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Sharp Shafer-Fink type inequalities for Gauss lemniscate functions

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Abstract

In this paper, we establish sharp Shafer-Fink type inequalities for Gauss lemniscate functions.

MSC: 26D07

Keywords: inequalities; lemniscate functions

1 Introduction and definitions

In geometry, the lemniscate of Bernoulli is a plane curve defined by two given points F_1 and F_2, known as foci, at distance $2a$ from each other as the locus of points P so that $PF_1 \cdot PF_2 = a^2$. This gives the equation $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$. In polar coordinates (r, θ), the equation becomes $r^2 = 2a \cos(2\theta)$. The arc length from the origin to a point on the Bernoulli lemniscate $r^2 = \cos(2\theta)$ is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1 - t^4}}, \quad |x| \le 1, \tag{1.1}$$

where $\operatorname{arcsl} x$ is called the arc lemniscate sine function studied by CF Gauss in 1797-1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1 + t^4}}, \quad x \in \mathbb{R}. \tag{1.2}$$

The functions (1.1) and (1.2) can be found in [1, p.259], [2, (2.5)-(2.6)], [3–7] and [8, Ch. 1].

Following Neuman [3], Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are defined by

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1 + x^4}}\right), \quad x \in \mathbb{R} \tag{1.3}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1 - x^4}}\right), \quad |x| < 1, \tag{1.4}$$

respectively.

For $0 \le x \le 1$, it is known in the literature that

$$\frac{3x}{2 + \sqrt{1 - x^2}} \le \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \le \arcsin x \le \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{1.5}$$

The first and second inequalities in equation (1.5) were established by Shafer (see, *e.g.*, [9, p.247]), while the third inequality was proved by Fink [10]. In recent years, Shafer-Fink's inequalities have attracted much attention of the mathematical community. By using the λ-method of Mitrinović and Vasić [9], Malešević [11] improved the upper bound for $\arcsin x$ and established the following inequality: For $0 \le x \le 1$,

$$\frac{3x}{2 + \sqrt{1 - x^2}} \le \arcsin x \le \frac{(\pi/(\pi - 2))x}{(2/(\pi - 2)) + \sqrt{1 - x^2}} \le \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{1.6}$$

In [12–15], other upper bounds for $\arcsin x$ were established: For $0 \le x \le 1$,

$$\frac{3x}{2 + \sqrt{1 - x^2}} \le \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \le \arcsin x$$

$$\le \frac{(\pi(2 - \sqrt{2})/(\pi - 2\sqrt{2}))(\sqrt{1 + x} - \sqrt{1 - x})}{(\sqrt{2}(4 - \pi)/(\pi - 2\sqrt{2})) + \sqrt{1 + x} + \sqrt{1 - x}}$$

$$\le \frac{\pi(\sqrt{2} + 1/2)(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \le \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{1.7}$$

In [16], Pan and Zhu gave some further generalizations of these results and obtained two new Shafer-Fink type double inequalities. In [17], Zhu provided a solution to an open problem posed by Oppenheim in [18]. At the same time, some Shafer-Fink inequalities were deduced from the solution of Oppenheim's problem. Chen and Cheung [19] provided a laconic proof to Oppenheim's problem. Recently, Qi and Guo [20, 21] presented a sharpening and generalizations of Shafer-Fink's inequality.

Related to the inverse sine inequality, the inverse tangent inequality is also of much interest. In the literature, we have

$$\frac{3x}{1 + 2\sqrt{1 + x^2}} < \arctan x < \frac{2x}{1 + \sqrt{1 + x^2}} \quad \text{for } x > 0. \tag{1.8}$$

The first inequality in equation (1.8) was presented without proof by Shafer [22]. Three proofs of it were later given in [23]. The second inequality in equation (1.8) can be found in, *e.g.*, [24, p.288]. Shafer's inequality (1.8) was recently sharpened and generalized by Qi *et al.* in [25]. For each $\theta > 0$, Chen *et al.* [26] determined the largest number θ_1 and the smallest number θ_2 such that the inequalities

$$\frac{\theta_1 x}{1 + \theta\sqrt{1 + x^2}} \le \arctan x \le \frac{\theta_2 x}{1 + \theta\sqrt{1 + x^2}} \tag{1.9}$$

are valid for all $x \ge 0$. Zhu [27, Theorems 1.9 and 1.10] established Shafer-Fink type inequalities for the inverse hyperbolic sine function.

Recently, numerous inequalities have been given for the lemniscate functions. For example, Neuman [5] proved the following inequalities:

$$\left(\frac{5}{3 + 2(1 - x^4)^{1/2}}\right)^{1/2} < \frac{\operatorname{arcsl} x}{x} < (1 - x^4)^{-1/10} \tag{1.10}$$

and

$$\left(\frac{5}{3 + 2(1 + x^4)^{1/2}}\right)^{1/2} < \frac{\text{arcslh}\,x}{x} < \left(1 + x^4\right)^{-1/10} \tag{1.11}$$

for $0 < |x| < 1$.

Chen [28, 29] established Wilker and Huygens type inequalities for Gauss lemniscate functions. For example, Chen [28] proved that for $0 < |x| < 1$,

$$2 + \frac{1}{20}x^3 \arctan x < \left(\frac{\text{arcsl}\,x}{x}\right)^2 + \frac{\text{arctl}\,x}{x} \tag{1.12}$$

with the best possible constant $\frac{1}{20}$. Chen [29] proved that for $0 < |x| < 1$,

$$\frac{2(\text{arcsl}\,x/x) + \text{arctl}\,x/x}{3} > \frac{3(\text{arcsl}\,x/x) + 2(\text{arctl}\,x/x)}{5} > 1. \tag{1.13}$$

In this paper, we establish sharp Shafer-Fink type inequalities for Gauss lemniscate functions.

The following lemma is required in our present investigation.

Lemma 1.1 ([30–32]) *Let* $-\infty < a < b < \infty$, *and let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$, *differentiable on* (a, b). *Let* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a, b), *then so are*

$$\big[f(x) - f(a)\big]/\big[g(x) - g(a)\big] \quad and \quad \big[f(x) - f(b)\big]/\big[g(x) - g(b)\big].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Remark 1.1 A generalization of the familiar trigonometric and hyperbolic functions was described by Lindqvist [33]. The generalized p-trigonometric functions occur as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. Recently, the p-trigonometric functions have been studied extensively, see for example [34–37] and their references. Very recently, Takeuchi [38] (see also [39]) introduced the (p, q)-trigonometric functions that coincide with the p-trigonometric functions for $p = q$ and are connected with the Dirichlet problem for the p, q-Laplacian. These (p, q)-trigonometric functions have been the subject of intense investigations (see, for example, [35, 39–42]). For $p, q > 1$ the function $\arcsin_{p,q}$ is defined in [38, 39] by

$$\arcsin_{p,q}(x) = \int_0^x \frac{dt}{(1 - t^q)^{1/p}}, \quad |x| \leq 1. \tag{1.14}$$

Similarly, for $p, q > 1$ the function $\text{arcsinh}_{p,q}$ is defined by [40]

$$\text{arcsinh}_{p,q}(x) = \int_0^x \frac{dt}{(1 + t^q)^{1/p}}, \quad x \geq 0. \tag{1.15}$$

Clearly,

$$\text{arcsl}\,x = \arcsin_{2,4}(x) \quad and \quad \text{arcslh}\,x = \text{arcsinh}_{2,4}(x).$$

2 Main results

Theorem 2.1 *For $0 < x < 1$,*

$$\frac{\alpha_1(\sqrt{1+x} - \sqrt{1-x})}{4 - \sqrt{1+x} - \sqrt{1-x}} < \operatorname{arcsl} x < \frac{\beta_1(\sqrt{1+x} - \sqrt{1-x})}{4 - \sqrt{1+x} - \sqrt{1-x}} \tag{2.1}$$

with the best possible constants

$$\alpha_1 = 2 \quad and \quad \beta_1 = \frac{2\sqrt{2}-1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = 2.39712057\ldots. \tag{2.2}$$

Here

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

is the beta function.

Proof For $0 < x < 1$, let

$$f(x) = \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = \operatorname{arcsl} x \quad and \quad f_2(x) = \frac{\sqrt{1+x} - \sqrt{1-x}}{4 - \sqrt{1+x} - \sqrt{1-x}}.$$

Then,

$$\frac{2f_1'(x)}{f_2'(x)} = \frac{(4 - \sqrt{1+x} - \sqrt{1-x})^2 \sqrt{1+x}\sqrt{1-x}}{\sqrt{1-x^4}(-1 + \sqrt{1-x} + \sqrt{1+x})} =: f_3(x).$$

Differentiation yields

$$f_3'(x) = \frac{4 - \sqrt{1+x} - \sqrt{1-x}}{\sqrt{1-x^2}\sqrt{1-x^4}(x^2+1)(-1 + \sqrt{1-x} + \sqrt{1+x})^2} f_4(x),$$

where

$$f_4(x) = \left(6x^3 + x^2 - 4x + 1\right)\sqrt{1-x} + \left(6x^3 - x^2 - 4x - 1\right)\sqrt{1+x}$$
$$- \left(x^3 - 3x\right)\sqrt{1-x^2} - 6x^3 + 6x.$$

Motivated by the investigations in [12], we are in a position to prove $f_4(x) > 0$ for $x \in (0,1)$. Let

$$A(x) = \begin{cases} \mu, & x = 0, \\ \frac{f_4(x)}{x^3}, & 0 < x < 1, \\ 0, & x = 1, \end{cases}$$

where μ is constant determined with limit:

$$\mu = \lim_{x \to 0} \frac{f_4(x)}{x^3} = \frac{27}{8}.$$

Using Maple we determine Taylor approximation for the function $A(x)$ by the polynomial of the fourth order:

$$P_1(x) = \frac{27}{8} - \frac{159}{128}x^2 - \frac{465}{1,024}x^4,$$

which has a bound of absolute error

$$\varepsilon_1 = \frac{1,719}{1,024}$$

for values $x \in [0,1]$. It is true that

$$A(x) - \left(P_1(x) - \varepsilon_1\right) \geq 0, \quad 0 \leq x \leq 1$$

and

$$P_1(x) - \varepsilon_1 = \frac{1,737}{1,024} - \frac{159}{128}x^2 - \frac{465}{1,024}x^4 > 0, \quad 0 < x < 1.$$

Hence, for $x \in (0,1)$ it is true that $A(x) > 0$ and therefore $f_4(x) > 0$ and $f_3'(x) > 0$ for $x \in (0,1)$. Therefore, the function $\frac{f_1'(x)}{f_2'(x)}$ is strictly increasing on $(0,1)$. By Lemma 1.1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0)}{f_2(x) - f_2(0)}$$

is strictly increasing on $(0,1)$. And hence,

$$2 = \lim_{x \to 0} f(x) < f(x) = \frac{\operatorname{arcsl} x}{\frac{\sqrt{1+x}-\sqrt{1-x}}{4-\sqrt{1+x}-\sqrt{1-x}}} < \lim_{x \to 1} f(x) = \frac{2\sqrt{2}-1}{4}B\left(\frac{1}{4},\frac{1}{2}\right)$$

for $0 < x < 1$. By rearranging terms in the last expression, Theorem 2.1 follows. □

Theorem 2.2 *For* $0 < x < 1$,

$$\frac{\alpha_2(\sqrt{1+x}-\sqrt{1-x})}{4-\sqrt{1+x}-\sqrt{1-x}} < \operatorname{arctlh} x < \frac{\beta_2(\sqrt{1+x}-\sqrt{1-x})}{4-\sqrt{1+x}-\sqrt{1-x}} \qquad (2.3)$$

with the best possible constants

$$\alpha_2 = 2 \quad and \quad \beta_2 = \frac{2\sqrt{2}-1}{4}B\left(\frac{1}{4},\frac{1}{4}\right) = 3.39004\ldots. \qquad (2.4)$$

Here $B(x,y)$ *denotes the beta function.*

Proof For $0 < x < 1$, let

$$F(x) = \frac{F_1(x)}{F_2(x)},$$

where

$$F_1(x) = \operatorname{arctlh} x \quad \text{and} \quad F_2(x) = \frac{\sqrt{1+x} - \sqrt{1-x}}{4 - \sqrt{1+x} - \sqrt{1-x}}.$$

Then,

$$\frac{2F_1'(x)}{F_2'(x)} = \frac{(4 - \sqrt{1+x} - \sqrt{1-x})^2 \sqrt{1+x}\sqrt{1-x}}{(1-x^4)^{3/4}(-1 + \sqrt{1+x} + \sqrt{1-x})} =: F_3(x).$$

Differentiation yields

$$F_3'(x) = \frac{4 - \sqrt{1+x} - \sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}(-1 + \sqrt{1-x} + \sqrt{1+x})^2(1-x^4)^{3/4}(x^2+1)} F_4(x),$$

where

$$F_4(x) = \left(11x^3 - x^2 - 4x - 1\right)\sqrt{1+x} + \left(11x^3 + x^2 - 4x + 1\right)\sqrt{1-x}$$
$$- 3x\left(x^2 - 1\right)\sqrt{1-x^2} - 12x^3 + 6x.$$

Motivated by the investigations in [12], we are in a position to prove $F_4(x) > 0$ for $x \in (0,1)$.
Let

$$B(x) = \begin{cases} \lambda, & x = 0, \\ \frac{F_4(x)}{x^3}, & 0 < x < 1, \\ 5\sqrt{2} - 6, & x = 1, \end{cases}$$

where λ is a constant determined by the limit

$$\lambda = \lim_{x \to 0} \frac{F_4(x)}{x^3} = \frac{43}{8}.$$

Using Maple we determine a Taylor approximation for the function $B(x)$ by the polynomial
of fourth order:

$$P_2(x) = \frac{43}{8} - \frac{191}{128}x^2 - \frac{609}{1{,}024}x^4,$$

which has a bound of the absolute error of

$$\varepsilon_2 = \frac{9{,}511}{1{,}024} - 5\sqrt{2}$$

for values $x \in [0,1]$. It is true that

$$B(x) - \left(P_2(x) - \varepsilon_2\right) \geq 0, \quad 0 \leq x \leq 1$$

and

$$P_2(x) - \varepsilon_2 = -\frac{4{,}007}{1{,}024} + 5\sqrt{2} - \frac{191}{128}x^2 - \frac{609}{1{,}024}x^4 > 0, \quad 0 \leq x \leq 1.$$

Hence, for $x \in (0,1)$ it is true that $B(x) > 0$ and therefore $F_4(x) > 0$ and $F_3'(x) > 0$ for $x \in (0,1)$. Therefore, the function $\frac{F_1'(x)}{F_2'(x)}$ is strictly increasing on $(0,1)$. By Lemma 1.1, the function

$$F(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x) - F_1(0)}{F_2(x) - F_2(0)}$$

is strictly increasing on $(0,1)$. And hence,

$$2 = \lim_{x \to 0} F(x) < F(x) = \frac{\operatorname{arctlh} x}{\frac{\sqrt{1+x} - \sqrt{1-x}}{4 - \sqrt{1+x} - \sqrt{1-x}}} < \lim_{x \to 1} F(x) = \frac{2\sqrt{2} - 1}{4} B\left(\frac{1}{4}, \frac{1}{4}\right)$$

for $0 < x < 1$. By rearranging terms in the last expression, Theorem 2.2 follows. □

Theorem 2.3 *For $0 < |x| < 1$,*

$$\frac{a_1}{4 + \sqrt{1 - x^4}} < \frac{\operatorname{arcsl} x}{x} < \frac{b_1}{4 + \sqrt{1 - x^4}} \tag{2.5}$$

and

$$\frac{a_2}{\frac{7}{3} + (1 - x^4)^{1/3}} < \frac{\operatorname{arcsl} x}{x} < \frac{b_2}{\frac{7}{3} + (1 - x^4)^{1/3}} \tag{2.6}$$

with the best possible constants

$$a_1 = 5, \qquad b_1 = B\left(\frac{1}{4}, \frac{1}{2}\right) = 5.2441151\ldots \tag{2.7}$$

and

$$a_2 = \frac{7}{12} B\left(\frac{1}{4}, \frac{1}{2}\right) = 3.0590671\ldots, \qquad b_2 = \frac{10}{3} = 3.3333333\ldots. \tag{2.8}$$

Here $B(x, y)$ denotes the beta function.

Proof For $0 < x < 1$, let

$$L_1(x) = \frac{4 + \sqrt{1 - x^4}}{x} \operatorname{arcsl} x.$$

Differentiation yields

$$L_1'(x) = \frac{x^4 + 4\sqrt{1 - x^4} + 1}{x^2 \sqrt{1 - x^4}} L_2(x),$$

where

$$L_2(x) = -\operatorname{arcsl} x + \frac{(4 + \sqrt{1 - x^4})x}{x^4 + 4\sqrt{1 - x^4} + 1}.$$

Elementary calculation shows that

$$L_2'(x) = \frac{24x^4(1 - \sqrt{1 - x^4})}{(x^4 + 4\sqrt{1 - x^4} + 1)^2 \sqrt{1 - x^4}} > 0, \quad 0 < x < 1.$$

Hence, $L_2(x) > L_2(0) = 0$ and $L_1'(x) > 0$ for $0 < x < 1$. Therefore, the function $L_1(x)$ is strictly increasing on $(0,1)$. And hence,

$$5 = \lim_{x\to 0} L_1(x) < L_1(x) = \frac{4 + \sqrt{1 - x^4}}{x} \operatorname{arcsl} x < \lim_{x\to 1} L_1(x) = B\left(\frac{1}{4}, \frac{1}{2}\right)$$

for $0 < x < 1$. Hence, inequality (2.5) holds with the best possible constants given in equation (2.7).

For $0 < x < 1$, let

$$M_1(x) = \frac{\frac{7}{3} + (1 - x^4)^{1/3}}{x} \operatorname{arcsl} x.$$

Differentiation yields

$$M_1'(x) = \frac{x^4 + 7(1 - x^4)^{2/3} + 3}{3x^2(1 - x^4)^{2/3}} M_2(x),$$

where

$$M_2(x) = -\operatorname{arcsl} x + \frac{x(1 - x^4)^{1/6}(7 + 3(1 - x^4)^{1/3})}{x^4 + 7(1 - x^4)^{2/3} + 3}.$$

Elementary calculation shows that

$$M_2'(x) = -\frac{2x^4}{3(1 - x^4)^{3/2}(x^4 + 7(1 - x^4)^{2/3} + 3)^2} M_3(x),$$

where

$$M_3(x) = 3x^8 - 66x^4 + 63 + \left(84 - 56x^4\right)\left(1 - x^4\right)^{2/3} - 147\left(1 - x^4\right)^{4/3}.$$

We claim that $M_3(x) > 0$ for $0 < x < 1$. By an elementary change of variable

$$x = \left(1 - t^3\right)^{1/4}, \quad 0 < t < 1, \tag{2.9}$$

we find that

$$M_3(x) > 0 \quad \text{for } 0 < x < 1 \quad \Longleftrightarrow \quad M_4(t) > 0 \quad \text{for } 0 < t < 1,$$

where

$$M_4(t) = 28t^2 + 60t^3 + 56t^5 + 3t^6 - 147t^4 = t^2(1 - t)\left(-3t^3 - 59t^2 + 88t + 28\right).$$

Obviously, $M_4(t) > 0$ for $0 < t < 1$. This proves the claim.

Hence, $M_2'(x) < 0$ for $0 < x < 1$. This implies that $M_2(x) < M_2(0) = 0$ and $M_1'(x) < 0$ for $0 < x < 1$. Therefore, the function $M_1(x)$ is strictly decreasing on $(0,1)$. And hence,

$$\frac{7}{12} B\left(\frac{1}{4}, \frac{1}{2}\right) = \lim_{x\to 1} M_1(x) < M_1(x) = \frac{\frac{7}{3} + (1 - x^4)^{1/3}}{x} \operatorname{arcsl} x < \lim_{x\to 0} M_1(x) = \frac{10}{3}$$

for $0 < x < 1$. Hence, inequality (2.6) holds with the best possible constants given in equation (2.8). $\qquad\square$

Remark 2.1 (i) There is no strict comparison between the two lower bounds in equations (2.5) and (2.6). Also, there is no strict comparison between the two upper bounds in equations (2.5) and (2.6).

(ii) The lower bound in equation (1.10) is sharper than the one in equation (2.5), since

$$\frac{5}{3 + 2(1 - x^4)^{1/2}} - \left(\frac{5}{4 + \sqrt{1 - x^4}}\right)^2 = \frac{5(1 - \sqrt{1 - x^4})^2}{(4 + \sqrt{1 - x^4})^2(3 + 2\sqrt{1 - x^4})} > 0, \quad 0 < |x| < 1.$$

There is no strict comparison between the two upper bounds in equations (1.10) and (2.5).

(iii) By two elementary changes of variable,

$$t = 1 - x^4, \quad 0 < x < 1 \quad \text{and} \quad t = u^{30}, \quad 0 < u < 1,$$

we find that

$$\frac{\frac{10}{3}}{\frac{7}{3} + (1 - x^4)^{1/3}} - \frac{1}{(1 - x^4)^{1/10}}$$

$$= \frac{10t^{1/10} - 7 - 3t^{1/3}}{(7 + 3t^{1/3})t^{1/10}} = \frac{10u^3 - 7 - 3u^{10}}{(7 + 3u^{10})u^3}$$

$$= -\frac{(3u^8 + 6u^7 + 9u^6 + 12u^5 + 15u^4 + 18u^3 + 21u^2 + 14u + 7)(1 - u)^2}{(7 + 3u^{10})u^3} < 0.$$

Hence, the upper bound in equation (2.6) is sharper than the one in equation (1.10). There is no strict comparison between the two lower bounds in equations (1.10) and (2.6).

Theorem 2.4 *For $x \neq 0$,*

$$\frac{a}{\frac{17}{3} + (1 + x^4)^{1/4}} < \frac{\text{arcslh}\, x}{x} < \frac{b}{\frac{17}{3} + (1 + x^4)^{1/4}} \tag{2.10}$$

with the best possible constants

$$a = \frac{1}{4}B\left(\frac{1}{4}, \frac{1}{4}\right) = 1.8540746\ldots \quad \text{and} \quad b = \frac{20}{3} = 6.6666666\ldots. \tag{2.11}$$

Here $B(x, y)$ denotes the beta function.

Proof The inequality (2.10) is obtained by considering the function $p(x)$ defined by

$$p(x) = \frac{(\frac{17}{3} + (1 + x^4)^{1/4})}{x} \text{arcslh}\, x, \quad x > 0.$$

Differentiation yields

$$p'(x) = -\frac{17(1 + x^4)^{3/4} + 3}{3x^2(1 + x^4)^{3/4}} \text{arcslh}\, x + \frac{(\frac{17}{3} + (1 + x^4)^{1/4})}{x} \frac{1}{\sqrt{1 + x^4}} = \frac{17(1 + x^4)^{3/4} + 3}{3x^2(1 + x^4)^{3/4}} q(x),$$

where

$$q(x) = -\operatorname{arcslh} x + \frac{x(1 + x^4)^{1/4}(17 + 3(1 + x^4)^{1/4})}{17(1 + x^4)^{3/4} + 3}.$$

Elementary calculation shows that

$$q'(x) = -\frac{x^4}{(1 + x^4)^{3/2}(17(1 + x^4)^{3/4} + 3)^2} r(x),$$

where

$$r(x) = -27 - 27x^4 - 51(1 + x^4)^{3/4} + 578(1 + x^4)^{3/2}.$$

We claim that $r(x) > 0$ for $x > 0$. By an elementary change of variable

$$t = (1 + x^4)^{1/4}, \quad x > 0, \tag{2.12}$$

we find that

$$r(x) > 0 \quad \text{for } x > 0 \quad \Longleftrightarrow \quad s(t) > 0 \quad \text{for } t > 1,$$

where

$$s(t) = t^3(-51 - 27t + 578t^3), \quad t > 1.$$

Obviously, $s(t) > 0$ for $t > 1$. This proves the claim.

Hence, $q'(x) < 0$ and $q(x) < q(0) = 0$ for $x > 0$. Therefore, $p'(x) < 0$ for $x > 0$, and we have

$$\frac{1}{4}B\left(\frac{1}{4}, \frac{1}{4}\right) = \lim_{x \to \infty} p(x) < p(x) = \frac{(\frac{17}{3} + (1 + x^4)^{1/4})\operatorname{arcslh} x}{x} < \lim_{x \to 0} p(x) = \frac{20}{3}.$$

Hence, the inequality (2.10) holds with the best possible constants given in equation (2.11). □

Remark 2.2 Inequality (1.11) is sharper than inequality (2.10).

Theorem 2.5 *For $x \neq 0$,*

$$\frac{\alpha}{\frac{2}{3} + (1 + x^4)^{1/4}} < \frac{\operatorname{arctl} x}{x} < \frac{\beta}{\frac{2}{3} + (1 + x^4)^{1/4}} \tag{2.13}$$

with the best possible constants

$$\alpha = \frac{1}{4}B\left(\frac{1}{4}, \frac{1}{2}\right) = 1.3110287\ldots \quad and \quad \beta = \frac{5}{3} = 1.6666666\ldots. \tag{2.14}$$

Here $B(x, y)$ denotes the beta function.

Proof The inequality (2.13) is obtained by considering the function $P(x)$ defined by

$$P(x) = \frac{(\frac{2}{3} + (1 + x^4)^{1/4})}{x} \operatorname{arctl} x, \quad x > 0.$$

Differentiation yields

$$P'(x) = -\frac{2(1 + x^4)^{3/4} + 3}{3x^2(1 + x^4)^{3/4}} \operatorname{arctl} x + \frac{(\frac{2}{3} + (1 + x^4)^{1/4})}{x} \frac{1}{(1 + x^4)^{3/4}} = \frac{2(1 + x^4)^{3/4} + 3}{3x^2(1 + x^4)^{3/4}} Q(x),$$

where

$$Q(x) = -\operatorname{arctl} x + \frac{(2 + 3(1 + x^4)^{1/4}x}{2(1 + x^4)^{3/4} + 3}.$$

Elementary calculation shows that

$$Q'(x) = -\frac{6x^4(2\sqrt{1 + x^4} + (1 + x^4)^{3/4} - 3)}{(1 + x^4)^{3/4}(2(1 + x^4)^{3/4} + 3)^2} < 0, \quad x > 0.$$

Hence, $Q(x) < Q(0) = 0$ for $x > 0$. Therefore, $P'(x) < 0$ for $x > 0$, and we have

$$\frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \lim_{x \to \infty} P(x) < P(x) = \frac{(\frac{2}{3} + (1 + x^4)^{1/4})}{x} \operatorname{arctl} x < \lim_{x \to 0} P(x) = \frac{5}{3}.$$

Hence, the inequality (2.13) holds with the best possible constants given in equation (2.14). □

Theorem 2.6 *For* $0 < |x| < 1$,

$$\frac{a_3}{\frac{7}{3} + \sqrt{1 - x^4}} < \frac{\operatorname{arctlh} x}{x} < \frac{b_3}{\frac{7}{3} + \sqrt{1 - x^4}} \tag{2.15}$$

and

$$\frac{a_4}{\frac{2}{3} + (1 - x^4)^{1/4}} < \frac{\operatorname{arctlh} x}{x} < \frac{b_4}{\frac{2}{3} + (1 - x^4)^{1/4}} \tag{2.16}$$

with the best possible constants

$$a_3 = \frac{10}{3} = 3.3333333\ldots, \qquad b_3 = \frac{7}{12} B\left(\frac{1}{4}, \frac{1}{4}\right) = 4.3261742\ldots \tag{2.17}$$

and

$$a_4 = \frac{1}{6} B\left(\frac{1}{4}, \frac{1}{4}\right) = 1.236049\ldots, \qquad b_4 = \frac{5}{3} = 1.666666\ldots. \tag{2.18}$$

Proof For $0 < x < 1$, let

$$J_1(x) = \frac{\frac{7}{3} + \sqrt{1 - x^4}}{x} \operatorname{arctlh} x.$$

Differentiation yields

$$J_1'(x) = \frac{3x^4 + 7\sqrt{1 - x^4} + 3}{3x^2\sqrt{1 - x^4}} J_2(x),$$

where

$$J_2(x) = -\operatorname{arctlh} x + \frac{(7 + 3\sqrt{1 - x^4})x}{(1 - x^4)^{1/4}(3x^4 + 7\sqrt{1 - x^4} + 3)}.$$

Elementary calculation shows that

$$J_2'(x) = \frac{3x^4}{(1 - x^4)^{7/4}(3x^4 + 7\sqrt{1 - x^4} + 3)^2} J_3(x),$$

where

$$J_3(x) = -25x^4 - 3x^8 + 28 + (42x^4 - 28)\sqrt{1 - x^4}.$$

We claim that $J_3(x) > 0$ for $0 < x < 1$. By an elementary change of variable

$$x = (1 - t^2)^{1/4}, \quad 0 < t < 1, \tag{2.19}$$

we find that

$$J_3(x) > 0 \quad \text{for } 0 < x < 1 \quad \Longleftrightarrow \quad J_4(t) > 0 \quad \text{for } 0 < t < 1,$$

where

$$J_4(t) = 14t + 31t^2 - 42t^3 - 3t^4 = t(1 - t)(3t^2 + 45t + 14).$$

Obviously, $J_4(t) > 0$ for $0 < t < 1$. This proves the claim.

Hence, $J_2'(x) > 0$ for $0 < x < 1$. This implies that $J_2(x) > J_2(0) = 0$ and $J_1'(x) > 0$ for $0 < x < 1$. Therefore, the function $J_1(x)$ is strictly increasing on $(0, 1)$. And hence,

$$\frac{10}{3} = \lim_{x \to 0} J_1(x) < J_1(x) = \frac{\frac{7}{3} + \sqrt{1 - x^4}}{x} \operatorname{arctlh} x < \lim_{x \to 1} J_1(x) = \frac{7}{12} B\left(\frac{1}{4}, \frac{1}{4}\right)$$

for $0 < x < 1$. Hence, inequality (2.15) holds with the best possible constants given in equation (2.17).

For $0 < x < 1$, let

$$T_1(x) = \frac{\frac{2}{3} + (1 - x^4)^{1/4}}{x} \operatorname{arctlh} x.$$

Differentiation yields

$$T_1'(x) = \frac{2(1 - x^4)^{3/4} + 3}{3x^2(1 - x^4)^{3/4}} T_2(x),$$

where

$$T_2(x) = -\operatorname{arctlh} x + \frac{x(2 + 3(1 - x^4)^{1/4})}{2(1 - x^4)^{3/4} + 3}.$$

Elementary calculation shows that

$$T_2'(x) = -\frac{6x^4(3 - 2\sqrt{1 - x^4} - (1 - x^4)^{3/4})}{(1 - x^4)^{3/4}(2(1 - x^4)^{3/4} + 3)^2} < 0, \quad 0 < x < 1,$$

which implies that $T_2(x) < T_2(0) = 0$ and $T_1'(x) < 0$ for $0 < x < 1$. Therefore, the function $T_1(x)$ is strictly decreasing on $(0,1)$. And hence,

$$\frac{1}{6}B\left(\frac{1}{4}, \frac{1}{4}\right) = \lim_{x \to 1} T_1(x) < T_1(x) = \frac{\frac{2}{3} + (1 - x^4)^{1/4}}{x} \operatorname{arctlh} x < \lim_{x \to 0} T_1(x) = \frac{5}{3}$$

for $0 < x < 1$. Hence, inequality (2.16) holds with the best possible constants given in equation (2.18). □

Remark 2.3 There is no strict comparison between the two lower bounds in equations (2.15) and (2.16). Also, there is no strict comparison between the two upper bounds in equations (2.15) and (2.16).

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