), T_s(x_1)) \le -\lambda_2 \|T_s(x_1) - T_s(x_2)\|^2 \le 0. \tag{3.4}$$

for all $(x_1, x_2) \in C \times C$ and $w_i^2 \in D(x_i)$, $i = 1, 2$. From (3.3) and (3.4), we have Theorem 2.10 hold.

It is easy to see that $I - \lambda A$ and $I - \eta B$ are nonexpansive mapping. Indeed, since A is a α-inverse strongly monotone mapping with $\lambda \in (0, 2\alpha)$, we have

$$\begin{aligned}\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2 &= \left\|x-y-\lambda(Ax-Ay)\right\|^2\\ &= \left\|x-y\right\|^2-2\lambda\langle x-y,Ax-Ay\rangle+\lambda^2\left\|Ax-Ay\right\|^2\\ &\leq \left\|x-y\right\|^2-2\alpha\lambda\left\|Ax-Ay\right\|^2+\lambda^2\left\|Ax-Ay\right\|^2\\ &= \left\|x-y\right\|^2+\lambda(\lambda-2\alpha)\left\|Ax-Ay\right\|^2\\ &\leq \left\|x-y\right\|^2.\end{aligned}$$

Thus $(I - \lambda A)$ is nonexpansive, so is $I - \eta B$. Since

$$\Phi_1(w_n^1,u_n,u)+\varphi_1(u)-\varphi_1(u_n)+\frac{1}{r_n}\langle u_n-x_n,u-u_n\rangle\geq 0,\quad \forall u\in C,$$

and Theorem 2.10, we have $u_n = T_{r_n}x_n$. Since

$$\Phi_2(w_n^2,v_n,v)+\varphi_2(v)-\varphi_2(v_n)+\frac{1}{s_n}\langle v_n-x_n,v-v_n\rangle\geq 0,\quad \forall v\in C,$$

and Theorem 2.10, we have $v_n = T_{s_n}x_n$. Let $z \in \mathbb{F}$, again by Theorem 2.10, we have $z = T_{r_n}z = T_{s_n}z = P_C(I-\lambda A)z = P_C(I-\eta B)z$. From nonexpansiveness of $\{T_{r_n}\}$, $\{T_{s_n}\}$, $\{I-\lambda A\}$, and $\{I - \eta B\}$, we have

$$\begin{aligned}\|z_n-z\| &= \left\|\delta_n(P_C(I-\lambda A)u_n-z)+(1-\delta_n)(P_C(I-\eta B)v_n-z)\right\|\\ &\leq \delta_n\left\|P_C(I-\lambda A)u_n-z\right\|+(1-\delta_n)\left\|P_C(I-\eta B)v_n-z\right\|\\ &\leq \delta_n\left\|T_{r_n}x_n-z\right\|+(1-\delta_n)\left\|T_{s_n}x_n-z\right\|\\ &\leq \|x_n-z\|.\end{aligned}\tag{3.5}$$

By (3.5), we have

$$\begin{aligned}\left\|y_n-z\right\| &= \left\|\alpha_n(z_n-z)+(1-\alpha_n)(S_nz_n-z)\right\|\\ &\leq \alpha_n\|z_n-z\|+(1-\alpha_n)\|S_nz_n-z\|\\ &\leq \|z_n-z\|\leq\|x_n-z\|.\end{aligned}\tag{3.6}$$

Next, we show that C_n is closed and convex for every $n \in \mathbb{N}$. It is obvious that C_n is closed. In fact, we know that, for $z \in C_n$,

$$\left\|y_n-z\right\|\leq\|x_n-z\| \text{ is equivalent to } \left\|y_n-x_n\right\|^2+2\langle y_n-x_n,x_n-z\rangle\leq 0.$$

So, we have that $\forall z_1, z_2 \in C_n$ and $t \in (0,1)$, it follows that

$$\begin{aligned}&\left\|y_n-x_n\right\|^2+2\langle y_n-x_n,x_n-(tz_1+(1-t)z_2)\rangle\\ &\quad= t(2\langle y_n-x_n,x_n-z_1\rangle+\left\|y_n-x_n\right\|^2)\\ &\qquad+(1-t)(2\langle y_n-x_n,x_n-z_2\rangle+\left\|y_n-x_n\right\|^2)\\ &\quad\leq 0,\end{aligned}$$

then, we have C_n is convex. By Theorem 2.10 and Lemma 2.3, we conclude that $\mathbb{F}$ is closed and convex. This implies that $P_{\mathbb{F}}$ is well defined. Next, we show that $\mathbb{F}\subset C_n$ for every $n \in \mathbb{N}$. Putting $q \in \mathbb{F}$, by (3.6), it is easy to see that $q \in C_n$, then we have $\mathbb{F}\subset C_n$ for all $n \in \mathbb{N}$. Since $x_n = P_{C_n}x_1$, for every $w \in C_n$, we have

$$\|x_n-x_1\|\leq\|w-x_1\|,\quad\forall n\in\mathbb{N}.\tag{3.7}$$

In particular, we have

$$\|x_n - x_1\| \leq \|P_{\mathbb{F}} x_1 - x_1\|. \tag{3.8}$$

Since C is bounded, we have $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{y_n\}$. Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x_1$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_n - x_1\| \, \|x_1 - x_{n+1}\|, \end{aligned}$$

it implies that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.$$

Hence, we have $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2, \end{aligned} \tag{3.9}$$

it implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.10}$$

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,$$

by (3.10), we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.11}$$

Since

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

by (3.10) and (3.11), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.12}$$

Next, we show that

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0. \tag{3.13}$$

By definition of y_n, we have

$$y_n - z_n = (1 - \alpha_n)(S_n z_n - z_n). \tag{3.14}$$

Claim that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.15}$$

Putting $M_n = P_C(I - \lambda A)u_n$ and $N_n = P_C(I - \eta B)v_n$, we have

$$\|z_n - x_n\| \le \delta_n \|M_n - x_n\| + (1 - \delta_n) \|N_n - x_n\| . \tag{3.16}$$

Let $z \in \mathbb{F}$. Since T_{r_n} is firmly nonexpansive mapping and $T_{r_n}x_n = u_n$, we have

$$\begin{aligned}\|z - u_n\|^2 &= \left\|T_{r_n}z - T_{r_n}x_n\right\|^2 \\ &\le \langle T_{r_n}z - T_{r_n}x_n, z - x_n\rangle \\ &= \frac{1}{2}(\|u_n - z\|^2 + \|x_n - z\|^2 - \|u_n - x_n\|^2).\end{aligned}$$

Hence

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.17}$$

Since T_{r_n} is firmly nonexpansive mapping and $T_{s_n}x_n = v_n$, by using the same method as (3.17), we have

$$\|v_n - z\|^2 \le \|x_n - z\|^2 - \|v_n - x_n\|^2. \tag{3.18}$$

By nonexpansiveness of S_n and (3.17), (3.18), we have

$$\begin{aligned}\left\|y_n - z\right\|^2 &\le \|z_n - z\|^2 \\ &\le \delta_n\|u_n - z\|^2 + (1 - \delta_n)\|v_n - z\|^2 \\ &\le \delta_n(\|x_n - z\|^2 - \|u_n - x_n\|^2) + (1 - \delta_n)(\|x_n - z\|^2 - \|v_n - x_n\|^2) \\ &= \|x_n - z\|^2 - \delta_n\|u_n - x_n\|^2 - (1 - \delta_n)\|v_n - x_n\|^2,\end{aligned}$$

it implies that

$$\begin{aligned}\delta_n\|u_n - x_n\|^2 &\le \|x_n - z\|^2 - \left\|y_n - z\right\|^2 - (1 - \delta_n)\|v_n - x_n\|^2 \\ &\le \|x_n - z\|^2 - \left\|y_n - z\right\|^2 \\ &\le (\|x_n - z\| + \left\|y_n - z\right\|) \left\|x_n - y_n\right\| ,\end{aligned}$$

by (3.12) and condition (i), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.19}$$

By using the same method as (3.19), we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.20}$$

Since

$$\begin{aligned}\left\|y_n - z\right\|^2 &\le \alpha_n\|z_n - z\|^2 + (1 - \alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\left(\delta_n\|M_n - z\|^2\right. \\ &\quad \left.+(1 - \delta)\|N_n - z\|^2\right)\end{aligned} \tag{3.21}$$

Claim that

$$\lim_{n\to\infty} \|Au_n - Az\| = \lim_{n\to\infty} \|Bv_n - Bz\| = 0.$$

By nonexpansiveness of P_C, we have

$$\begin{aligned}
\left\|y_n - z\right\|^2 &\le \|z_n - z\|^2 \\
&\le \delta_n \left\|P_C(I-\lambda A)u_n - P_C(I-\lambda A)z\right\|^2 \\
&\quad + (1-\delta_n)\left\|P_C(I-\eta B)v_n - P_C(I-\eta B)z\right\|^2 \\
&\le \delta_n \left\|(I-\lambda A)u_n - (I-\lambda A)z\right\|^2 + (1-\delta_n)\left\|(I-\eta B)v_n - (I-\eta B)z\right\|^2 \\
&\le \delta_n \left\|(u_n - \lambda A u_n - (z-\lambda Az)\right\|^2 + (1-\delta_n)\left\|(v_n - \eta B v_n - (z-\eta Bz)\right\|^2 \\
&= \delta_n \left\|(u_n - z) - \lambda(Au_n - Az)\right\|^2 + (1-\delta_n)\left\|(v_n - z) - \eta(Bv_n - Bz)\right\|^2 \\
&= \delta_n (\|u_n - z\|^2 + \lambda^2\|Au_n - Az\|^2 - 2\lambda\langle u_n - z, Au_n - Az\rangle) \\
&\quad + (1-\delta_n)(\|v_n - z\|^2 + \eta^2\|Bv_n - Bz\|^2 - 2\eta\langle v_n - z, Bv_n - Bz\rangle) \\
&\le \delta_n (\|u_n - z\|^2 + \lambda^2\|Au_n - Az\|^2 - 2\lambda\alpha\|Au_n - Az\|^2) \\
&\quad + (1-\delta_n)(\|v_n - z\|^2 + \eta^2\|Bv_n - Bz\|^2 - 2\eta\beta\|Bv_n - Bz\|^2) \\
&\le \delta_n (\|x_n - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_n - Az\|^2) \\
&\quad + (1-\delta_n)(\|x_n - z\|^2 + \eta(\eta - 2\beta)\|Bv_n - Bz\|^2) \\
&= \|x_n - z\|^2 - \delta_n\lambda(2\alpha - \lambda)\|Au_n - Az\|^2 \\
&\quad - (1-\delta_n)\eta(2\beta - \eta)\|Bv_n - Bz\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
\delta_n\lambda(2\alpha - \lambda)\|Au_n - Az\|^2 &\le \|x_n - z\|^2 - \left\|y_n - z\right\|^2 \\
&\quad - (1-\delta_n)\eta(2\beta - \eta)\|Bv_n - Bz\|^2 \\
&\le (\|x_n - z\| + \left\|y_n - z\right\|)\left\|y_n - x_n\right\|,
\end{aligned} \tag{3.22}$$

by conditions (*i*), (*ii*), $\lambda \in (0, 2\alpha)$ and (3.12), it implies that

$$\lim_{n\to\infty} \|Au_n - Az\| = 0. \tag{3.23}$$

By using the same method as (3.23), we have

$$\lim_{n\to\infty} \|Bv_n - Bz\| = 0. \tag{3.24}$$

By nonexpansiveness of T_{r_n}, we have

$$\begin{aligned}
\|M_n - z\|^2 &= \left\|P_C(u_n - \lambda Au_n) - P_C(z - \lambda Az)\right\|^2 \\
&\le \langle (u_n - \lambda Au_n) - (z - \lambda Az), M_n - z\rangle \\
&= \frac{1}{2}(\left\|(u_n - \lambda Au_n) - (z - \lambda Az)\right\|^2 + \|M_n - z\|^2 - \left\|(u_n - \lambda Au_n)\right. \\
&\quad \left. -(z - \lambda Az) - (M_n - z)\right\|^2) \\
&\le \frac{1}{2}(\|u_n - z\|^2 + \|M_n - z\|^2 - \left\|(u_n - M_n) - \lambda(Au_n - Az)\right\|^2) \\
&= \frac{1}{2}(\left\|T_{r_n}x_n - T_{r_n}z\right\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 \\
&\quad + 2\lambda\langle u_n - M_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2) \\
&\le \frac{1}{2}(\|x_n - z\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2\lambda\langle u_n - M_n, Au_n - Az\rangle \\
&\quad - \lambda^2\|Au_n - Az\|^2).
\end{aligned}$$

Hence, we have

$$\begin{aligned}\|M_n - z\|^2 \le \|x_n - z\|^2 - \|u_n - M_n\|^2 + 2\lambda\langle u_n - M_n, Au_n - Az\rangle \\ - \lambda^2\|Au_n - Az\|^2.\end{aligned} \tag{3.25}$$

By using the same method as (3.25), we have

$$\|N_n - z\|^2 \le \|x_n - z\|^2 - \|v_n - N_n\|^2 + 2\eta\langle v_n - N_n, Bv_n - Bz\rangle - \eta^2\|Bv_n - Bz\|^2. \tag{3.26}$$

Substitute (3.25) and (3.26) in (3.21), we have

$$\begin{aligned}\|y_n - z\|^2 &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)(\delta_n\|M_n - z\|^2 \\ &\quad + (1-\delta_n)\|N_n - z\|^2) \\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\big(\delta_n(\|x_n - z\|^2 - \|u_n - M_n\|^2 \\ &\quad + 2\lambda\langle u_n - M_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2) + (1+\delta_n)(\|x_n - z\|^2 - \|v_n - N_n\|^2 \\ &\quad + 2\eta\langle v_n - N_n, Bv_n - Bz\rangle - \eta^2\|Bv_n - Bz\|^2)\big) \\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\big(\delta_n\|x_n - z\|^2 - \delta_n\|u_n - M_n\|^2 \\ &\quad + 2\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle + (1-\delta_n)\|x_n - z\|^2 - (1-\delta_n)\|v_n - N_n\|^2 \\ &\quad + 2\eta(1-\delta_n)\langle v_n - N_n, Bv_n - Bz\rangle\big) \\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\big(\|x_n - z\|^2 - \delta_n\|u_n - M_n\|^2 \\ &\quad + 2\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle - (1-\delta_n)\|v_n - N_n\|^2 \\ &\quad + 2\eta(1-\delta_n)\langle v_n - N_n, Bv_n - Bz\rangle\big) \\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\delta_n\|u_n - M_n\|^2 \\ &\quad + 2(1-\alpha_n)\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle - (1-\delta_n)(1-\alpha_n)\|v_n - N_n\|^2 \\ &\quad + 2\eta(1-\delta_n)(1-\alpha_n)\langle v_n - N_n, Bv_n - Bz\rangle \\ &= \|x_n - z\|^2 - (1-\alpha_n)\delta_n\|u_n - M_n\|^2 \\ &\quad + 2(1-\alpha_n)\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle - (1-\delta_n)(1-\alpha_n)\|v_n - N_n\|^2 \\ &\quad + 2\eta(1-\delta_n)(1-\alpha_n)\langle v_n - N_n, Bv_n - Bz\rangle,\end{aligned}$$

it implies that

$$\begin{aligned}(1-\alpha_n)\delta_n\|u_n - M_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\quad + 2(1-\alpha_n)\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle - (1-\delta_n)(1-\alpha_n)\|v_n - N_n\|^2 \\ &\quad + 2\eta(1-\delta_n)(1-\alpha_n)\langle v_n - N_n, Bv_n - Bz\rangle \\ &\le (\|x_n - z\| + \|y_n - z\|)\,\|y_n - x_n\| \\ &\quad + 2(1-\alpha_n)\lambda\delta_n\langle u_n - M_n, Au_n - Az\rangle \\ &\quad + 2\eta(1-\delta_n)(1-\alpha_n)\langle v_n - N_n, Bv_n - Bz\rangle,\end{aligned} \tag{3.27}$$

from (3.12), (3.23), (3.24) and conditions (*i*) and (*ii*), we have

$$\lim_{n\to\infty}\|u_n - M_n\| = 0. \tag{3.28}$$

By using the same method as (3.28), we have

$$\lim_{n\to\infty}\|v_n - N_n\| = 0. \tag{3.29}$$

By (3.19) and (3.28), we have

$$\lim_{n\to\infty}\|M_n - x_n\| = 0. \tag{3.30}$$

By (3.20) and (3.29), we have

$$\lim_{n\to\infty} \|N_n - x_n\| = 0. \tag{3.31}$$

From (3.16), (3.30) and (3.31), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.32}$$

From (3.12) and (3.32), we have

$$\lim_{n\to\infty} \left\|y_n - z_n\right\| = 0. \tag{3.33}$$

From (3.14), (3.33) and condition (*i*), we have (3.13).
Let $a \in (0,1)$, by (3.10) there exists $N_0 \in \mathbb{N}$ such

$$\|x_{n+1} - x_n\| < a^n, \ \forall n \geq N_0. \tag{3.34}$$

Thus, for any number $n, p \in \mathbb{N}$, $p > 0$, we have

$$\left\|x_{n+p} - x_n\right\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k \leq \frac{a^n}{1-a}. \tag{3.35}$$

Since $a \in (0,1)$, we have $\lim_{n\to\infty} a^n = 0$. By (3.35), we have $\{x_n\}$ is Cauchy sequence in Hilbert space. Let $\lim_{n\to\infty} x_n = x^*$. Since $T : C \to CB(H)$ be $\mathcal{H}$ -Lipschitz continuous with constant μ_1 and (1.10), we have

$$\left\|w_n^1 - w_{n+1}^1\right\| \leq (1 + \frac{1}{n})\mathcal{H}(T(x_n), T(x_{n+1})) \leq (1 + \frac{1}{n})\mu_1 \|x_{n+1} - x_n\|. \tag{3.36}$$

By (3.34) and for any number $n, p \in \mathbb{N}$, $p > 0$, we have

$$\begin{aligned} \left\|w_{n+p}^1 - w_n^1\right\| &\leq \sum_{k=n}^{n+p-1} \left\|w_{k+1}^1 - w_k^1\right\| \\ &\leq \sum_{k=n}^{n+p-1} \left(1 + \frac{1}{k}\right) \mu_1 \|x_{k+1} - x_k\| \\ &\leq \sum_{k=n}^{n+p-1} 2\mu_1 a^k \\ &\leq 2\mu_1 \frac{a^n}{1-a}. \end{aligned} \tag{3.37}$$

Since $a \in (0,1)$, we have $\lim_{n\to\infty} a^n = 0$. By (3.37), we have $\{w_n^1\}$ is cauchy sequence in Hilbert space. Let $\lim_{n\to\infty} w_n^1 = w_1^*$. Next, we will prove that $w_1^* \in T(x^*)$. Since $w_n^1 \in T(x_n)$, we have

$$
\begin{aligned}
d(w_n^1, T(x^*)) &\leq \max\left\{d(w_n^1, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1)\right\} \\
&\leq \max\left\{\sup_{z \in T(x_n)} d(z, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1)\right\} \\
&= \mathcal{H}(T(x_n), T(x^*)).
\end{aligned} \tag{3.38}
$$

Since

$$
\begin{aligned}
d(w_1^*, T(x^*)) &\leq \|w_1^* - w_n^1\| + d(w_n^1, T(x^*)) \\
&\leq \|w_1^* - w_n^1\| + \mathcal{H}(T(x_n), T(x^*)) \\
&\leq \|w_1^* - w_n^1\| + \mu_1 \|x_n - x^*\|,
\end{aligned}
$$

by $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} w_n^1 = w_1^*$, we have $d(w_1^*, T(x^*)) = 0$, this implies that $w_1^* \in T(x^*)$. By using the same method as above, we have $\lim_{n\to\infty} w_n^2 = w_2^*$ and $w_2^* \in D(x^*)$.

Let $\omega(x_n)$ be the set of all weakly ω-limit of $\{x_n\}$. We shall show that $\omega(x_n) \subset \mathbb{F}$. Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converse weakly to q. Since $\{x_n\}$ is a Cauchy sequence in Hilbert space, we have $x_{n_i} \to q$ as $\{i \to \infty\}$, it implies that $x_n \to q$ as $n \to \infty$. Since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} x_n = q$, we have $x^* = q$, then we have $w_1^* \in T(q)$ and $w_2^* \in D(q)$. From (3.19) and $x_n \to q$ as $n \to \infty$, we have $u_n \to q$ as $n \to \infty$.

By $u_n = T_{r_n} x_n$, we have

$$
\Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \quad \forall u \in C,
$$

by (3.19), $(H1)$ and lower semicontinuity of ϕ_1, we have

$$
\Phi_1(w_1^*, q, u) + \varphi_1(u) - \varphi_1(q) \geq 0, \ \forall u \in C,
$$

then, we have

$$
q \in (GEP)_s(\Phi_1, \varphi_1). \tag{3.39}
$$

By using the same method as (3.39), we have

$$
q \in (GEP)_s(\Phi_2, \varphi_2). \tag{3.40}
$$

Since $\kappa < a \leq \alpha_1^{n,j}, \ \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \ldots, N-1, \ \kappa < c \leq \alpha_1^{n,N} \leq 1, \ \kappa \leq \alpha_3^{n,N} \leq d < 1$ and $\kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,\ldots, N$. Without loss of generality, we may assume $\alpha_1^{n_i,j} \to \alpha_1^j \in (\kappa, 1)$ as $i \to \infty$, $\alpha_3^{n_i,j} \to \alpha_3^j \in (\kappa, 1)$ and $\alpha_2^{n_i,j} \to \alpha_2^j \in (\kappa, 1)$ as $i \to \infty$, $\forall j = 1,2,\ldots,N$.

Let S be S-mapping generated by T_1, T_2 ..., T_N. and β_1, β_2,...,β_N, where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$. By Lemma 2.4, we have S is nonexpansive and $F(S) = \cap_{i=1}^N F(T_i)$.

By Lemma 2.6, we have

$$\lim_{k\to\infty} \left\|S_{n_i}z_{n_i} - Sz_{n_i}\right\| = 0. \tag{3.41}$$

By (3.13) and (3.41), we have

$$\lim_{n\to\infty} \left\|z_{n_i} - Sz_{n_i}\right\| = 0. \tag{3.42}$$

Since $x_{n_i} \rightharpoonup q$ as $i \to \infty$ and (3.32), we have $z_{n_i} \rightharpoonup q$ as $i \to \infty$. By $z_{n_i} \rightharpoonup q$ as $i \to \infty$, (3.42) and Lemma 2.7, we have

$$q \in \bigcap_{i=1}^{N} F(T_i). \tag{3.43}$$

Next, we define $Q : C \to C$ by

$$Qx = \delta P_C(I - \lambda A)x + (1 - \delta)P_C(I - \eta B)x. \tag{3.44}$$

By Lemma 2.2, we have

$$F(Q) = F(P_C(I - \lambda A)) \cap (P_C(I - \eta B)) = F(G_1) \cap F(G_2). \tag{3.45}$$

From (3.44), we have

$$\begin{aligned}
\|Qx_n - x_n\| &\le \|Qx_n - z_n\| + \|z_n - x_n\| \\
&\le \big\|\delta P_C(I - \lambda A)x_n + (1 - \delta)P_C(I - \eta B)x_n \\
&\quad -\delta_n P_C(I - \lambda A)u_n - (1 - \delta_n)P_C(I - \eta B)v_n \|+\| z_n - x_n\big\| \\
&= \big\|\delta P_C(I - \lambda A)x_n - \delta P_C(I - \lambda A)u_n + \delta P_C(I - \lambda A)u_n \\
&\quad + (1 - \delta)P_C(I - \eta B)x_n - (1 - \delta)P_C(I - \eta B)v_n + (1 - \delta)P_C(I - \eta B)v_n \\
&\quad -\delta_n P_C(I - \lambda A)u_n - (1 - \delta_n)P_C(I - \eta B)v_n \|+\| z_n - x_n\big\| \\
&\le \delta \left\|P_C(I - \lambda A)x_n - P_C(I - \lambda A)u_n\right\| + |\delta - \delta_n| \left\|P_C(I - \lambda A)u_n\right\| \\
&\quad + (1 - \delta) \left\|P_C(I - \eta B)x_n - P_C(I - \eta B)v_n\right\| + \left|(1 - \delta) - (1 - \delta_n)\right| \left\|P_C(I - \eta B)v_n\right\| \\
&\quad + \|z_n - x_n\| \\
&\le \delta \|x_n - u_n\| + |\delta - \delta_n| \left\|P_C(I - \lambda A)u_n\right\| \\
&\quad + (1 - \delta) \|x_n - v_n\| + |\delta_n - \delta| \left\|P_C(I - \eta B)v_n\right\| \\
&\quad + \|z_n - x_n\|
\end{aligned}$$

by condition (*i*), (3.19), (3.20), and (3.32), we have

$$\lim_{n\to\infty} \|Qx_n - x_n\| = 0. \tag{3.46}$$

Since $x_{n_i} \rightharpoonup q$ as $i \to \infty$ and Lemma 2.7, we have

$$q \in F(Q) = F(G_1) \cap F(G_2). \tag{3.47}$$

From (3.39), (3.40), (3.43), and (3.47), we have $q \in \mathbb{F}$. Hence $\omega(x_n) \subset \mathbb{F}$. Hence, by Lemma 2.8 and (3.8), it implies that $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$. This completes the proof.

Remark 3.2. If we take $T \equiv D, w_n^1 = w_n^2\ u_n = v_n \forall n \in \mathbb{N}, \Phi_1 \equiv \Phi_2$ and $\phi_1 = \phi_2$, then the Algorithm 1.3 reduces to the following algorithm:

$$\begin{cases} w_n^1 \in T(x_n),\ \left\| w_n^1 - w_{n+1}^1 \right\| \le \left(1 + \frac{1}{n}\right) \mathcal{H}\left(T(x_n), T(x_{n+1})\right), \\ \Phi_1\left(w_n^1, u_n, u\right) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \ge 0, \forall u \in C, \\ z_n = P_C(I - \lambda A)u_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \left\{ z \in C_{n+1} : \left\| y_n - z \right\| \le \|x_n - z\| \right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \ge 1, \end{cases} \tag{3.48}$$

under the same conditions of Theorem 3.1, we have the sequence $\{x_n\}$ generated by algorithm (3.48) converges strongly to $P_{\mathbb{F}} x_1$ where $\mathbb{F} = \cap_{i=1}^N F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap F(G_1)$, where $G_1 : C \to C$ is defined by $G_1(x) = P_C(x - \lambda A x)$ $\forall x \in C$ and $P_{\mathbb{F}} x_1$ is a solution of $\langle Ax^*, x - x^* \rangle \ge 0$

4 Application

In this section, by using our main result we prove a strong convergence theorem of the sequence $\{x_n\}$ generated by Algorithm 4.1 as follows:

Algorithm 4.1. *Let T_i, $i = 1,2,...,N$, be κ_i-pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1,2,..., N\}$ and let S_n be the S-mappings generated by $T_1, T_2,..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \le \alpha_1^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \le \alpha_1^{n,N} \le 1, \kappa \le \alpha_3^{n,N} \le d < 1, \kappa \le \alpha_2^{n,j} \le e < 1$ for all $j = 1,2,...,N$. Let $x_1 \in C = C_1$ and $w_1^i \in T^i(x_1)$, there exists sequence $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C,\ \forall i = 1, 2, \dots, N$ such that*

$$\begin{cases} w_n^i \in T^i(x_n),\ \left\| w_n^i - w_{n+1}^i \right\| \le \left(1 + \frac{1}{n}\right) \mathcal{H}\left(T(x_n), T(x_{n+1})\right)\ \forall i = 1, 2, \dots, N. \\ \Phi_i\left(w_n^i, u_n^i, u\right) + \varphi_i(u) - \varphi_i(u_n^i) + \frac{1}{r_n^i}\left\langle u_n^i - x_n, u - u_n^i\right\rangle \ge 0,\ \forall u \in C,\ \ \forall i = 1, 2, \dots, N. \\ z_n = \sum_{i=1}^N \delta_n^i P_C(I - \lambda_i A_i) u_n^i, \ \ \text{where } \sum_{i=1}^N \delta_n^i = 1, \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \left\{ z \in C_{n+1} : \left\| y_n - z \right\| \le \|x_n - z\| \right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \ge 1. \end{cases} \tag{4.1}$$

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Theorem 4.2. *Let C be a nonempty bounded, closed, and convex subset of Hilbert space H and let ϕ_i : be a lower semicontinuous and convex function, for all i = 1,2,..., N. Let $T^i : C \to CB(H)$ be $\mathcal{H}$ -Lipschitz continuous with constant μ_i, $\Phi_i : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying (H1)-(H3) and $A_i : C \to H$ be a α_i-inverse strongly monotone mappings $\forall i = 1,2,..., N$ and let T_i, $i = 1,2,...,N$, be κ_i-pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1,2,..., N\}$ with $\mathbb{F} = \cap_{i=1}^N F(T_i) \cap_{i=1}^N (GEP)_s(\Phi_i, \varphi_i) \cap_{i=1}^N F(G_i)$, where $G_i : C \to C$ is defined by $G_i(x) = P_C(x - \lambda_i A_i x)$ $\forall x \in C$, i = 1,2,...,N. Let S_n be the S-mappings generated by $T_1, T_2,..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = \left(\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}\right) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \le \alpha_1^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \le \alpha_1^{n,N} \le 1, \kappa \le \alpha_3^{n,N} \le d < 1, \kappa \le \alpha_2^{n,j} \le e < 1$ for all j = 1,2,..., N and let $\{x_n\}, \{u_n^i\}, \{w_n^i\},\quad \forall i = 1, 2, \dots, N$, be sequences generated by (4.1), where $\{\alpha_n\}$ is* *a sequence in* $[0,1]$, $r_n^i, \lambda_i \in [a,b] \subset (0, 2\alpha)\ \forall i = 1, 2, \ldots, N$ *and* $n \in \mathbb{N}$ *and suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \delta_n^i = \delta^i \in (0,1),\ \forall i = 1, 2, \ldots, N$.

(ii) $0 \le \kappa \le \alpha_n < 1,\ \forall n \ge 1,$

(iii) $\sum_{n=1}^{\infty} \left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right| < \infty, \sum_{n=1}^{\infty} \left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right| < \infty$, *for all* $j \in \{1,2,\ldots,N\}$.

(iv) *There exists* λ^i, $\forall i = 1,2,\ldots, N$ *such that*

$$\Phi_1\left(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\right) + \Phi_1\left(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\right) \le -\lambda^i \left\|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\right\|^2, \qquad (4.2)$$

for all $i = 1, 2, \ldots, N, \left(r_1^i, r_2^i\right) \in \Theta^i \times \Theta^i, w_k^i \in T^i(x_k)$ *for* $k = 1,2$ *where* $\Theta^i = \left\{r_n^i : n \ge 1\right\}$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathbb{F}}x_1$ *and* $P_{\mathbb{F}}x_1$ *is a solutions of (4.3):*

$$\left\langle A_i x^*, x - x^*\right\rangle \ge 0, \forall i = 1, 2, \ldots, N. \qquad (4.3)$$

Competing interests

The author declares that they have no competing interests.

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