

RESEARCH

Open Access

A sum analogous to the high-dimensional Kloosterman sums and its upper bound estimate

Yijun Li^{1,2} and Di Han^{1*}

*Correspondence: handi515@com

¹Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China

Full list of author information is available at the end of the article

Abstract

The main purpose of this paper is, using the properties of Gauss sums and the estimate for the generalized exponential sums, to study the upper bound estimate problem of one kind sums analogous to the high-dimensional Kloosterman sums and to give some interesting mean value formula and an upper bound estimate for it.

MSC: 11L05

Keywords: a sum analogous to the high-dimensional Kloosterman sums; Gauss sums; upper bound estimate; mean value

1 Introduction

For any integer $q \geq 3$, the high-dimensional Kloosterman sums $K(c_1, c_2, \dots, c_k, m; q)$ are defined as follows:

$$K(c_1, c_2, \dots, c_k, m; q) = \sum_{a_1=1}^q \cdots \sum_{a_k=1}^q e\left(\frac{c_1 a_1 + \cdots + c_k a_k + m \bar{a}_1 \cdots \bar{a}_k}{q}\right),$$

where $e(x) = e^{2\pi i x}$, $\sum_{a_i=1}^q$ denotes the summation over all integers $1 \leq a_i \leq q$ such that $(a_i, q) = 1$, c_i and m are integers with $(m, q) = 1$, \bar{a}_i denotes the solution of the congruent equation $x \cdot a_i \equiv 1 \pmod{q}$ ($i = 1, 2, \dots, k$).

There are several results on the properties of the Kloosterman sums $K(c_1, c_2, \dots, c_k, m; q)$. For example, see [1, 2] and [3]. Related works can also be found in [4–8] and [9].

In this paper, we consider a sum analogous to the high-dimensional Kloosterman sums as follows:

$$S(c_1, c_2, \dots, c_k, m, \chi; q) = \sum_{a_1=1}^q \cdots \sum_{a_k=1}^q \chi(c_1 a_1 + \cdots + c_k a_k + m \bar{a}_1 \cdots \bar{a}_k), \quad (1.1)$$

where χ is a Dirichlet character mod q .

If $k = 1$ and $q = p$ (an odd prime), then for any integer a with $(a, p) = 1$, applying the Fermat little theorem, one can deduce $a^{p-2} \equiv \bar{a} \pmod{p}$. So, the sum (1.1) becomes

$$\sum_{a=1}^{p-1} \chi(ca + m\bar{a}).$$

It is a special case of the general polynomial character sums

$$\sum_{a=N+1}^{N+M} \chi(f(a)),$$

where M and N are any positive integers and $f(x)$ is a polynomial. Let χ be a q th-order character mod p . If $f(x)$ is not a perfect q th power mod p , then from Weil's classical work (see [10]), we can deduce the estimate

$$\sum_{x=N+1}^{N+M} \chi(f(x)) \ll p^{\frac{1}{2}} \ln p,$$

where ' \ll ' constant depends only on the degree of $f(x)$. Some related results can also be found in [11–13] and [14].

Now we are concerned with the upper bound estimate problem of (1.1). Regarding this contents, it seems that nobody has yet studied it, at least we have not seen any related result before. The problem is interesting because it can reflect some new properties of character sums. The main purpose of this paper is, using the analytic methods and the properties of Gauss sums, to study this problem and give a sharp upper bound estimate for (1.1). That is, we prove the following conclusions.

Theorem 1 *Let p be an odd prime, let k be a positive integer with $k \geq 2$, and let χ be any non-principal character mod p . Then for any integers c_1, c_2, \dots, c_k and m with $(c_1 c_2 \cdots c_k m, p) = 1$, we have the identity*

$$|S(c_1, c_2, \dots, c_k, m, \chi; p)| = \begin{cases} p^{\frac{k}{2}} & \text{if } (k+1, p-1) = 1, \\ 0 & \text{if } (k+1)|(p-1) \text{ and } \chi^{\frac{p-1}{k+1}} \neq \chi_0, \end{cases}$$

where χ_0 denotes the principal character mod p .

Theorem 2 *Let p be an odd prime, let k be a positive integer with $k \geq 2$, and let χ be any non-principal character mod p . Then for any integers c_1, c_2, \dots, c_k and m with $(c_1 c_2 \cdots c_k m, p) = 1$, we have the estimate*

$$|S(c_1, c_2, \dots, c_k, m, \chi; p)| \leq (k+1) \cdot p^{\frac{k}{2}}.$$

Theorem 3 *Let p and q be two odd primes, let r be any q th non-residue mod p . Then for any integers c_1, c_2, \dots, c_{q-1} with $(c_1 c_2 \cdots c_{q-1}, p) = 1$, we have the identity*

$$\sum_{i=0}^{q-1} \left[\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \cdots + c_{q-1} a_{q-1} + r^i \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{q-1}}{p} \right) \right]^2 = \begin{cases} q^2 \cdot p^{q-1} & \text{if } q|(p-1), \\ q \cdot p^{q-1} & \text{if } (q, p-1) = 1. \end{cases}$$

If $p \equiv 1 \pmod{4}$, then the above formula also holds for $q = 2$, where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol.

Taking $p = 4m + 1$ and $q = 2$ in Theorem 3, note that $2|(p - 1)$, we may immediately deduce the following.

Corollary *Let p be an odd prime with $p \equiv 1 \pmod{4}$, then we have the identity*

$$p = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + r\bar{a}}{p} \right) \right)^2 + \left(\sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b + s\bar{b}}{p} \right) \right)^2,$$

where r and s are any two integers such that $\left(\frac{r}{p}\right) \cdot \left(\frac{s}{p}\right) = -1$.

This gives another proof for a classical work in elementary number theory (i.e., see [15] Theorems 4-11): For any prime p with $p \equiv 1 \pmod{4}$, there exist two positive integers x and y such that $p = x^2 + y^2$.

2 Several lemmas

To complete the proof of our theorems, we need the following basic lemmas.

Lemma 1 *Let p be an odd prime, let χ be any non-principal character mod p , and let k be any positive integer such that $(k, p - 1) = 1$ or $k|p - 1$. Then for any integer m with $(m, p) = 1$, we have the identity*

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = \begin{cases} \bar{\chi}^r(m) \cdot \tau(\chi^r) & \text{if } (k, p - 1) = 1, \\ 0 & \text{if } k|(p - 1) \text{ and } \chi^{\frac{p-1}{k}} \neq \chi_0, \\ \bar{\chi}_1(m) \cdot \sum_{i=0}^{k-1} \bar{\chi}_k^i(m) \tau(\chi_1 \chi_k^i) & \text{if } k|(p - 1) \text{ and } \chi^{\frac{p-1}{k}} = \chi_0, \end{cases}$$

where $r \cdot k \equiv 1 \pmod{p - 1}$, χ_0 denotes the principal character mod p , χ_k denotes any k -order character mod p and $\chi_1^k = \chi$.

Proof If $(k, p - 1) = 1$, then there exists one integer r with $(r, p - 1) = 1$ such that $r \cdot k \equiv 1 \pmod{p - 1}$. This time, for any integer a with $(a, p) = 1$, we have $a^{rk} \equiv a \pmod{p}$. If a passes through a reduced residue system mod p , then a^r also passes through a reduced residue system mod p . Therefore, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi(a^r) e\left(\frac{ma^{rk}}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi^r(a) e\left(\frac{ma}{p}\right) = \bar{\chi}^r(m) \cdot \tau(\chi^r). \end{aligned} \tag{2.1}$$

If $k > 1$ and $k|(p - 1)$ with $\chi^{\frac{p-1}{k}} \neq \chi_0$, then there must exist an integer n with $(n, p) = 1$ such that $\chi^{\frac{p-1}{k}}(n) \neq 1$. For this n , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi\left(a \cdot n^{\frac{p-1}{k}}\right) e\left(\frac{m(a \cdot n^{\frac{p-1}{k}})^k}{p}\right) \\ &= \chi\left(n^{\frac{p-1}{k}}\right) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k n^{p-1}}{p}\right) = \chi^{\frac{p-1}{k}}(n) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) \end{aligned}$$

or

$$(1 - \chi^{\frac{p-1}{k}}(n)) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = 0.$$

Since $\chi^{\frac{p-1}{k}}(n) \neq 1$, from the above identity, we have

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = 0. \tag{2.2}$$

If $\chi^{\frac{p-1}{k}} = \chi_0$, then χ must be a k th character mod p , so there exists one character χ_1 mod p such that $\chi = \chi_1^k$. Let χ_k be a k -order character mod p (i.e., $\chi_k^k = \chi_0$), then for any integer a with $(a, p) = 1$, note that

$$1 + \chi_k(a) + \chi_k^2(a) + \dots + \chi_k^{k-1}(a) = \begin{cases} k & \text{if } a \text{ is a } k\text{th residue mod } p, \\ 0, & \text{otherwise.} \end{cases}$$

From the properties of Gauss sums, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi_1^k(a) e\left(\frac{ma^k}{p}\right) = \sum_{a=1}^{p-1} \chi_1(a^k) e\left(\frac{ma^k}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_1(a) (1 + \chi_k(a) + \chi_k^2(a) + \dots + \chi_k^{k-1}(a)) e\left(\frac{ma}{p}\right) \\ &= \bar{\chi}_1(m) \cdot \sum_{i=0}^{k-1} \bar{\chi}_k^i(m) \tau(\chi_1 \chi_k^i). \end{aligned} \tag{2.3}$$

Now Lemma 1 follows from (2.1), (2.2) and (2.3). □

Lemma 2 *Let p and q be two odd primes with $q|(p-1)$, and let χ_q be any q -order character mod p . Then for any integers c_1, c_2, \dots, c_{q-1} and m with $(mc_1c_2 \dots c_{q-1}, p) = 1$, we have the identities*

$$\begin{aligned} \text{(I)} \quad & \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \dots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \dots + c_{q-1} a_{q-1} + m \bar{a}_1 \bar{a}_2 \dots \bar{a}_{q-1}}{p} \right) \\ &= \left(\frac{n}{p}\right) \cdot \tau^{q-1}(\chi_2) \left(\sum_{i=0}^{q-1} \chi_q^i(n) \frac{\tau^q(\chi_2 \chi_q^i)}{\tau^q(\chi_2)} \right); \\ \text{(II)} \quad & \sum_{a=1}^{p-1} \left(\frac{a + m \bar{a}}{p} \right) = \begin{cases} \frac{\bar{\chi}_1(m)}{\tau(\chi_2)} (\tau^2(\chi_1) + \frac{m}{p} \tau^2(\bar{\chi}_1)) & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases} \end{aligned}$$

where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre symbol, $n = mc_1c_2 \dots c_{q-1}$ and $\chi_1^2 = \chi_2$.

Proof If q is an odd prime, then $\chi_2^q = \chi_2$ and $\bar{\chi}_2 = \chi_2$, so applying (2.3) and the properties of Gauss sums, we have

$$\begin{aligned} & \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \cdots + c_{q-1} a_{q-1} + m \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{q-1}}{p} \right) \\ &= \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{a_1 + a_2 + \cdots + a_{q-1} + n \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{q-1}}{p} \right) \\ &= \frac{1}{\tau(\chi_2)} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a_1 + \cdots + a_{q-1}) + bn \bar{a}_1 \cdots \bar{a}_{q-1}}{p}\right) \\ &= \frac{1}{\tau(\chi_2)} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-2}=1}^{p-1} \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b(1 + a_1 + \cdots + a_{q-2})}{p}\right) \\ &\quad \times \sum_{a_{q-1}=1}^{p-1} \chi_2(a_{q-1}) e\left(\frac{bn \bar{a}_{q-1} \bar{a}_1 \cdots \bar{a}_{q-2}}{p}\right) \\ &= \frac{1}{\tau(\chi_2)} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-2}=1}^{p-1} \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b(1 + a_1 + \cdots + a_{q-2})}{p}\right) \\ &\quad \times \sum_{c=1}^{p-1} \chi_2(c) (1 + \chi_q(c) + \cdots + \chi_q^{q-1}(c)) e\left(\frac{bnc \bar{a}_1 \cdots \bar{a}_{q-2}}{p}\right) \\ &= \left(\frac{n}{p}\right) \cdot \tau^{q-1}(\chi_2) \left(\sum_{i=0}^{q-1} \chi_q^i(n) \frac{\tau^q(\chi_2 \chi_q^i)}{\tau^q(\chi_2)}\right). \end{aligned}$$

This proves formula (I).

To prove formula (II), note that if $p \equiv 3 \pmod{4}$, then χ_2 must be an odd character mod p (i.e., $\chi_2(-1) = -1$) so that

$$\begin{aligned} \sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p}\right) &= \sum_{a=1}^{p-1} \left(\frac{p - a + m\overline{p-a}}{p}\right) \\ &= \left(\frac{-1}{p}\right) \sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p}\right) = - \sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p}\right) \end{aligned}$$

or

$$\sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p}\right) = 0.$$

If $p \equiv 1 \pmod{4}$, then there exists one character $\chi_1 \pmod{p}$ such that $\chi_1^2 = \chi_2$. Note that $\chi_1^3 = \bar{\chi}_1$ and $\chi_2 \bar{\chi}_1 = \chi_1$; from the properties of Gauss sums, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p}\right) &= \frac{1}{\tau(\chi_2)} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a + m\bar{a})}{p}\right) \\ &= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) \sum_{a=1}^{p-1} \chi_2(\bar{a}) e\left(\frac{b + mb\bar{a}^2}{p}\right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi_1^2(a) e\left(\frac{mba^2}{p}\right) \\
 &= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi_1(a) (1 + \chi_2(a)) e\left(\frac{mba}{p}\right) \\
 &= \frac{\bar{\chi}_1(m)}{\tau(\chi_2)} \left(\tau^2(\chi_1) + \left(\frac{m}{p}\right) \tau^2(\bar{\chi}_1) \right).
 \end{aligned}$$

This proves Lemma 2. □

3 Proof of the theorems

In this section, we complete the proof of our theorems. First we prove Theorems 1 and 2. Let $n = mc_1c_2 \cdots c_k$, $(k + 1, p - 1) = d$. If $d = 1$, we can assume $r(k + 1) \equiv 1 \pmod{p - 1}$, then from Lemma 1, the properties of a reduced residue system mod p and Gauss sums, we have

$$\begin{aligned}
 &S(c_1, c_2, \dots, c_k, m, \chi; p) \\
 &= \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi(a_1 + \cdots + a_k + mc_1c_2 \cdots c_k \cdot \bar{a}_1 \cdots \bar{a}_k) \\
 &= \frac{1}{\tau(\bar{\chi})} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \bar{\chi}(b) e\left(\frac{b(a_1 + \cdots + a_k + n\bar{a}_1 \cdots \bar{a}_k)}{p}\right) \\
 &= \frac{1}{\tau(\bar{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{a_1=1}^{p-1} \sum_{b=1}^{p-1} \bar{\chi}(b\bar{a}_1) e\left(\frac{b(1 + \cdots + a_k) + nb\bar{a}_1^{k+1}\bar{a}_2 \cdots \bar{a}_k}{p}\right) \\
 &= \frac{1}{\tau(\bar{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \bar{\chi}(b) e\left(\frac{b + b(a_2 + \cdots + a_k)}{p}\right) \\
 &\quad \times \sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\bar{a}_1^{k+1}\bar{a}_2 \cdots \bar{a}_k}{p}\right) \\
 &= \frac{1}{\tau(\bar{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \bar{\chi}(b) e\left(\frac{b + b(a_2 + \cdots + a_k)}{p}\right) \\
 &\quad \times \sum_{a_1=1}^{p-1} \bar{\chi}^r(a_1) e\left(\frac{nb\bar{a}_1\bar{a}_2 \cdots \bar{a}_k}{p}\right) \\
 &= \frac{\tau(\bar{\chi}^r)}{\tau(\bar{\chi})} \sum_{b=1}^{p-1} \bar{\chi}(b) e\left(\frac{b}{p}\right) \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi^r(nb\bar{a}_2 \cdots \bar{a}_k) e\left(\frac{b(a_2 + \cdots + a_k)}{p}\right) \\
 &= \chi^r(n) \cdot \frac{\tau^{k+1}(\bar{\chi}^r)}{\tau(\bar{\chi})} \\
 &= \chi^r(mc_1c_2 \cdots c_k) \cdot \frac{\tau^{k+1}(\bar{\chi}^r)}{\tau(\bar{\chi})}. \tag{3.1}
 \end{aligned}$$

If $d > 1$ and $\chi^{\frac{p-1}{d}} \neq \chi_0$, then from the method of proving (2.2), we have

$$\sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\bar{a}_1^{k+1}\bar{a}_2 \cdots \bar{a}_k}{p}\right) = 0.$$

From this identity and the method of proving (3.1), we may immediately deduce that if $\chi^{\frac{p-1}{d}} \neq \chi_0$, then

$$S(c_1, c_2, \dots, c_k, m, \chi; p) = 0. \tag{3.2}$$

If $d > 1$ and $\chi^{\frac{p-1}{d}} = \chi_0$, then χ must be a d th character mod p , so there exists a character $\chi_1 \pmod p$ such that $\chi = \chi_1^d$. Let χ_d be a d -order character mod p , then we have

$$\begin{aligned} & \sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\bar{a}_1^{k+1}\bar{a}_2 \cdots \bar{a}_k}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_1(a^d) e\left(\frac{nb\bar{a}^{d \cdot \frac{k+1}{d}} \bar{a}_2 \cdots \bar{a}_k}{p}\right) \\ &= \sum_{a=1}^{p-1} \bar{\chi}_1(a) (1 + \chi_d(a) + \chi_d^2(a) + \cdots + \chi_d^{d-1}(a)) e\left(\frac{nba^{\frac{k+1}{d}} \bar{a}_2 \cdots \bar{a}_k}{p}\right) \\ &= \sum_{i=0}^{d-1} \sum_{a=1}^{p-1} \bar{\chi}_1(a) \chi_d^i(a) e\left(\frac{nba^{\frac{k+1}{d}} \bar{a}_2 \cdots \bar{a}_k}{p}\right). \end{aligned} \tag{3.3}$$

Let $(\frac{k+1}{d}, p-1) = d_1$, then repeat the process of proving (2.1), (2.2) and (2.3). Combining (3.1), (3.2) and (3.3), we may immediately deduce the estimate

$$|S(c_1, c_2, \dots, c_k, m, \chi; p)| \leq (k+1) \cdot p^{\frac{k}{2}}. \tag{3.4}$$

Now note that $|\tau(\chi)| = |\tau(\chi^r)| = \sqrt{p}$, Theorems 1 and 2 follow from (3.1), (3.2) and (3.4).

Now we prove Theorem 3. If $q \geq 3$, we separate q into two cases $(q, p-1) = 1$ and $(q, p-1) = q$. If $(q, p-1) = q$, then note that for any q th non-residue $r \pmod p$, we have

$$\sum_{h=0}^{q-1} \chi_q^l(r^h) = \begin{cases} q & \text{if } q|l, \\ 0 & \text{if } (q, l) = 1. \end{cases}$$

From (I) of Lemma 2, we can deduce that

$$\begin{aligned} & \sum_{i=0}^{q-1} \left[\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \cdots + c_{q-1} a_{q-1} + r^i \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{q-1}}{p} \right) \right]^2 \\ &= p^{q-1} \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \sum_{h=0}^{q-1} \chi_q^{i+j} (c_1 c_2 \cdots c_{q-1} r^h) \frac{\tau^q(\chi_2 \chi_q^i)}{\tau^q(\chi_2)} \cdot \frac{\tau^q(\chi_2 \chi_q^j)}{\tau^q(\chi_2)} \\ &= q \cdot p^{q-1} \sum_{i=0}^{q-1} \sum_{\substack{j=0 \\ i+j=0,q}}^{q-1} \frac{\tau^q(\chi_2 \chi_q^i)}{\tau^q(\chi_2)} \cdot \frac{\tau^q(\bar{\chi}_2 \bar{\chi}_q^j)}{\tau^q(\chi_2)} \\ &= q^2 \cdot p^{q-1}. \end{aligned}$$

If $(q, p - 1) = 1$, then from the method of proving (2.1) and the properties of Gauss sums, we can deduce that

$$\begin{aligned} & \sum_{i=0}^{q-1} \left[\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \cdots + c_{q-1} a_{q-1} + r^i \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{q-1}}{p} \right) \right]^2 \\ &= \sum_{i=0}^{q-1} p^{q-1} = q \cdot p^{q-1}. \end{aligned}$$

If $q = 2$ and $p \equiv 1 \pmod{4}$, then applying (II) of Lemma 2, we have

$$\begin{aligned} \left(\sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p} \right) \right)^2 &= p \cdot \left| 1 + \left(\frac{m}{p} \right) \cdot \frac{\tau^2(\bar{\chi}_1)}{\tau^2(\chi_1)} \right|^2 \\ &= 2p + \left(\frac{m}{p} \right) \cdot \frac{\tau^2(\bar{\chi}_1)}{\tau^2(\chi_1)} + \left(\frac{m}{p} \right) \cdot \frac{\tau^2(\chi_1)}{\tau^2(\bar{\chi}_1)}. \end{aligned} \tag{3.5}$$

Therefore, from (3.5) we can deduce that

$$\left(\sum_{a=1}^{p-1} \left(\frac{a + \bar{a}}{p} \right) \right)^2 + \left(\sum_{a=1}^{p-1} \left(\frac{a + r\bar{a}}{p} \right) \right)^2 = 4p.$$

This proves Theorem 3.

To prove the corollary, note that

$$\sum_{a=1}^{p-1} \left(\frac{a + m\bar{a}}{p} \right) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + m\bar{a}}{p} \right) + \sum_{a=\frac{p+1}{2}}^{p-1} \left(\frac{a + m\bar{a}}{p} \right) = 2 \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + m\bar{a}}{p} \right).$$

From (3.5) we may immediately deduce the identity

$$p = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + r\bar{a}}{p} \right) \right)^2 + \left(\sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b + s\bar{b}}{p} \right) \right)^2,$$

where r and s are any two integers such that $\left(\frac{r}{p}\right) \cdot \left(\frac{s}{p}\right) = -1$.

This completes the proof of our corollary.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YL carried out the upper bound estimate problem of one kind sums analogous to the high-dimensional Kloosterman sums. DH participated in the research and summary of the study.

Author details

¹Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China. ²Department of Mathematics, Xi'an Shiyou University, Xi'an, Shaanxi, P.R. China.

Acknowledgements

The authors would like to thank the referee for carefully examining this paper and providing a number of important comments. This work is supported by the N.S.F. of P.R. China (11071194, 61202437), and by the Youth Science and Technology Innovation Foundation of Xi'an Shiyou University (2012QN012).

References

1. Chowla, S: On Kloosterman's sum. *Norske Vid. Selsk. Forh., Trondheim* **40**, 70-72 (1967)
2. Malyshev, AV: A generalization of Kloosterman sums and their estimates. *Vestn. Leningr. Univ.* **15**, 59-75 (1960) (in Russian)
3. Zhang, W: On the fourth power mean of the general Kloosterman sums. *Indian J. Pure Appl. Math.* **35**, 237-242 (2004)
4. Cochrane, T, Zheng, Z: Bounds for certain exponential sums. *Asian J. Math.* **4**, 757-774 (2000)
5. Cochrane, T, Zheng, Z: Upper bounds on a two-term exponential sums. *Sci. China Ser. A* **44**, 1003-1015 (2001)
6. Cochrane, T, Zheng, Z: Pure and mixed exponential sums. *Acta Arith.* **91**, 249-278 (1999)
7. Cochrane, T, Pinner, C: A further refinement of Mordell's bound on exponential sums. *Acta Arith.* **116**, 35-41 (2005)
8. Cochrane, T, Pinner, C: Using Stepanov's method for exponential sums involving rational functions. *J. Number Theory* **116**, 270-292 (2006)
9. Weil, A: On some exponential sums. *Proc. Natl. Acad. Sci. USA* **34**, 204-207 (1948)
10. Burgess, DA: On Dirichlet characters of polynomials. *Proc. Lond. Math. Soc.* **13**, 537-548 (1963)
11. Apostol, TM: *Introduction to Analytic Number Theory*. Springer, New York (1976)
12. Granville, A, Soundararajan, K: Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *J. Am. Math. Soc.* **20**, 357-384 (2007)
13. Zhang, W, Yao, W: A note on the Dirichlet characters of polynomials. *Acta Arith.* **115**, 225-229 (2004)
14. Zhang, W, Yi, Y: On Dirichlet characters of polynomials. *Bull. Lond. Math. Soc.* **34**, 469-473 (2002)
15. Zhang, W, Li, H: *Elementary Number Theory*. Shaanxi Normal University Press, Xi'an (2008)

doi:10.1186/1029-242X-2013-130

Cite this article as: Li and Han: A sum analogous to the high-dimensional Kloosterman sums and its upper bound estimate. *Journal of Inequalities and Applications* 2013 **2013**:130.

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com
