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Convergence and stability of modified Ishikawa iteration sequence with errors

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Technology, Guangzhou, 510520,
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available at the end of the article**Abstract**

We show some stability and convergence theorems of the modified Ishikawa iterative sequence with errors for a strongly successively pseudocontractive and strictly asymptotically pseudocontractive mapping in a real Banach space. Additionally, we prove that if T is a uniformly Lipschitzian strongly accretive mapping, the modified Ishikawa iteration sequence with errors converges strongly to the unique solution of the equation $Tx = f$. The main results of this paper improve and extend the known results in the current literature.

MSC: Primary 47H09; 47H10; secondary 47J25**Keywords:** modified Ishikawa iteration sequence with errors; strongly successively pseudocontractive mapping; uniformly Lipschitzian mapping; almost T -stability; weak T -stability

1 Introduction

Developments in fixed point theory reflect that the iterative construction of fixed points is proposed and vigorously analyzed for various classes of maps in different spaces. The class of pseudocontractive mappings in their relation with iteration procedures has been studied by several researchers under suitable conditions; for more details, see [1–3] and the references therein. Also, the class of nonexpansive mappings via iteration methods has extensively been studied in this regard; see Tan and Xu [4]. The class of strongly pseudocontractive mappings has been studied by many researchers (see [5–7]) under certain conditions. Stability results established in metric space, normed linear space, and Banach space settings are available in the literature. There are several authors whose contributions are of colossal value in the study of stability of the fixed point iterative procedures: Imoru and Olatinwo [8], Olatinwo [9], Haghi *et al.* [10], Olatinwo and Postolache [11]. Reich and Zaslavski [12] in Chapter 2 established the existence and uniqueness of a fixed point for a generic mapping, convergence of the iterates of a nonexpansive mapping, stability of the fixed point under small perturbations of a mapping, convergence of Krasnosel'skii-Mann iterations of nonexpansive mappings, generic power convergence of order preserving mappings, and existence and uniqueness of positive eigenvalues and eigenvectors of order-preserving linear operators. They also studied the convergence of iterates of nonexpansive mappings in the presence of computational errors in this chapter. Harker and Hicks in [13] showed how a stability sequence could arise in practice and demonstrated the importance of investigating the stability of various iterative sequences for some kinds of nonlinear mappings.

The purpose in this paper is to study the modified Ishikawa iteration sequence with errors converging strongly to a fixed point of the uniformly Lipschitzian strongly successively pseudocontractive mapping under the lack of some conditions. On the other hand, the authors show that the modified Ishikawa iteration sequence with errors converges strongly to the unique solution of the equation $Tx = f$ if T is a Lipschitzian strongly accretive mapping. The results of this paper improve and extend some recent results.

2 Preliminaries

Throughout this paper, we assume that E is a real Banach space with dual E^* . Suppose that $\langle \cdot, \cdot \rangle$ is the dual pair between E and E^* , and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}, \quad \forall x \in E.$$

First, we recall some concepts. A mapping $T : E \rightarrow E$ is said to be:

- (i) uniformly Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad \text{for all } x, y \in E, n \geq 1;$$

- (ii) strongly successively pseudocontractive if for every $x, y \in E$ there exist $t > 1$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \frac{1}{t} \|x - y\|^2 \quad \text{for all } n \geq 1; \tag{2.1}$$

- (iii) strongly pseudocontractive if for every $x, y \in E$ there exist $t > 1$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{t} \|x - y\|^2. \tag{2.2}$$

Example 2.1 Let $E = R = (-\infty, \infty)$ with the usual norm. Take $K = [0, 1]$ and define $T : K \rightarrow K$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{9} & \text{if } x = 1, \\ x - \frac{1}{3^{n+1}} & \text{if } \frac{1}{3^{n+1}} \leq x < \frac{1}{3} \left(\frac{1}{3^{n+1}} + \frac{1}{3^n} \right), \\ \frac{1}{3^n} - x & \text{if } \frac{1}{3} \left(\frac{1}{3^{n+1}} + \frac{1}{3^n} \right) \leq x < \frac{1}{3^n} \end{cases}$$

for all $n \geq 0$. Then $F(T) = \{0\}$ and T is not continuous at $x = 1$. We can verify that

$$Tx \leq \frac{1}{3}x, \quad x \in K.$$

Thus T^2 is continuous in K and $T^2 K \subset [0, 3^{-n}]$ for all $n \geq 1$. Then for any $x \in K$, there exists $j(x - 0) \in J(x - 0)$ satisfying

$$\langle T^n x - T^n 0, j(x - 0) \rangle = T^n x \cdot x \leq \frac{1}{3} \|x\|^2$$

for all $n \geq 1$. That is, T is a strongly successively pseudocontractive mapping.

Lemma 2.2 (see [14, Lemma 1.1]) *Let E be a Banach space and $x, y \in E$. Then $\|x\| \leq \|x + \gamma y\|$ for all $\gamma > 0$ if and only if there exists $j(x) \in J(x)$ such that $\operatorname{Re}\langle y, j(x) \rangle \geq 0$.*

In the sequel, let $k = \frac{t-1}{t}$, where t is the constant appearing in (2.1). It follows from (2.1) that

$$\left(((1-k)I - T^n)x - ((1-k)I - T^n)y, j(x-y) \right) \geq 0. \tag{2.3}$$

Therefore, it follows from Lemma 2.2 and (2.3) that the definition of a strongly successively pseudocontractive mapping is equivalent to the following definition.

Definition 2.3 T is strongly successively pseudocontractive if there exists $t > 1$ such that

$$\|x - y\| \leq \|x - y + s[((1-k)I - T^n)x - ((1-k)I - T^n)y]\| \tag{2.4}$$

for all $x, y \in E$, $s > 0$ and $n \geq 1$.

Definition 2.4 Let $T : E \rightarrow E$ be a mapping. For a given $x_0 \in E$. $\{a_n\}$, $\{b_n\}$ are sequences in $[0, 1]$, and $\{u_n\}$, $\{v_n\}$ are sequences in E . The sequence $\{x_n\} \subset E$ defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T^n z_n + u_n, \\ z_n = (1 - b_n)x_n + b_n T^n x_n + v_n, \quad n = 0, 1, \dots, \end{cases} \tag{2.5}$$

is said to be a modified Ishikawa iteration sequence with errors.

The following lemmas will be needed in proving our main results.

Lemma 2.5 (see [15]) *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{c_n\}$ be nonnegative real sequences satisfying the inequality*

$$\lambda_{n+1} \leq (1 + \mu_n)\lambda_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then (i) $\lim_{n \rightarrow \infty} \lambda_n$ exists, and (ii) in particular, if $\{\lambda_n\}$ has a subsequence $\{\lambda_{n_k}\}$ converging to 0, then $\lim_{n \rightarrow \infty} \lambda_n = 0$.

From Lemma 2.5 we have the following.

Lemma 2.6 *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{d_n\}$ be nonnegative real sequences satisfying*

$$\lambda_{n+1} \leq (1 - t_n)\lambda_n + \mu_n \lambda_n + c_n + d_n, \quad \forall n \geq 1, \tag{2.6}$$

where $\{t_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} t_n = \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $d_n = o(t_n)$. Then $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof Since $d_n = o(t_n)$, there exists a natural number n_0 such that $d_n \leq \frac{t_n \lambda_n}{2}$ for $n \geq n_0$. It follows from (2.6) that

$$\lambda_{n+1} \leq \left(1 - \frac{t_n}{2}\right) \lambda_n + \mu_n \lambda_n + c_n \leq (1 + \mu_n)\lambda_n + c_n, \quad \forall n \geq n_0.$$

By Lemma 2.5, we see that $\lim_{n \rightarrow \infty} \lambda_n$ exists. Therefore, there exists $D > 0$ such that $\lambda_n \leq D$.

Let $\lim_{n \rightarrow \infty} \lambda_n = \delta$, then $\delta = 0$. Indeed, if $\delta > 0$, there exists a natural number N_0 such that $\lambda_n \geq (\delta/2)$, $d_n \leq (t_n \lambda_n)/2$ for $n \geq N_0$. It follows from (2.6) that

$$\lambda_{n+1} \leq \lambda_n - t_n \lambda_n + \mu_n \lambda_n + c_n + d_n \leq \lambda_n - \frac{t_n \lambda_n}{2} + D\mu_n + c_n.$$

This implies $\frac{\delta t_n}{4} \leq \frac{\lambda_n t_n}{2} \leq \lambda_n - \lambda_{n+1} + D\mu_n + c_n, \forall n \geq N_0$. Therefore,

$$\frac{\delta}{4} \sum_{i=N_0}^n t_i \leq \lambda_{N_0} + D \sum_{i=N_0}^n \mu_i + \sum_{i=N_0}^n c_i.$$

Note that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, and we have $\sum_{n=1}^{\infty} t_n < \infty$, a contradiction with $\sum_{n=1}^{\infty} t_n = \infty$. Then $\lim_{n \rightarrow \infty} \lambda_n = 0$. This completes the proof of Lemma 2.6. \square

Let E be a Banach space and T a self-map of E . Suppose $x_0 \in E$ and $x_{n+1} = f(x_n, T)$ defines an iteration procedure which yields a sequence of points $\{x_n\} \subset E$. Let $F(T) = \{p \in E : Tp = p\} \neq \emptyset$ denote the fixed point of T and let $\{x_n\}$ converge to a fixed point p of T . Let $\{y_n\} \subset E$ and let $\varepsilon_n = \|y_{n+1} - f(y_n, T)\|$ be a sequence in $[0, \infty)$.

Definition 2.7 (see [13, 16, 17]) If $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ implies that $\lim_{n \rightarrow \infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(x_n, T)$ is said to be T -stable. If $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies $\lim_{n \rightarrow \infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(x_n, T)$ is said to be almost T -stable.

3 Main results

Theorem 3.1 Let $T : E \rightarrow E$ be a uniformly Lipschitzian and strongly successively pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by (2.5) and $\{a_n\}, \{b_n\} \subset [0, 1]$ and $\{u_n\}, \{v_n\} \subset E$ satisfying the conditions:

- (i) $\|u_n\| = o(a_n), \|v_n\| \rightarrow 0$ as $n \rightarrow \infty; \sum_{n=1}^{\infty} a_n = \infty$;
- (ii) $\lim_{n \rightarrow \infty} \sup a_n < k / ((L + 1)^3 + 2)$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$.

Suppose $\{y_n\} \subset E$ and define $\{\varepsilon_n\}$ by

$$s_n = (1 - b_n)y_n + b_n T^n y_n + v_n,$$

$$\varepsilon_n = \|y_{n+1} - (1 - a_n)y_n - a_n T^n s_n - u_n\|, \quad n \geq 0.$$

Then the following assertions hold:

- (1) $\{x_n\}$ converges strongly to a unique fixed point of T in E ;
- (2) $\{x_n\}$ is almost T -stable;
- (3) if $\lim_{n \rightarrow \infty} \varepsilon_n / a_n = 0$ implies $\lim_{n \rightarrow \infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(x_n, T)$ is said to be a weakly T -stable. Thus $\{x_n\}$ is also weakly T -stable.

Proof We will show that $F(T)$ is a singleton. Indeed, if $p_1, p_2 \in F(T)$, by the definition of strongly successively pseudocontractive of T , there exists a $j(p_1 - p_2) \in J(p_1 - p_2)$ such that

$$\|p_1 - p_2\|^2 = \langle Tp_1 - Tp_2, j(p_1 - p_2) \rangle = \langle T^n p_1 - T^n p_2, j(p_1 - p_2) \rangle \leq \frac{1}{t} \|p_1 - p_2\|^2,$$

and, since $t > 1$, this implies that $p_1 = p_2$.

It follows from (2.5) that

$$\begin{aligned} \|z_n - p\| &= \|(1 - b_n)(x_n - p) + b_n(T^n x_n - p) + v_n\| \\ &\leq (1 - b_n)\|x_n - p\| + b_n\|T^n x_n - p\| + \|v_n\| \\ &\leq \|x_n - p\| + \|T^n x_n - p\| + \|v_n\| \\ &\leq (L + 1)\|x_n - p\| + \|v_n\|, \end{aligned} \tag{3.1}$$

$$\|x_n - T^n z_n\| \leq \|x_n - p\| + L\|z_n - p\| \leq (1 + L(L + 1))\|x_n - p\| + L\|v_n\|. \tag{3.2}$$

It follows from (2.5), (3.1), and (3.2) that

$$\begin{aligned} \|T^n x_{n+1} - T^n z_n\| &\leq L\|x_{n+1} - z_n\| = L\|(x_n - z_n) + a_n(T^n z_n - x_n) + u_n\| \\ &\leq L\|x_n - z_n\| + La_n\|T^n z_n - x_n\| + L\|u_n\| \\ &= L\|b_n(x_n - T^n x_n) - v_n\| + La_n\|T^n z_n - x_n\| + L\|u_n\| \\ &\leq Lb_n\|x_n - T^n x_n\| + L\|v_n\| + La_n\|T^n z_n - x_n\| + \|u_n\| \\ &\leq Lb_n\|x_n - T^n x_n\| + L\|v_n\| + La_n\|T^n z_n - z_n\| + L\|u_n\| \\ &\leq [L(L + 1)b_n + L(L^2 + L + 1)a_n]\|x_n - p\| \\ &\quad + L(1 + L)\|v_n\| + L\|u_n\|. \end{aligned} \tag{3.3}$$

It follows from (2.5) that

$$\begin{aligned} x_n &= x_{n+1} + a_n x_n - a_n T^n z_n - u_n \\ &= x_{n+1} + (2 - k)a_n x_n - a_n T^n z_n - (1 - k)a_n x_n - u_n \\ &= x_{n+1} + (2 - k)a_n(x_{n+1} + a_n(x_n - T^n z_n) - u_n) - a_n T^n z_n - (1 - k)a_n x_n - u_n \\ &= (1 + a_n)x_{n+1} + (1 - k)a_n x_{n+1} - a_n T^n x_{n+1} + a_n T^n x_{n+1} - a_n T^n z_n \\ &\quad + (2 - k)a_n^2(x_n - T^n z_n) - (1 - k)a_n x_n - (2 - k)a_n u_n - u_n \\ &= (1 + a_n)x_{n+1} + a_n(I - T^n - kI)x_{n+1} - (1 - k)a_n x_n \\ &\quad + (2 - k)a_n^2(x_n - T^n z_n) + a_n(T^n x_{n+1} - T^n z_n) - [(2 - k)a_n + 1]u_n. \end{aligned}$$

For $p \in F(T)$, we have $p = (1 + a_n)p + a_n(I - T^n - kI)p - (1 - k)a_n p$. Therefore, we get

$$\begin{aligned} x_n - p &= (1 + a_n)(x_{n+1} - p) + a_n[(I - T^n - kI)x_{n+1} - (I - T^n - kI)p] \\ &\quad - (1 - k)a_n(x_n - p) + (2 - k)a_n^2(x_n - T^n z_n) \\ &\quad + a_n(T^n x_{n+1} - T^n z_n) - [(2 - k)a_n + 1]u_n. \end{aligned} \tag{3.4}$$

It follows from (3.4) and (2.4) that

$$\begin{aligned} \|x_n - p\| &= \|(1 + a_n)(x_{n+1} - p) + a_n[(I - T^n - kI)x_{n+1} - (I - T^n - kI)p] \\ &\quad - (1 - k)a_n(x_n - p) + (2 - k)a_n^2(x_n - T^n z_n) \end{aligned}$$

$$\begin{aligned}
 & + a_n(T^n x_{n+1} - T^n z_n) - [(2-k)a_n + 1]u_n \| \\
 \geq & (1 + a_n)\|x_{n+1} - p\| - (1-k)a_n\|x_n - p\| \\
 & - (2-k)a_n^2\|x_n - T^n z_n\| \\
 & - a_n\|T^n x_{n+1} - T^n z_n\| - [(2-k)a_n + 1]\|u_n\| \\
 \geq & (1 + a_n)\|x_{n+1} - p\| - (1-k)a_n\|x_n - p\| \\
 & - (2-k)a_n^2[(1+L+L^2)\|x_n - p\| + L\|v_n\|] \\
 & - a_n[L(L+1)b_n + L(L^2+L+1)a_n]\|x_n - p\| \\
 & - La_n\|u_n\| - L(L+1)a_n\|v_n\| - [(2-k)a_n + 1]\|u_n\|.
 \end{aligned}$$

Since $(1 + a_n)^{-1} \leq 1$, $(1 + a_n)^{-1} \leq 1 - a_n + a_n^2$, and $(1 + (1-k)a_n)(1 - a_n + a_n^2) = 1 - ka_n + ka_n^2 + (1-k)a_n^3 \leq 1 - ka_n + ka_n^2 + (1-k)a_n^2 = 1 - ka_n + a_n^2$, we have

$$\begin{aligned}
 \|x_{n+1} - p\| & \leq \frac{1 + (1-k)a_n}{1 + a_n} \|x_n - p\| \\
 & + [L(L+1)a_n b_n + (L^3 + 3L^2 + 3L + 2)a_n^2]\|x_n - p\| \\
 & + (L+3)\|u_n\| + L(L+3)a_n\|v_n\| \\
 & \leq (1 + (1-k)a_n)(1 - a_n + a_n^2)\|x_n - p\| \\
 & + [L(L+1)a_n b_n + (L^3 + 3L^2 + 3L + 2)a_n^2]\|x_n - p\| \\
 & + (L+3)\|u_n\| + L(L+3)a_n\|v_n\| \\
 & \leq (1 - ka_n)\|x_n - p\| \\
 & + [L(L+1)a_n b_n + (L^3 + 3L^2 + 3L + 3)a_n^2]\|x_n - p\| \\
 & + (L+3)\|u_n\| + L(L+3)a_n\|v_n\|. \tag{3.5}
 \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} \sup a_n < k/((L+1)^3 + 2)$, then there exists a natural number N_1 such that $\gamma = \sup_{n \geq N_1} a_n < k/((L+1)^3 + 2)$. It follows from (3.5) that

$$\begin{aligned}
 \|x_{n+1} - p\| & \leq [1 - (k - \gamma((L+1)^3 + 2))a_n]\|x_n - p\| \\
 & + L(L+1)a_n b_n\|x_n - p\| + (L+3)\|u_n\| + L(L+3)a_n\|v_n\| \tag{3.6}
 \end{aligned}$$

holds for $n \geq N_1$. Let $\lambda_n := \|x_n - p\|$, $\mu_n := L(L+1)a_n b_n$, $c_n = 0$, $d_n := (L+3)\|u_n\| + L(L+3)a_n\|v_n\|$, $t_n := (k - \gamma((L+1)^3 + 2))a_n$. Thus, (3.6) becomes

$$\lambda_{n+1} \leq (1 - t_n)\lambda_n + \mu_n \lambda_n + d_n, \quad \forall n \geq N_1.$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty$, $d_n = o(t_n)$, it follows from Lemma 2.6 that we have $\lim_{n \rightarrow \infty} \lambda_n = 0$. That is, $\{x_n\}$ converges strongly to p .

Next, we prove the conclusion (2). Let

$$p_n = (1 - a_n)y_n + a_n T^n s_n + u_n. \tag{3.7}$$

For $p \in F(T)$, we have

$$\begin{aligned} \|y_{n+1} - p\| &= \|y_{n+1} - (1 - a_n)y_n - a_n T^n s_n - u_n + (1 - a_n)y_n + a_n T^n s_n + u_n - p\| \\ &\leq \varepsilon_n + \|p_n - p\|. \end{aligned} \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned} y_n &= p_n + a_n y_n - a_n T^n s_n - u_n \\ &= (1 + a_n)p_n + a_n(I - T^n - kI)p_n - (1 - k)a_n y_n \\ &\quad + (2 - k)a_n^2(y_n - T^n s_n) + a_n(T^n p_n - T^n s_n) - [(2 - k)a_n + 1]u_n. \end{aligned}$$

By using a similar method to proving (3.6), we can prove that

$$\begin{aligned} \|p_n - p\| &\leq [1 - (k - \gamma((L + 1)^3 + 2))a_n] \|y_n - p\| \\ &\quad + L(L + 1)a_n b_n \|y_n - p\| + (L + 3)\|u_n\| + L(L + 3)a_n \|v_n\|. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) for $n \geq N_0$ we get

$$\begin{aligned} \|y_{n+1} - p\| &\leq [1 - (k - \gamma((L + 1)^3 + 2))a_n] \|y_n - p\| \\ &\quad + L(L + 1)a_n b_n \|y_n - p\| + (L + 3)\|u_n\| \\ &\quad + L(L + 3)a_n \|v_n\| + \varepsilon_n. \end{aligned} \tag{3.10}$$

If $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, setting $\lambda_n := \|y_n - p\|$, $\mu_n := L(L + 1)a_n b_n$, $c_n = \varepsilon_n$, $d_n := (L + 3)\|u_n\| + L(L + 3)a_n \|v_n\|$, $t_n := (k - \gamma((L + 1)^3 + 2))a_n$ in Lemma 2.6, we have $y_n \rightarrow p$ as $n \rightarrow \infty$, i.e., $\{x_n\}$ is almost T -stable.

If $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{a_n} = 0$, setting $\lambda_n := \|y_n - p\|$, $\mu_n := L(L + 1)a_n b_n$, $c_n = 0$, $d_n := (L + 3)\|u_n\| + L(L + 3)a_n \|v_n\| + \varepsilon_n$, $t_n := (k - \gamma((L + 1)^3 + 2))a_n$ in Lemma 2.6, we have $y_n \rightarrow p$ as $n \rightarrow \infty$, i.e., $\{x_n\}$ is weakly T -stable. This completes the proof. \square

Similar to the proof of Theorem 3.1, we have the following.

Theorem 3.2 *Let $T : E \rightarrow E$ be a uniformly Lipschitzian and strictly asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by (2.5). Assume that $\{a_n\}, \{b_n\} \subset [0, 1]$ and $\{u_n\}, \{v_n\} \subset E$ satisfy the conditions:*

- (i) $\|u_n\| = o(a_n)$, $\|v_n\| \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) there exists $\delta \in (0, 1 - k)$ such that $\lim_{n \rightarrow \infty} \sup a_n < \delta / ((L + 1)^3 + 2)$;
- (iii) $\sum_{n=1}^{\infty} a_n b_n < \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$.

Suppose $\{y_n\} \subset E$ and define $\{\varepsilon_n\}$ by

$$\begin{aligned} s_n &= (1 - b_n)y_n + b_n T^n y_n + v_n, \\ \varepsilon_n &= \|y_{n+1} - (1 - a_n)y_n - a_n T^n s_n - u_n\|, \quad n \geq 0. \end{aligned}$$

Then the following assertions hold:

- (1) $\{x_n\}$ converges strongly to a unique fixed point of T in E ;
- (2) $\{x_n\}$ is both almost T -stable and weakly T -stable.

Theorem 3.3 Let $T : E \rightarrow E$ be a Lipschitzian and strongly accretive mapping. Let $\{a_n\}, \{b_n\} \subset [0, 1]$, and $\{u_n\}, \{v_n\} \subset E$ satisfy the conditions:

- (i) $\|u_n\| = o(a_n)$, $\|v_n\| \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\lim_{n \rightarrow \infty} \sup a_n < k / ((L + 1)^3 + 2)$, where k is the constant of strongly accretive mapping T , and L is the Lipschitzian constant of mapping $I - T$;
- (iii) $\sum_{n=1}^{\infty} a_n b_n < \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$.

For arbitrary $x_0 \in E$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n(f + (I - T)z_n) + u_n, \\ z_n = (1 - b_n)x_n + b_n(f + (I - T)x_n) + v_n, \quad \forall n \geq 0, \end{cases} \quad (3.11)$$

converges strongly to a solution p of $Tx = f$.

Proof From the result of [5], we obtain the existence of a solution for $Tx = f$. Since T is strongly accretive with a constant $k \in (0, 1)$, we can prove that the solution of $Tx = f$ is unique. Define $Sx = f + (I - T)x$, then S is a strongly pseudocontractive mapping and has a fixed point p , and it is also a Lipschitzian mapping with a constant L . For all $x, y \in E$, there exists $s > 0$ such that

$$\|x - y\| \leq \|x - y + s[(1 - k)I - S]x - ((1 - k)I - S)y\|.$$

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. \square

Remark 3.4 (1) Theorem 3.3 extends the main result of [6] from a uniformly smooth Banach space to a real Banach space and without the boundedness assumption of $D(T) = R(T)$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$; (2) Theorem 3.3 extends and improves the corresponding results of [7] by removing the assumptions $b_n \leq a_n$ and $\sum_{n=1}^{\infty} a_n^2 < \infty$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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