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Boundedness of solutions for semilinear Duffing's equation with asymmetric nonlinear term

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Abstract

In this paper we study the following second-order periodic system:

$$x'' + V'(x) + p(x)f(t) = 0,$$

where $V(x)$ has a singularity. Under some assumptions on the $V(x)$, $p(x)$ and $f(t)$, by Ortega's small twist theorem, we obtain the existence of quasi-periodic solutions and boundedness of all the solutions.

Keywords: boundedness of solutions; singularity; small twist theorem

1 Introduction and main result

In the early 1960s, Littlewood [1] asked whether or not the solutions of the Duffing-type equations,

$$x'' + g(x,t) = 0, \tag{1.1}$$

are bounded for all time, *i.e.*, whether there are resonances that might cause the amplitude of the oscillations to increase without bound.

The first positive result of boundedness of solutions in the superlinear case (*i.e.*, $\frac{g(x,t)}{x} \to \infty$ as $|x| \to \infty$) was due to Morris [2]. By means of KAM theorem, Morris proved that every solution of differential equation (1.1) is bounded if $g(x,t) = 2x^3 - p(t)$, where $p(t)$ is piecewise continuous and periodic. This result relies on the fact that the nonlinearity $2x^3$ can guarantee the twist condition of KAM theorem. Later, several authors (see [3, 4]) improved the result of (1.1) and obtained a similar result for a large class of superlinear functions $g(x,t)$.

When $g(x,t)$ satisfies

$$0 \le k \le \frac{g(x,t)}{x} \le K \le +\infty, \quad \forall x \in R,$$

i.e., differential equation (1.1) is *semilinear*, similar results also hold. But the proof is more difficult since there may be a resonant case.

Liu [5] studied the following equation:

$$\left(\Phi_p\left(x'\right)\right)' + \alpha\Phi_p\left(x^+\right) + \beta\Phi_p\left(x^-\right) = f(x,t),$$

where $f(x,t)$ is 2π-periodic in t and has limits $f_\pm(t)$ as $x \to \pm\infty$. Under some reasonable assumptions on $f(x,t)$, Liu [5] proved the existence of quasi-periodic solutions and the boundedness of solutions. Later, Cheng and Xu [6] studied a more general equation

$$\left(\Phi_p\left(x'\right) + \Psi\left(x'\right)\right)' + \alpha\Phi_p\left(x^+\right) + \beta\Phi_p\left(x^-\right) = \phi(x,t), \tag{1.2}$$

where $\phi(x,t)$ is 2π-periodic in t. They defined a new function $\bar{\phi}(t,x) = \frac{\phi(t,x)}{|x|^{p-1-\sigma}}$, where $\sigma \in (0,p)$, $\bar{\phi}(t,x)$ has limits $\phi_\pm(t)$ and the similar property to $f(x,t)$ in [5]. Then the authors proved the boundedness of solutions for (1.2). We observe that $\phi(x,t)$ in [6] is unbounded while $f(x,t)$ in [5] is bounded and that is the major difference between [5] and [6]. The idea in [5, 6] is to change the original problem to a Hamiltonian system and then use a twist theorem of area-preserving mapping to the Poincaré map.

Recently, Capietto *et al.* [7] studied the following equation:

$$x'' + V'(x) = F(x,t), \tag{1.3}$$

where $F(x,t) = p(t)$ is a π-periodic function, $V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\nu} - 1$, $x_+ = \max\{x,0\}$, $x_- = \max\{-x,0\}$ and ν is a positive integer. Under the Lazer-Leach assumption that

$$1 + \frac{1}{2}\int_0^\pi p(t_0+\theta)\sin\theta\,d\theta > 0, \quad \forall t_0 \in R, \tag{1.4}$$

they proved the boundedness of solutions and the existence of a quasi-periodic solution by the Moser twist theorem. It was the first time that the equation of the boundedness of all solutions was treated in case of a singular potential.

Motivated by the papers [5–7], we observe that $F(x,t) = p(t)$ in (1.3) is smooth and bounded, so a natural question is to find sufficient conditions on $F(x,t)$ such that all solutions of (1.3) are bounded when $F(x,t)$ is unbounded. The purpose of this paper is to deal with this problem.

We consider the following equation:

$$x'' + V'(x) + p(x)f(t) = 0, \tag{1.5}$$

where

$$V = \frac{1}{2}x_+^2 + \frac{1}{1-x_-^2} - 1, \quad x > -1. \tag{1.6}$$

In order to state our main results, we give some notations and assumptions. Let $f(t)$ be a π-periodic function and

$$\lim_{x\to+\infty}\frac{p(x)}{|x|^\alpha} = 1, \qquad P(x) = \int_0^x p(s)\,ds, \tag{1.7}$$

where $0 < \alpha < 1$. We suppose that the following Lazer-Leach assumption holds:

$$\int_0^{\pi} f(t_0 + \theta)(\sin\theta)^{1+\alpha}\, d\theta > 0, \quad \forall t_0 \in R. \tag{1.8}$$

Our main result is the following theorem.

Theorem 1 *Under assumptions* (1.6)-(1.8), *all the solutions of* (1.5) *are defined for all* $t \in (-\infty, +\infty)$, *and for each solution* $x(t)$, *we have* $\sup_{t\in R}(|x(t)| + |x'(t)|) < +\infty$.

The main idea of our proof is acquired from [8]. The proof of Theorem 1 is based on a small twist theorem due to Ortega [9]. Hypotheses (1.6)-(1.8) of our theorem are used to prove that the Poincaré mapping of (1.5) satisfies the assumptions of Ortega's theorem.

Moreover, we have the following theorem on solutions of Mather type.

Theorem 2 *Assume that* $f(t) \in C$ *satisfies* (1.8); *then, there is* $\epsilon_0 > 0$ *such that for any* $\omega \in (\frac{1}{\pi}, \frac{1}{\pi+\epsilon_0})$, *equation* (1.5) *has a solution* $(x_\omega(t), x'_\omega(t))$ *of Mather type with rotation number* ω. *More precisely,*

Case 1: $\omega = \frac{p}{q}$ *is rational. The solutions* $(x_\omega(t + 2i\pi), x'_\omega(t + 2i\pi))$, $1 \le i \le q - 1$, *are independent periodic solutions of periodic* $q\pi$; *moreover, in this case,*

$$\lim_{q\to\infty} \min_{t\in R}\bigl(\left|x_\omega(t)\right| + \left|x'_\omega(t)\right|\bigr) = +\infty.$$

Case 2: ω *is irrational. The solution* $(x_\omega(t), x'_\omega(t))$ *is either a usual quasi-periodic solution or a generalized one.*

2 Proof of the theorem

2.1 Action-angle variables and some estimates

Observe that (1.5) is equivalent to the following Hamiltonian system:

$$x' = \frac{\partial H}{\partial y}, \qquad y' = -\frac{\partial H}{\partial x} \tag{2.1}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + V(x) + P(x)f(t).$$

In order to introduce action and angle variables, we first consider the auxiliary autonomous equation

$$x' = y, \qquad y' = -V'(x), \tag{2.2}$$

which is an integrable Hamiltonian system with the Hamiltonian function

$$H_1(x, y, t) = \frac{1}{2}y^2 + V(x).$$

The closed curves $H_1(x, y, t) = h > 0$ are just the integral curves of (2.2).

Denote by $T_0(h)$ the time period of the integral curve Γ_h of (2.2) defined by $H_1(x,y,t) = h$ and by I the area enclosed by the closed curve Γ_h for every $h > 0$. Let $-1 < -\alpha_h < 0 < \beta_h$ be such that $V(-\alpha_h) = V(\beta_h) = h$. It is easy to see that

$$I_0(h) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2\bigl(h - V(s)\bigr)}\,ds, \quad \forall h > 0,$$

and

$$T_0(h) = I_0'(h) = 2\int_{\alpha_h}^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}\,ds, \quad \forall h > 0.$$

By a direct computation, we get

$$\begin{aligned} I_0(h) &= 2\int_0^{\beta_h} \sqrt{2\bigl(h - V(s)\bigr)}\,ds + 2\int_{-\alpha_h}^{0} \sqrt{2\bigl(h - V(s)\bigr)}\,ds \\ &= \pi h + 2\int_0^{\alpha_h} \sqrt{2\bigl(h - V(-s)\bigr)}\,ds, \end{aligned}$$

so

$$T_0(h) = \pi + \int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\,ds.$$

We then have

$$I_0(h) = I_-(h) + I_+(h), \qquad T_0(h) = T_-(h) + T_+(h),$$

where

$$\begin{aligned} I_-(h) &= 2\int_0^{-\alpha_h} \sqrt{2\bigl(h - V(s)\bigr)}\,ds, \qquad I_+(h) = \pi h, \\ T_-(h) &= 2\int_0^{-\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\,ds, \qquad T_+(h) = \pi. \end{aligned}$$

We now give the estimates on the functions I_- and T_-.

Lemma 1 *We have*

$$h^n \left| \frac{d^n T_-(h)}{dh^n} \right| \le C h^{-\frac{1}{2}},$$

and

$$h^n \left| \frac{d^n I_-(h)}{dh^n} \right| \le C h^{\frac{1}{2}},$$

where $n = 0, 1, \ldots, 6$, $h \to +\infty$. *Note that here and below we always use* C, C_0 *or* C_0' *to indicate some constants.*

Proof Now we estimate the first inequality. We choose $\frac{V(s)}{h} = \eta$ as the new variable of integration, then we have

$$T_{-}(h) = \int_{-\alpha_h}^{0} \frac{1}{\sqrt{2(h - V(s))}} \, ds = \int_0^1 \frac{\sqrt{h}}{V'(s(\eta, h))} \frac{1}{\sqrt{2(1-\eta)}} \, d\eta.$$

Since $V(s) = \frac{1}{1-s^2} - 1$ and $\frac{V(s)}{h} = \eta$, we have $s = \sqrt{\frac{\eta h}{1+\eta h}}$. By a direct computation, we have

$$V'(s) = \frac{2s}{(1-s^2)^2} = \frac{2\sqrt{\eta h}(1+\eta h)^2}{\sqrt{1+\eta h}},$$

then we get

$$T_{-}^{(n)}(h) = \frac{(-\frac{3}{2})!}{(-\frac{3}{2} - n)!} \int_0^1 \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{\frac{3}{2}+n}} \, d\eta, \quad n = 0, 1, \ldots, 6.$$

When $0 \le \eta \le h^{-1}$ and h is sufficient large, there exits C_0 such that $1 - \eta > C_0$, so we have

$$\begin{aligned} \int_0^{h^{-1}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{\frac{3}{2}+n}} \, d\eta &\le C \int_0^{h^{-1}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}} \, d\eta \\ &\le \frac{C}{C_0} \int_0^{h^{-1}} \eta^{n-\frac{1}{2}} \, d\eta \le C h^{-\frac{1}{2}-n}. \end{aligned} \tag{2.3}$$

Since $h^{-\frac{2}{3}} \le \eta \le 1$, we have

$$h^{\frac{1}{3}} < 1 + h^{\frac{1}{3}} \le 1 + \eta h \le 1 + h,$$

then

$$\begin{aligned} \int_{h^{-\frac{2}{3}}}^1 \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{\frac{3}{2}+n}} \, d\eta &\le C \int_{h^{-\frac{2}{3}}}^1 \frac{\eta^n h^n}{\sqrt{2\eta(1-\eta)} h^n (1+\eta h)^n (1+\eta h)^{\frac{3}{2}}} \, d\eta \\ &\le C \int_{h^{-\frac{2}{3}}}^1 \frac{1}{\sqrt{2\eta(1-\eta)} h^n (1+\eta h)^{\frac{3}{2}}} \, d\eta \\ &\le C \int_{h^{-\frac{2}{3}}}^1 \frac{1}{\sqrt{2\eta(1-\eta)} h^n h^{\frac{1}{2}}} \, d\eta \\ &\le C h^{-\frac{1}{2}-n} \int_0^1 \frac{1}{\sqrt{2\eta(1-\eta)}} \, d\eta \le C h^{-\frac{1}{2}-n}. \end{aligned} \tag{2.4}$$

Observing that there is $C_0 > 0$ such that $\sqrt{1-\eta} \ge C_0$ when $h^{-1} \le \eta \le h^{-\frac{2}{3}}$ and $h \to +\infty$, we have

$$\begin{aligned} &\int_{h^{-1}}^{h^{-\frac{2}{3}}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{\frac{3}{2}+n}} \, d\eta \\ &\quad \le C_1 h^{-\frac{3}{2}-n} \int_{h^{-1}}^{h^{-\frac{2}{3}}} \frac{1}{\sqrt{2\eta(1-\eta)}\eta^{\frac{3}{2}}} \, d\eta \end{aligned}$$

$$\begin{aligned}&\le \frac{C_1}{C_0}h^{-\frac{3}{2}-n}\int_{h^{-1}}^{h^{-\frac{2}{3}}}\frac{1}{\eta^2}\,d\eta = \frac{C_1}{C_0}h^{-\frac{3}{2}-n}\frac{1}{\eta}\Big|_{h_{-1}}^{h^{-\frac{2}{3}}}\\&= \frac{C_1}{C_0}h^{-\frac{3}{2}-n}\bigl(h-h^{\frac{2}{3}}\bigr) \le Ch^{-\frac{1}{2}-n}.\end{aligned} \tag{2.5}$$

By (2.3)-(2.5) we have $T_-^{(n)}(h) \le Ch^{-\frac{1}{2}-n}$, $n = 0,1,\dots,6$.

The proof of the second inequality is similar to that of the first one, so we only give a brief proof. We choose $\frac{V(s)}{h} = \eta$ as the new variable of integration, so we have

$$\frac{\partial s}{\partial h} = \frac{\eta}{V'}, \qquad s = \sqrt{\frac{\eta h}{1+\eta h}}$$

and

$$V'(s) = \frac{2s}{(1-s^2)^2} = \frac{2\sqrt{\eta h}(1+\eta h)^2}{\sqrt{1+\eta h}}.$$

By a direct computation, we have

$$I_-(h) = 2\int_{-\alpha_h}^{0}\sqrt{2\bigl(h-V(s)\bigr)}\,ds = h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}(1+\eta h)^{\frac{3}{2}}}\,d\eta. \tag{2.6}$$

By (2.6), we can easily get

$$\begin{aligned}I_-^{(n)}(h) = I_{-1}^{(n)}(h) + I_{-2}^{(n)}(h) &= n\frac{(-\frac{3}{2})!}{(-\frac{3}{2}-n+1)!}\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^{n-1}}{(1+\eta h)^{\frac{3}{2}+n-1}}\,d\eta\\&\quad+\frac{(-\frac{3}{2})!}{(-\frac{3}{2}-n)!}h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^n}{(1+\eta h)^{\frac{3}{2}+n}}\,d\eta,\end{aligned} \tag{2.7}$$

where $n = 0,1,\dots,6$.

By a similar way to that in estimating $T_-^{(n)}(h)$, we get

$$I_{-1}^{(n)}(h) \le Ch^{\frac{1}{2}-n}, \qquad I_{-2}^{(n)}(h) \le Ch^{\frac{1}{2}-n},$$

which means that

$$I_-^{(n)}(h) \le Ch^{\frac{1}{2}-n}, \quad n = 0,1,\dots,6.$$

Thus Lemma 1 is proved. □

Remark 1 It follows from the definitions of $T_+(h)$, $T_-(h)$ and Lemma 1 that

$$\lim_{h\to+\infty} T_-(h) = 0, \qquad \lim_{h\to+\infty} T_+(h) = \pi.$$

Thus the time period $T_0(h)$ is dominated by $T_+(h)$ when h is sufficiently large. From the relation between $T_-(h)$ and $I_-(h)$, we know $I_0(h)$ is dominated by $I_+(h)$ when h is sufficiently large.

Remark 2 It also follows from the definition of $I(h)$, $I_-(h)$, $I_+(h)$ and Remark 1 that

$$\left|h^n \frac{d^n I_0(h)}{dh^n}\right| \le C_0 I_0(h) \quad \text{for } n \ge 1.$$

Remark 3 Note that $h = h_0(I_0)$ is the inverse function of I_0. By Remark 2, we have

$$\left|I^n \frac{d^n h(I)}{dI^n}\right| \le C_0 h(I) \quad \text{for } n \ge 1.$$

We now carry out the standard reduction to the action-angle variables. For this purpose, we define the generating function $S(x,I) = \int_C \sqrt{2(h - V(s))}\,ds$, where C is the part of the closed curve Γ_h connecting the point on the y-axis and point (x,y).

We define the well-know map $(\theta, I) \to (x,y)$ by

$$y = \frac{\partial S}{\partial x}(x,I), \qquad \theta = \frac{\partial S}{\partial I}(x,I),$$

which is symplectic since

$$dx \wedge dy = dx \wedge (S_{xx}\,dx + S_{xI}\,dI) = S_{xI}\,dx \wedge dI,$$

$$d\theta \wedge dI = (S_{Ix}\,dx + S_{II}\,dI) \wedge dI = S_{Ix}\,dx \wedge dI.$$

From the above discussion, we can easily get

$$\theta = \begin{cases} \frac{\pi}{T_0(h(x,y))}\left(\frac{T_-(h(x,y))}{2} + \arcsin\frac{x}{\sqrt{2(h(x,y))}}\right) & \text{if } x > 0, y > 0, \\ \frac{\pi}{T_0(h(x,y))}\left(\frac{T_-(h(x,y))}{2} + \pi + \arcsin\frac{x}{\sqrt{2(h(x,y))}}\right) & \text{if } x > 0, y < 0, \\ \frac{\pi}{T_0(h(x,y))}\left(\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y)+1-(1-s^2)^{-1})}}\,ds\right) & \text{if } x < 0, y > 0, \\ \frac{\pi}{T_0(h(x,y))}\left(T_0(h(x,y)) - \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y)+1-(1-s^2)^{-1})}}\,ds\right) & \text{if } x < 0, y < 0 \end{cases} \tag{2.8}$$

and

$$I(x,y) = I_0\big(h(x,y)\big) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2\big(h(x,y) - V(s)\big)}\,ds. \tag{2.9}$$

In the new variables (θ, I), system (2.1) becomes

$$\theta' = \frac{\partial H}{\partial I}, \qquad I' = -\frac{\partial H}{\partial \theta}, \tag{2.10}$$

where

$$H(\theta, I, t) = \pi h_0(I) + \pi P\big(x(I,\theta)\big) f(t). \tag{2.11}$$

In order to estimate $\pi P(I,\theta)$, we need the estimate on the functions $x(I,\theta)$.

Lemma 2 *For I sufficient large and $-\alpha_h \le x < 0$, the following estimates hold*:

$$\left|I^n \frac{\partial^n x(I,\theta)}{\partial I^n}\right| \le c\sqrt{I} \quad \textit{for } 0 \le n \le 6.$$

The lemma was first proved in [3], later Capietto *et al.* [7] gave a different proof; using the method of induction-hypothesis, Jiang and Fang [10] also gave another proof. So, for concision, we omit the proof.

2.2 New action and angle variables

Now we are concerned with Hamiltonian system (2.10) with the Hamiltonian function $H(\theta, I, t)$ given by (2.11). Note that

$$I\,d\theta - H\,dt = -(H\,dt - I\,d\theta).$$

This means that if one can solve I from (2.11) as a function of H (θ and t as parameters), then

$$\frac{dH}{d\theta} = -\frac{\partial I}{\partial t}(t, H, \theta), \qquad \frac{dt}{d\theta} = \frac{\partial I}{\partial H}(t, H, \theta) \tag{2.12}$$

is also a Hamiltonian system with the Hamiltonian function I and now the action, angle and time variables are H, t and θ.

From (2.11) and Lemma 1, we have

$$\frac{\partial H}{\partial I} \to 1 \quad \text{as } I \to +\infty.$$

So, we assume that I can be written as

$$I = I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right),$$

where R satisfies $|R| < \frac{H}{\pi}$. Recalling that h_0 is the inverse function of I_0, we have

$$\frac{H}{\pi} + R(H, t, \theta) = h_0(I),$$

which implies that

$$R(H, t, \theta) = P\bigl(x(I, \theta)\bigr)f(t).$$

As a consequence, R is implicitly defined by

$$R(H, t, \theta) = P\left[x\left(I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right), \theta\right)\right]f(t). \tag{2.13}$$

Now we give the estimates of R. By a similar way to that in estimating Lemma 2.3 in [7], we have the following lemma.

Lemma 3 *The function* $R(H, t, \theta)$ *satisfies the following estimates*:

$$\left|\frac{\partial^{m+l} R(H, t, \theta)}{\partial H^m\,\partial t^l}\right| \le H^{\frac{\alpha+1}{2}} \quad \textit{for } m + l \le 6.$$

Moreover, by the implicit function theorem, there exists a function $R_1 = R_1(t,H,\theta)$ such that

$$R(H,t,\theta) = P\bigl(x(H,\theta)\bigr)f(t) + R_1(H,t,\theta).$$

Since

$$\begin{aligned} R_1(H,t,\theta) &= R(H,t,\theta) - P\bigl(x(H,\theta)\bigr)f(t) \\ &= P\left\{x\left[I_0\left(\frac{H}{\pi} + R(H,t,\theta)\right),\theta\right]\right\}f(t) - P\bigl(x(H,\theta)\bigr)f(t) \\ &= \int_0^1 p\bigl\{x\bigl[H + s(\pi R + I_-),\theta\bigr]\bigr\} \\ &\quad \cdot \frac{\partial x}{\partial I}\bigl(H + s(\pi R + I_-),\theta\bigr) \cdot (\pi R + I_-)f(t)\,ds. \end{aligned}$$

By Lemmas 1 and 3, we have the estimates on $R_1(H,t,\theta)$.

Lemma 4 $|\frac{\partial^{k+l} R_1(H,t,\theta)}{\partial^k H\,\partial^l t}| < H^{\frac{\alpha}{2}}$ *for* $k + l \leq 6$.

For the estimate of $I(\frac{H}{\pi} + R)$, we need the estimate on $I_-(\frac{H}{\pi} + R)$. By Lemma 1 and noticing that $|R| < \frac{H}{\pi}$, we have the following lemma.

Lemma 5 $|\frac{\partial^{k+l} I_-(\frac{H}{\pi}+R)}{\partial^k H\,\partial^l t}| < H^{\frac{1}{2}}$ *for* $k + l \leq 6$.

Now the new Hamiltonian function $I = I(t,H,\theta)$ is written in the form

$$\begin{aligned} I = I_0\left(\frac{H}{\pi} + R\right) &= I_+\left(\frac{H}{\pi} + R\right) + I_-\left(\frac{H}{\pi} + R\right) \\ &= H + \pi R(H,t,\theta) + I_-\left(\frac{H}{\pi} + R\right) \\ &= H + \pi P\bigl(x(H,\theta)\bigr)f(t) + \pi R_1(H,t,\theta) + I_-\left(\frac{H}{\pi} + R\right). \end{aligned}$$

System (2.12) is of the form

$$\begin{cases} \frac{dt}{d\theta} = \frac{\partial I}{\partial H} = 1 + \pi \frac{\partial x}{\partial H}(H,\theta)p(x(H,\theta))f(t) + \pi \frac{\partial R_1}{\partial H}(H,t,\theta) + \frac{\partial I_-}{\partial H}(H,t,\theta), \\ \frac{dH}{d\theta} = -\frac{\partial I}{\partial t} = -\pi P(x(\theta,H))f'(t) - \pi \frac{\partial R_1}{\partial t}(t,H,\theta) - \frac{\partial I_-}{\partial t}(H,t,\theta). \end{cases} \tag{2.14}$$

Introduce a new action variable $\rho \in [1,2]$ and a parameter $\epsilon > 0$ by $H = \epsilon^{-2}\rho$. Then $H \gg 1 \Leftrightarrow 0 < \epsilon \ll 1$. Under this transformation, system (2.14) is changed into the form

$$\begin{cases} \frac{dt}{d\theta} = \frac{\partial I}{\partial H} = 1 + \pi \frac{\partial x}{\partial H}(H,\theta)p(x(H,\theta))f(t) + \pi \frac{\partial R_1}{\partial H}(H,t,\theta) + \frac{\partial I_-}{\partial H}(H,t,\theta), \\ \frac{d\rho}{d\theta} = -\frac{\partial I}{\partial t} = -\epsilon^2[\pi P(x(\theta,H))f'(t) + \pi \frac{\partial R_1}{\partial t}(H,t,\theta) + \frac{\partial I_-}{\partial t}(H,t,\theta)], \end{cases} \tag{2.15}$$

which is also a Hamiltonian system with the new Hamiltonian function

$$\Gamma(t,\rho,\theta;\epsilon) = \rho + \pi\epsilon^{-2}P\bigl(x\bigl(\theta,\epsilon^{-2}\rho\bigr)\bigr)f(t) + \pi\epsilon^{-2}R_1\bigl(\epsilon^{-2}\rho,\theta,t\bigr) + \epsilon^{-2}I_-\bigl(\epsilon^{-2}\rho,\theta,t\bigr).$$

Obviously, if $\epsilon \ll 1$, the solution $(t(\theta, t_0, \rho_0), \rho(\theta, t_0, \rho_0))$ of (2.15) with the initial data $(t_0, \rho_0) \in R \times [1,2]$ is defined in the interval $\theta \in [0, 2\pi]$ and $\rho(\theta, t_0, \rho_0) \in [\frac{1}{2}, 3]$. So, the Poincaré map of (2.15) is well defined in the domain $R \times [1,2]$.

Lemma 6 ([8] Lemma 5.1) *The Poincaré map of* (2.15) *has the intersection property.*

The proof is similar to the corresponding one in [8].

For convenience, we introduce the notation $O_k(1)$ and $o_k(1)$. We say a function $f(t, \rho, \theta, \epsilon) \in O_k(1)$ if f is smooth in (t, ρ) and for $k_1 + k_2 \leq k$,

$$\left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1} \partial \rho^{k_2}} f(t, \rho, \theta, \epsilon) \right| \leq C,$$

for some constant $C > 0$ which is independent of the arguments t, ρ, θ, ϵ.

Similarly, we say $f(t, \rho, \theta, \epsilon) \in o_k(1)$ if f is smooth in (t, ρ) and for $k_1 + k_2 \leq k$,

$$\lim_{\epsilon \to 0} \left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1} \partial \rho^{k_2}} f(t, \rho, \theta, \epsilon) \right| = 0,$$

uniformly in (t, ρ, θ).

2.3 Poincaré map and twist theorems

We will use Ortega's small twist theorem to prove that the Poincaré map P has an invariant closed curve if ϵ is sufficiently small. Let us first recall the theorem in [9].

Lemma 7 (Ortega's theorem) *Let $A = \mathbb{S}^1 \times [a,b]$ be a finite cylinder with universal cover $\mathbb{A} = \mathbb{R} \times [a,b]$. The coordinate in $\mathbb{A}$ is denoted by (τ, ν). Consider the map*

$$\bar{f} : A \to \mathbb{S} \times \mathbb{R}.$$

We assume that the map has the intersection property. Suppose that $f : A \to \mathbb{R} \times \mathbb{R}$, $(\tau_0, \nu_0) \to (\tau_1, \nu_1)$ is a lift of $\bar{f}$ and it has the form

$$\begin{cases} \tau_1 = \tau_0 + 2N\pi + \delta l_1(\tau_0, \nu_0) + \delta \tilde{g}_1(\tau_0, \nu_0), \\ \nu_1 = \nu_0 + \delta l_2(\tau_0, \nu_0) + \delta \tilde{g}_2(\tau_0, \nu_0), \end{cases} \tag{2.16}$$

where N is an integer, $\delta \in (0,1)$ is a parameter. The functions l_1, l_2, $\tilde{g}_1$ and $\tilde{g}_2$ satisfy

$$\begin{aligned} &l_1 \in C^6(A), \qquad l_1(\tau_0, \nu_0) > 0, \qquad \frac{\partial l_1}{\partial \nu_0}(\tau_0, \nu_0) > 0, \quad \forall (\tau_0, \nu_0) \in A, \\ &l_2(\cdot,\cdot), \tilde{g}_1(\cdot,\cdot,\epsilon), \tilde{g}_2(\cdot,\cdot,\epsilon) \in C^5(A). \end{aligned} \tag{2.17}$$

In addition, we assume that there is a function $I : A \to R$ satisfying

$$I \in C^6(A), \qquad \frac{\partial I}{\partial \nu_0}(\tau_0, \nu_0) > 0, \quad \forall (\tau_0, \nu_0) \in A \tag{2.18}$$

and

$$l_1(\tau_0, \nu_0) \cdot \frac{\partial I}{\partial \tau_o}(\tau_0, \nu_0) + l_2(\tau_0, \nu_0) \cdot \frac{\partial I}{\partial \nu_0}(\tau_0, \nu_0) = 0, \quad \forall (\tau_0, \nu_0) \in A. \tag{2.19}$$

Moreover, suppose that there are two numbers $\tilde{a}$ *and* $\tilde{b}$ *such that* $a < \tilde{a} < \tilde{b} < b$ *and*

$$I_M(a) < I_m(\tilde{a}) \le I_M(\tilde{a}) < I_m(\tilde{b}) \le I_M(\tilde{b}) < I_m(b), \tag{2.20}$$

where

$$I_M(r) = \max_{\rho \in S^1} I(\rho_o, \tau_o), \qquad I_m(r) = \min_{\rho \in S^1} I(\rho_o, \tau_o).$$

Then there exist $\epsilon > 0$ *and* $\Delta > 0$ *such that if* $\delta < \Delta$ *and*

$$\|\tilde{g}_1(\cdot,\cdot,\epsilon)\|_{C^5(A)} + \|\tilde{g}_2(\cdot,\cdot,\epsilon)\|_{C^5(A)} < \epsilon,$$

the mapping $\bar{f}$ *has an invariant curve in* Γ_A, *the constant* ϵ *is independent of* δ.

We make the *ansatz* that the solution of (2.15) with the initial condition $(t(0), \rho(0)) = (t_0, \rho_0)$ is of the form

$$t = t_0 + \theta + \epsilon^{1-\alpha}\Sigma_1(t_0, \rho_0, \theta; \epsilon), \qquad \rho = \rho_0 + \epsilon^{1-\alpha}\Sigma_2(t_0, \rho_0, \theta; \epsilon).$$

Then the Poincaré map of (2.15) is

$$P: \quad t_1 = t_0 + 2\pi + \epsilon^{1-\alpha}\Sigma_1(t_0, \rho_0, 2\pi; \epsilon), \qquad \rho_1 = \rho_0 + \epsilon^{1-\alpha}\Sigma_2(t_0, \rho_0, 2\pi; \epsilon). \tag{2.21}$$

The functions Σ_1 and Σ_2 satisfy

$$\begin{cases} \Sigma_1 = \pi\epsilon^{\alpha-1}\int_0^\theta \frac{\partial x}{\partial H}(\epsilon^{-2}\rho, \theta) p(x(\epsilon^{-2}\rho, \theta)) f(t)\, d\theta \\ \qquad + \pi\epsilon^{\alpha-1}\int_0^\theta (\frac{\partial R_1}{\partial H}(\epsilon^{-2}\rho, t, \theta) + \frac{\partial I_-}{\partial H}(\epsilon^{-2}\rho, t, \theta))\, d\theta, \\ \Sigma_2 = -\pi\epsilon^{\alpha+1}\int_0^\theta P(x(\epsilon^{-2}\rho, \theta)) f'(t)\, d\theta \\ \qquad - \epsilon^{\alpha+1}\int_0^\theta (\pi\frac{\partial R_1}{\partial t}(\epsilon^{-2}\rho, t, \theta) - \frac{\partial I_-}{\partial t}(\epsilon^{-2}\rho, t, \theta))\, d\theta, \end{cases} \tag{2.22}$$

where $t = t_0 + \theta + \epsilon^{1-\alpha}\Sigma_1$, $\rho = \rho_0 + \epsilon^{1-\alpha}\Sigma_2$. By Lemmas 4, 6 and 7, we know that

$$|\Sigma_1| + |\Sigma_2| \le C \quad \text{for } \theta \in [0, 2\pi]. \tag{2.23}$$

Hence, for $\rho_0 \in [1, 2]$, we may choose ϵ sufficiently small such that

$$\rho_0 + \epsilon\Sigma_2 \ge \frac{\rho_0}{2} \ge \frac{1}{2}. \tag{2.24}$$

Moreover, we can prove that

$$\Sigma_1, \Sigma_2 \in O_6(1). \tag{2.25}$$

Similar to the way of estimating R_1, by a direct calculation, we have the following lemma.

Lemma 8 *The following estimates hold*:

$$P\bigl(x\bigl(\epsilon^{-2}\rho,\theta\bigr)\bigr)-P\bigl(x\bigl(\epsilon^{-2}\rho_0,\theta\bigr)\bigr)\in\epsilon^{-\alpha}O_6(1),$$
$$\frac{\partial x}{\partial H}\bigl(\epsilon^{-2}\rho,\theta\bigr)p\bigl(x\bigl(\epsilon^{-2}\rho,\theta\bigr)\bigr)-\frac{\partial x}{\partial H}\bigl(\epsilon^{-2}\rho_0,\theta\bigr)p\bigl(x\bigl(\epsilon^{-2}\rho_0,\theta\bigr)\bigr)\in\epsilon^{2-\alpha}O_6(1).$$

Now we turn to give an asymptotic expression of the Poincaré map of (2.14), that is, we study the behavior of the functions Σ_1 and Σ_2 at $\theta=\pi$ as $\epsilon\to0$. In order to estimate Σ_1 and Σ_2, we need to introduce the following definition and lemma. Let

$$\Theta_+(I)=\operatorname{meas}\bigl\{\theta\in[0,\pi],x(H_0,\theta)>0\bigr\},\qquad\Theta_-(I)=T_0-\Theta_+(I),$$

where $H_0=\epsilon^{-2}\rho_0$.

Lemma 9

$$\Theta_+(I)=\pi+\epsilon O_6(1),\qquad\Theta_-(I)=\epsilon O_6(1).$$

Proof This lemma was proved in [7], so we omit the details. □

For estimate Σ_1 and Σ_2, we need the estimates of x and x_H.

We recall that when $x<0$, we have

$$\bigl|x(H_0,\theta)\bigr|=O_6(1),\qquad\bigl|x_H(H_0,\theta)\bigr|=\epsilon^2O_5(1).$$

When $x>0$, by the definition of θ, we have

$$\arcsin\frac{x(H_0,\theta)}{\sqrt{2h}}=\frac{T_0(h)}{\pi}\theta-\frac{T_-(h)}{2}=\theta+\epsilon^2O_5(1),$$

which yields that

$$x(H_0,\theta)=\sqrt{\frac{2H_0}{\pi}}\sin\theta+O_5(1),$$
$$x_H(H_0,\theta)=\sqrt{\frac{1}{2H_0\pi}}\sin\theta+\epsilon^2O_5(1).$$

Now we can give the estimates of Σ_1 and Σ_2.

Lemma 10 *The following estimates hold true*:

$$\Sigma_1(t_0,\rho_0,2\pi;\epsilon)=\left(\frac{\pi}{2\rho_0}\right)^{\frac{\alpha-1}{2}}\int_0^\pi(\sin\theta)^{1+\alpha}f(t_0+\theta)\,d\theta+o_6(1),$$
$$\Sigma_2(t_0,\rho_0,2\pi;\epsilon)=-\pi^{\frac{1-\alpha}{2}}(2\rho_0)^{\frac{\alpha+1}{2}}\int_0^\pi(\sin\theta)^{1+\alpha}f'(t_0+\theta)\,d\theta+o_6(1)$$

for $\epsilon\to0$.

Proof Firstly we consider Σ_1. By Lemmas 3, 4, 8 and (2.22), we have

$$\begin{aligned}\Sigma_1(t_0,\rho_0,2\pi;\epsilon) &= \pi\epsilon^{\alpha-1}\int_0^\pi \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho,\theta\big)p\big(x\big(\epsilon^{-2}\rho,\theta\big)\big)f(t)\,d\theta \\ &\quad + \epsilon^{\alpha-1}\int_0^\pi \pi\frac{\partial R_1}{\partial H}\big(x\big(\epsilon^{-2}\rho,\theta\big),t\big) + \frac{\partial I_-}{\partial H}\big(x\big(\epsilon^{-2}\rho,\theta\big),t\big)\,d\theta \\ &= \pi\epsilon^{\alpha-1}\int_0^\pi \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho_0,\theta\big)p\big(x\big(\epsilon^{-2}\rho_0,\theta\big)\big)f(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1) \\ &= \pi\epsilon^{\alpha-1}\int_{\Theta_+} \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho_0,\theta\big)p\big(x\big(\epsilon^{-2}\rho_0,\theta\big)\big)f(t_0+\theta)\,d\theta \\ &\quad + \pi\epsilon^{\alpha-1}\int_{\Theta_-} \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho_0,\theta\big)p\big(x\big(\epsilon^{-2}\rho_0,\theta\big)\big)f(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1).\end{aligned}$$

Since $\lim_{x\to+\infty}\frac{p(x)}{|x|^\alpha} = 1$ and $\epsilon \to 0$ means $x \to \infty$, we have

$$\begin{aligned}&\pi\epsilon^{\alpha-1}\int_{\Theta_+} \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho_0,\theta\big)p\big(x\big(\epsilon^{-2}\rho_0\big),\theta\big)f(t_0+\theta)\,d\theta \\ &\quad = \pi\epsilon^{\alpha-1}\int_{\Theta_+} \frac{\partial x}{\partial H}\big(\theta,\epsilon^{-2}\rho\big)|x|^\alpha f(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1).\end{aligned} \tag{2.26}$$

By the measure of Θ_-, we have

$$\pi\epsilon^{\alpha-1}\int_{\Theta_-} \frac{\partial x}{\partial H}\big(\epsilon^{-2}\rho_0,\theta\big)p\big(x\big(\epsilon^{-2}\rho_0,\theta\big)\big)f(t_0+\theta)\,d\theta = \epsilon^\alpha O_6(1). \tag{2.27}$$

By (2.26) and (2.27), we have

$$\begin{aligned}\Sigma_1(t_0,\rho_0,2\pi;\epsilon) &= \pi\epsilon^{\alpha-1}\int_{\Theta_+} \frac{\partial x}{\partial H}\big(\theta,\epsilon^{-2}\rho\big)|x|^\alpha f(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1) \\ &= \pi\epsilon^{\alpha-1}\int_0^\pi \frac{\partial x}{\partial H}\big(\theta,\epsilon^{-2}\rho\big)|x|^\alpha f(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1) \\ &= \left(\frac{\pi}{2\rho_0}\right)^{\frac{1-\alpha}{2}}\int_0^\pi (\sin\theta)^{\alpha+1} f(t_0+\theta)\,d\theta + o_6(1).\end{aligned}$$

Now we consider Σ_2. By Lemmas 3, 4, 8 and (2.22), we have

$$\begin{aligned}\Sigma_2(t_0,\rho_0,2\pi;\epsilon) &= -\pi\epsilon^{\alpha+1}\int_0^\pi P\big(x\big(\theta,\epsilon^{-2}\rho\big)\big)f'(t)\,d\theta \\ &\quad - \epsilon^{\alpha+1}\int_0^\pi \left[\pi\frac{\partial R_1}{\partial t}\big(x\big(\theta,\epsilon^{-2}\rho\big),t\big) + \frac{\partial I_-}{\partial t}\big(x\big(\theta,\epsilon^{-2}\rho\big),t\big)\right]d\theta \\ &= -\pi\epsilon^{\alpha+1}\int_0^\pi P\big(x\big(\theta,\epsilon^{-2}\rho_0\big)\big)f'(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1) \\ &= -\pi\epsilon^{\alpha+1}\int_{\Theta_+} P\big(x\big(\theta,\epsilon^{-2}\rho_0\big)\big)f'(t_0+\theta)\,d\theta \\ &\quad - \pi\epsilon^{\alpha+1}\int_{\Theta_-} P\big(x\big(\theta,\epsilon^{-2}\rho_0\big)\big)f'(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1).\end{aligned}$$

By (1.7) and $\epsilon \to 0$ means $x \to \infty$, we have

$$-\pi \epsilon^{\alpha+1} \int_{\Theta_+} P\bigl(x\bigl(\theta, \epsilon^{-2}\rho_0\bigr)\bigr) f'(t_0+\theta)\,d\theta$$
$$= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_{\Theta_+} \bigl|x\bigl(\theta, \epsilon^{-2}\rho_0\bigr)\bigr|^{\alpha} x\bigl(\theta, \epsilon^{-2}\rho_0\bigr) f'(t_0+\theta)\,d\theta + \epsilon^{\alpha} O_6(1). \tag{2.28}$$

By the measure of Θ_-, we have

$$-\pi \epsilon^{\alpha+1} \int_{\Theta_-} P\bigl(x\bigl(\theta, \epsilon^{-2}\rho_0\bigr)\bigr) f'(t_0+\theta)\,d\theta = \epsilon^{\alpha} O_6(1). \tag{2.29}$$

By (2.28) and (2.29), we have

$$\begin{aligned} \Sigma_2 &= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_{\Theta_+} \bigl|x\bigl(\theta, \epsilon^{-2}\rho_0\bigr)\bigr|^{\alpha} x\bigl(\theta, \epsilon^{-2}\rho_0\bigr) f'(t_0+\theta)\,d\theta + \epsilon^{\alpha} O_6(1) \\ &= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_{0}^{\pi} \bigl|x\bigl(\theta, \epsilon^{-2}\rho_0\bigr)\bigr|^{\alpha} x\bigl(\theta, \epsilon^{-2}\rho_0\bigr) f'(t_0+\theta)\,d\theta + \epsilon^{\alpha} O_6(1) \\ &= -\frac{1}{\alpha+1} \pi^{\frac{1-\alpha}{2}} (2\rho_0)^{\frac{\alpha+1}{2}} \int_0^{\pi} (\sin\theta)^{1+\alpha} f'(t_0+\theta)\,d\theta + o_6(1). \end{aligned}$$

Thus Lemma 10 is proved. □

2.4 Proof of Theorem 1

Let

$$\Psi_1(t_0, \rho_0) = \left(\frac{\pi}{2\rho_0}\right)^{\frac{1-\alpha}{2}} \int_0^{\pi} (\sin\theta)^{1+\alpha} f(t_0+\theta)\,d\theta,$$
$$\Psi_2(t_0, \rho_0) = -\frac{1}{\alpha+1} \pi^{\frac{1-\alpha}{2}} (2\rho_0)^{\frac{\alpha+1}{2}} \int_0^{\pi} (\sin\theta)^{1+\alpha} f'(t_0+\theta)\,d\theta.$$

Then there are two functions ϕ_1 and ϕ_2 such that the Poincaré map of (2.15), given by (2.21), is of the form

$$\begin{aligned} P:\quad & t_1 = t_0 + 2\pi + \epsilon^{1-\alpha}\Psi_1(t_0, \rho_0) + \epsilon^{1-\alpha}\phi_1, \\ & \rho_1 = \rho_0 + \epsilon^{1-\alpha}\Psi_2(t_0, \rho_0) + \epsilon^{1-\alpha}\phi_2, \end{aligned} \tag{2.30}$$

where $\phi_1, \phi_2 \in o_6(1)$.

Since $\int_0^{\pi} p(t_0+\theta)\sin\theta\,d\theta > 0$, $\forall t_0 \in R$, we have

$$\Psi_1 > 0, \qquad \frac{\partial \Psi_1}{\partial \rho_0} \neq 0.$$

Let

$$L = \frac{\rho_0^{-\frac{1+\alpha}{2}}}{\int_0^{\pi} (\sin\theta)^{1+\alpha} f(t_0+\theta)\,d\theta}.$$

Then

$$\frac{\partial L}{\partial t_0}\Psi_1(t_0,\rho_0)+\frac{\partial L}{\partial \rho_0}\Psi_2(t_0,\rho_0)=0.$$

The other assumptions of Ortega's theorem are easily verified. Hence, there is an invariant curve of P in the annulus $(t_0,\rho_0)\in S^1\times[1,2]$, which implies the boundedness of our original equation (1.5). Then Theorem 1 is proved.

2.5 Proof of Theorem 2

We apply Aubry-Mather theory. By Theorem B in [11] and the monotone twist property of the Poincaré map P guaranteed by $\frac{\partial \Psi_1}{\partial \rho_0}<0$, it is straightforward to check that Theorem 2 is correct.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of three authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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References

1. Littlewood, J: Unbounded solutions of $y'' + g(y) = p(t)$. J. Lond. Math. Soc. **41**, 133-149 (1966)
2. Morris, GR: A case of boundedness of Littlewood's problem on oscillatory differential equations. Bull. Aust. Math. Soc. **14**, 71-93 (1976)
3. Levi, M: Quasiperiodic motions in superquadratic time-periodic potential. Commun. Math. Phys. **144**, 43-82 (1991)
4. Dieckerhoff, R, Zehnder, E: Boundedness of solutions via the twist theorem. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **14**, 79-95 (1987)
5. Liu, B: Boundedness of solutions for equations with *p*-Laplacian and an asymmetric nonlinear term. J. Differ. Equ. **207**, 73-92 (2004)
6. Cheng, C, Xu, J: Boundedness of solutions for a second-order differential equations. Nonlinear Anal. **7**, 1993-2004 (2008)
7. Capietto, A, Dambrosio, W, Liu, B: On the boundedness of solutions to a nonlinear singular oscillator. Z. Angew. Math. Phys. **60**(6), 1007-1034 (2009)
8. Liu, B: Quasi-periodic solutions of forced isochronous oscillators at resonance. J. Differ. Equ. **246**, 3471-3495 (2009)
9. Ortega, R: Boundedness in a piecewise linear oscillator and a variant of the small twist theorem. Proc. Lond. Math. Soc. **79**, 381-413 (1999)
10. Jiang, S, Fang, F: Lagrangian stability of a class of second-order periodic systems. Abstr. Appl. Anal. **2011**, Article ID 106214 (2011)
11. Pei, ML: Aubry-Mather sets for finite-twist maps of a cylinder and semilinear Duffing equations. J. Differ. Equ. **113**, 106-127 (1994)

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