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Uniqueness of difference operators of meromorphic functions

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Abstract

In this article, we investigate the uniqueness problems of difference operators of two meromorphic functions. Uniqueness of a meromorphic function f^n and its difference operator with the same 1-points and poles is also proved.

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1. Introduction and main results

In this article, a meromorphic function always means meromorphic in the whole complex plane, and c always means a non-zero constant. For a meromorphic function $f(z)$, we define its shift by $f(z + c)$, and define its difference operators by

$$\Delta_c f(z) = f(z + c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N}, \quad n \geq 2.$$

We adopt the standard notations of the Nevanlinna theory of meromorphic functions such as $T(r, f)$, $m(r, f)$, $N(r, f)$ and $\overline{N}(r, f)$ as explained in [1-3]. In addition, we use $N_2\left(r, \frac{1}{f}\right)$ to denote the counting function of the zeros of $f(z)$ such that simple zeros are counted once and multiple zeros twice.

Let $f(z)$ and $a(z)$ be two meromorphic functions. We say that $a(z)$ is a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f) = o(T(r, f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. For a small function $a(z)$ related to $f(z)$, we define

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f - a}\right)}{T(r, f)}.$$

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ IM, provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros (ignoring multiplicities), and we say that $f(z)$ and $g(z)$ share $a(z)$ CM, provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities.

Recently, a number of articles including [4-10] have focused on value distribution in difference analogues of meromorphic functions. In particular, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shifts or their difference operators (see, e.g., [8-13]). Some of them (see, e.g., [9,10]) dealt with the uniqueness problems on the case that shifts or difference polynomials of two entire functions share a small function. Our aim in this article is to investigate the uniqueness problems of difference operators of meromorphic functions.

To begin the statement of our results, we recall the following Theorem A (see [3]), which is an improvement of an original result of Shibazaki [14].

Theorem A ([[3], Theorem 9.16]). *Let f and g be non-constant entire functions. If f' and g' share 1 CM, and* $\delta(0, f) + \delta(0, g) > 1$, *then* $f \equiv g$ *or* $f'g' \equiv 1$.

Qi and Liu [9] considered the case that shifts of two entire functions share a small function. They proved

Theorem B ([[9], Theorem 5]). *Suppose that f and g are two entire functions of finite order, and let a and b be distinct small functions related to f and g such that* $\delta(a) = \delta(a, f) + \delta(a, g) > 1$. *If* $f(z + c_1)$ *and* $g(z + c_2)$ *share b CM, then exactly one of the following assertions holds*

(i) $f(z) \equiv g(z + c)$, *where* $c = c_2 - c_1$;
(ii) $f(z + c_1) \equiv (a - b)e^h + a$, $g(z + c_2) \equiv (a - b)e^{-h} + a$, *where h(z) is an entire function.*

As a difference analogue of Theorem A, we prove the following Theorem 1.1, whose proof is omitted as it is similar as the proof of Theorem 1.2.

Theorem 1.1. *Let f(z) and g(z) be entire functions of finite order, and let* $a(z)(\not\equiv 0)$ *be a small entire function with respect to f(z) and g(z). Suppose that* $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ *such that* $\Delta_{c_1}f(z) \cdot \Delta_{c_2}g(z) \not\equiv 0$. *If* $\Delta_{c_1}f(z)$ *and* $\Delta_{c_2}g(z)$ *share a CM, and* $\delta(0) = \delta(0, f) + \delta(0, g) > 1$, *then one of the following assertions holds:*

(i) $\Delta_{c_1}f(z) \equiv \Delta_{c_2}g(z)$;
(ii) $\Delta_{c_1}f(z) \equiv -a(z)e^{h(z)}$, $\Delta_{c_2}g(z) \equiv -a(z)e^{-h(z)}$, *where h(z) is a polynomial.*

We consider the case that $c_1 = c_2$ and obtain the following Theorem 1.2, which is an extension of Theorem 1.1 for this case.

Theorem 1.2. *Let f(z) and g(z) be entire functions of finite order, and let a(z) and b(z) be small entire functions with respect to f(z) and g(z). Suppose that* $c \in \mathbb{C}\backslash\{0\}$ *such that* $\Delta_c(f - b) \cdot \Delta_c(g - b) \cdot (a - \Delta_c b) \not\equiv 0$. *If* $\Delta_c f(z)$ *and* $\Delta_c g(z)$ *share a CM, and* $\delta(b) = \delta(b, f) + \delta(b, g) > 1$, *then one of the following assertions holds:*

(i) $\Delta_c f(z) \equiv \Delta_c g(z)$;
(ii) $\Delta_c f \equiv (\Delta_c b - a)e^{h(z)} + \Delta_c b$, $\Delta_c g \equiv (\Delta_c b - a)e^{-h(z)} + \Delta_c b$, *where h(z) is a polynomial.*

Example 1. We list three examples to show that there exist entire functions satisfying the cases in Theorems 1.1 and 1.2.

(1) Let $f(z) = (z + 1)e^z$, $g(z) = ze^z$, and $c_1 = c_2 = 2\pi i$. Then $\delta(0) = \delta(0,f) + \delta(0,g) > 1$, and for any $a \in \mathbb{C}$, $\Delta_{c_1}f(z)$ and $\Delta_{c_2}g(z)$ share a CM, and we have $\Delta_{c_1}f(z) \equiv \Delta_{c_2}g(z)$. This example satisfies Theorems 1.1(i) and 1.2(i).

(2) Let $f(z) = e^z$, $g(z) = -e(e+1)e^{-z}$, $c_1 = 2$, and $c_2 = 1$. Then $\delta(0) = \delta(0, f) + \delta(0, g) > 1$, and for $a = 1 - e^2$, $\Delta_{c_1}f(z)$ and $\Delta_{c_2}g(z)$ share a CM, and we have $\Delta_{c_1}f(z) \equiv -ae^z$ and $\Delta_{c_2}g(z) \equiv -ae^{-z}$. This example satisfies Theorem 1.1(ii).

(3) Let $f(z) = -e^z + 2$, $g(z) = e \cdot e^{-z} + 2$, and $c = 1$. Then $\delta(2) = \delta(2, f) + \delta(2, g) > 1$, and for $a = e - 1$, $\Delta_c f(z)$ and $\Delta_c g(z)$ share a CM, and we have $\Delta_c f(z) \equiv -ae^z$ and $\Delta_c g(z) = -ae^{-z}$. This example satisfies Theorem 1.2(ii).

Now an interesting question is whether the deficiency condition in Theorem 1.1 can be replaced by other condition or not. Considering this question, we prove the following result, which is an improvement of Theorem 1.4 in [12].

Theorem 1.3. *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Let $f(z)$ and $g(z)$ be entire functions of finite order $\rho(f)$ and $\rho(g)$, respectively. Suppose that a and b are distinct complex constants. If $\Delta_{c_2}g(z)$and $\Delta_{c_2}g(z)$share a CM, and $\Delta_{c_1}f(z) - b$ and $\Delta_{c_2}g(z) - b$ have at least* $\max\{[\rho(f)], [\rho(g)], 1\}$ *distinct common zeros of multiplicity ≥ 2, then $\Delta_{c_1}f(z) \equiv \Delta_{c_2}g(z)$.*

Example 2. We give examples (1) and (2) to show that there exist entire functions satisfying Theorem 1.3. Moreover, Example (3) shows that the condition related to common zeros cannot be omitted in Theorem 1.3.

(1) Let $f(z) = e^{z \log 2} \sin^2(2\pi z)$, $g(z) = e^{z \log 2} \sin^2(2\pi z) + e^{2\pi i z}$, and $c = 1$. Then $\Delta_c f(z) \equiv \Delta_c g(z) \equiv e^{z \log 2} \sin^2(2\pi z)$. In this example, $\Delta_c f(z)$ and $\Delta_c g(z)$ have infinitely many distinct common zeros of multiplicity 2.

(2) Let $f(z) = e^{z \log 2} \sin^2(2\pi z)$, $g(z) = \frac{1}{3}e^{z\log 2}\sin^2(2\pi z)$, and $c_1 = 1$, $c_2 = 2$. Then $\Delta_{c_1}f(z) \equiv \Delta_{c_2}g(z) \equiv e^{z\log 2}\sin^2(2\pi z)$. In this example, $\Delta_{c_1}f(z)$ and $\Delta_{c_2}g(z)$ have infinitely many distinct common zeros of multiplicity 2.

(3) Let $f(z) = ze^z - z$, $g(z) = ze^{-z} - z$, and $c = 2\pi i$. Then $\Delta_c f(z) = 2\pi i(e^z - 1)$ and $\Delta_c g(z) = 2\pi i(1 - e^z)e^{-z}$ share $-2\pi i$ CM. However, $\max\{[\rho(f)],[\rho(g)],1\} = 1$, while $\Delta_c f(z)$ and $\Delta_c g(z)$ have only simple common zeros and $\Delta_c f(z) \not\equiv \Delta_c g(z)$.

In 2008, Yang and Zhang [15] considered the uniqueness problems on the meromorphic function f^n sharing values with its first derivative. One of their results can be stated as follows.

Theorem C ([[15], Theorem 4.3]). *Let $f(z)$ be a non-constant meromorphic function and $n \geq 12$ be an integer. Let $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a non-zero constant.

To replace F' by $\Delta_c F$ in Theorem C, we prove the following Theorem 1.4.

Theorem 1.4. *Let $f(z)$ be a non-constant meromorphic function of finite order and $n \geq 9$ be an integer. Let $F(z) = f(z)^n$. If $F(z)$ and $\Delta_c F$ share 1, ∞ CM, then $F(z) = \Delta_c F$.*

For the entire functions case, using the same method as in the proof of Theorem 1.4, we get the following result.

Theorem 1.5. *Let $f(z)$ be a non-constant entire function of finite order and $n \geq 6$ be an integer. Let $F(z) = f(z)^n$. If $F(z)$ and $\Delta_c F$ share 1 CM, then $F(z) = \Delta_c F$.*

Example 3. The following example (1) satisfies Theorem 1.4, while example (2) satisfies both Theorems 1.4 and 1.5.

(1) Let $f(z) = e^{\frac{z}{9}\log 2}/\sin(2\pi z)$, $n = 9$ and $c = 1$. Then we see that $F(z) = f(z)^n = e^{z \log 2}/\sin^9(2\pi z)$ and $\Delta_c F(z) = F(z + c) - F(z) = e^{z \log 2}/\sin^9(2\pi z)$ share 1, ∞ CM, and $F(z) \equiv \Delta_c F$.

(2) Let $f(z) = 2^{\frac{1}{9}} e^z$, $n = 9$ and $c = \frac{1}{9}\log 2$. Then we see that $F(z) = f(z)^n = 2e^{9z}$ and $\Delta_c F(z) = F(z + c) - F(z) = 2e^{9z}$ share 1 CM, and $F(z) \equiv \Delta_c F$.

But we still wonder whether the lower bound of n in our results is sharp or not.

2. Proof of Theorem 1.2

Lemma 2.1 ([[7], Lemma 2.3]). *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f),$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

The proof of Theorem 1.2 is based on a result in [15], which can be read as follows:

Lemma 2.2 ([[15], Theorem 3.1]). *Let $f_j(z)(j = 1,2,3)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z) \equiv 1.$$

If $f_1(z)$ is not a constant, and

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + \sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \quad r \in I,$$

where $0 \le \lambda < 1$, $T(r) = \max_{1 \le j \le 3}\{T(r, f_j)\}$, and I has infinite linear measure, then either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Proof of Theorem 1.2. By the condition that $\delta(b) = \delta(b, f) + \delta(b, g) > 1$, we see easily that $\delta(b, f) > 0$ and $\delta(b, g) > 0$. Then for any given ε such that $0 < \varepsilon < \min\left\{\frac{\delta(b,f)}{2}, \frac{\delta(b,g)}{2}, \frac{\delta(b) - 1}{2}\right\}$, we have

$$(\delta(b,f) - \varepsilon)T(r,f) \le m\left(r, \frac{1}{f - b}\right), \tag{2.1}$$

and

$$(\delta(b,g) - \varepsilon)T(r,g) \le m\left(r, \frac{1}{g - b}\right). \tag{2.2}$$

From Lemma 2.1, we see that

$$m(r, \Delta_c f) \le m\left(r, \frac{\Delta_c f}{f}\right) + m(r, f) = m(r, f) + S(r, f). \tag{2.3}$$

Thus,

$$T(r, \Delta_c f) = m(r, \Delta_c f) \le m(r, f) + S(r, f) = T(r, f) + S(r, f). \tag{2.4}$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &= m\left(r, \frac{\Delta_c(f-b)}{f-b} \cdot \frac{1}{\Delta_c f - \Delta_c b}\right) \\ &\le m\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) + S(r, f). \end{aligned} \tag{2.5}$$

Combining (2.1) and (2.5) gives

$$\begin{aligned} (\delta(b, f) - \varepsilon)T(r, f) &\le m\left(r, \frac{1}{f-b}\right) \le m\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) + S(r, f) \\ &\le T(r, \Delta_c f - \Delta_c b) + S(r, f) \\ &\le T(r, \Delta_c f) + S(r, f). \end{aligned} \tag{2.6}$$

Hence, by (2.4) and (2.6), we have $S(r, \Delta_c f) = S(r, f)$. Similarly, $S(r, \Delta_c g) = S(r, g)$. From (2.1), (2.4) and (2.5), we have

$$\begin{aligned} (\delta(b, f) - \varepsilon)T(r, \Delta_c f) &\le (\delta(b, f) - \varepsilon)T(r, f) + S(r, f) \\ &\le m\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &\le m\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) + S(r, f) \\ &\le T(r, \Delta_c f) - N\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) + S(r, f). \end{aligned}$$

That is

$$N\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) \le (1 - \delta(b, f) + \varepsilon)T(r, \Delta_c f) + S(r, f). \tag{2.7}$$

By the same reasoning, we have

$$N\left(r, \frac{1}{\Delta_c g - \Delta_c b}\right) \le (1 - \delta(b, g) + \varepsilon)T(r, \Delta_c g) + S(r, g). \tag{2.8}$$

Set $I_1 = \{r: T(r, \Delta_c f) \ge T(r, \Delta_c g)\} \subseteq (0, \infty)$, and $I_2 = (0, \infty) \setminus I_1$. Then there exists at least one of I_i $(i = 1, 2)$, which has infinite logarithmic measure. Without loss of generality, we may suppose I_1 has infinite logarithmic measure.

Since $\Delta_c f(z)$ and $\Delta_c g(z)$ share a CM, we have

$$\frac{\Delta_c f(z) - a}{\Delta_c g(z) - a} = e^{h(z)}, \tag{2.9}$$

where $h(z)$ is a polynomial.

We rewrite (2.9) as

$$F_1(z) + F_2(z) + F_3(z) \equiv 1,$$

where

$$F_1(z) = \frac{\Delta_c f - \Delta_c b}{a - \Delta_c b}, F_2(z) = -\frac{\Delta_c g - \Delta_c b}{a - \Delta_c b} e^{h(z)}, \quad F_3(z) = e^{h(z)}.$$

Set $T(r) = \max_{1\le j\le 3}\{T(r, F_j)\}$ and $S(r) = o(T(r))$. Then

$$T(r) \geq T(r, F_1) = T(r, \Delta_c f) + S(r, f).$$

We deduce from the definition of $F_j(z)$ $(j = 1, 2, 3)$ that

$$\begin{aligned} &\overline{N}(r, F_1) = \overline{N}(r, F_2) = \overline{N}\left(r, \frac{1}{a - \Delta_c b}\right) = S(r, f), \\ &\overline{N}(r, F_3) = 0, \quad N_2\left(r, \frac{1}{F_3}\right) = 0. \end{aligned} \tag{2.10}$$

From (2.7) and (2.8), we have

$$\begin{aligned} &N_2\left(r, \frac{1}{F_1}\right) + N_2\left(r, \frac{1}{F_2}\right) \\ &\leq N\left(r, \frac{1}{\Delta_c f - \Delta_c b}\right) + N\left(r, \frac{1}{\Delta_c g - \Delta_c b}\right) \\ &\leq (2 - \delta(b) + 2\varepsilon)T(r, \Delta_c f) + S(r, \Delta_c f), \quad r \in I_1, \end{aligned} \tag{2.11}$$

where I_1 has infinite logarithmic measure.

Then we get from (2.10) and (2.11) that

$$\begin{aligned} \sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} \overline{N}(r, F_j) &\leq (2 - \delta(b) + 2\varepsilon)T(r, \Delta_c f) + S(r, f) \\ &\leq (2 - \delta(b) + 2\varepsilon)T(r) + S(r), \quad r \in I_1, \end{aligned}$$

where I_1 has infinite logarithmic measure, and also infinite linear measure.

It is obvious that F_1 is not a constant. By Lemma 2.2, we have $F_2(z) \equiv 1$ or $F_3(z) \equiv 1$. If $F_2(z) = 1$, we have

$$\Delta_c g \equiv (\Delta_c b - a)e^{-h(z)} + \Delta_c b, \quad \Delta_c f \equiv (\Delta_c b - a)e^{h(z)} + \Delta_c b.$$

If $F_3(z) \equiv 1$, we have $e^{h(z)} \equiv 1$. By (2.9), the conclusion holds.

3. Proof of Theorem 1.3

Lemma 3.1 ([[6], Theorem 2.1]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a non-zero complex constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z + c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

Proof of Theorem 1.3. Since $\Delta_{c_1} f(z)$ and $\Delta_{c_2} g(z)$ share a CM, we have

$$\frac{\Delta_{c_1} f(z) - a}{\Delta_{c_2} g(z) - a} = e^{h(z)}, \tag{3.1}$$

where $h(z)$ is a polynomial.

Then by (3.1) and Lemma 3.1, we deduce that

$$\begin{aligned} T(r, e^{h(z)}) &= T\left(r, \frac{\Delta_{c_1} f(z) - a}{\Delta_{c_2} g(z) - a}\right) \\ &\le T(r, \Delta_{c_1} f(z)) + T(r, \Delta_{c_2} g(z)) + O(1) \\ &\le T(r, f(z + c_1)) + T(r, f(z)) + T(r, g(z + c_2)) + T(r, g(z)) + O(1) \\ &\le 2T(r, f) + 2T(r, g) + S(r, f) + S(r, g). \end{aligned} \tag{3.2}$$

It follows from (3.2) that $\rho(e^{h(z)}) \le \max\{\rho(f), \rho(g)\}$. Clearly, $\rho(e^{h(z)}) = \deg h(z)$ is an integer. Thus, we deduce that $\deg h(z) \le \max\{[\rho(f)], [\rho(g)]\}$.

Suppose that $h(z)$ is a constant. Notice that $\Delta_{c_1} f(z) - b$ and $\Delta_{c_2} g(z) - b$ have at least a common zero. Suppose that a point $z_0 \in \mathbb{C}$ satisfies $\Delta_{c_1} f(z_0) = \Delta_{c_2} g(z_0) = b$. It follows from (3.1) that $e^{h(z)} \equiv e^{h(z_0)} = 1$. Thus, $\Delta_{c_1} f(z) \equiv \Delta_{c_2} g(z)$.

Suppose that $h(z)$ is not a constant. By taking the derivative in (3.1), we get

$$\Delta_{c_1} f'(z) - \Delta_{c_2} g'(z) e^{h(z)} - h'(z)(\Delta_{c_2} g(z) - a) e^{h(z)} \equiv 0. \tag{3.3}$$

Set $k = \max\{[\rho(f)], [\rho(g)], 1\}$. Then $k \ge \max\{[\rho(f)], [\rho(g)]\} \ge \deg h(z)$. Notice that $\Delta_{c_1} f(z) - b$ and $\Delta_{c_2} g(z) - b$ have at least k distinct common zeros of multiplicity ≥ 2. Suppose that z_j ($j = 1,2,\dots, k$) satisfies

$$\begin{aligned} &\Delta_{c_1} f(z_j) = \Delta_{c_2} g(z_j) = b, \\ &\Delta_{c_1} f'(z_j) = \Delta_{c_2} g'(z_j) = 0. \end{aligned} \tag{3.4}$$

Then we get from (3.3) and (3.4) that $h'(z_j) = 0$, $j = 1,2,\dots, k$. It implies that $h(z)$ is a polynomial with $\deg h(z) \ge k + 1$, which is a contradiction since $k \ge \deg h(z)$.

4. Proof of Theorem 1.4

Since $f(z)$ is a non-constant meromorphic function of finite order, then $F(z) = f(z)^n$ is a non-constant meromorphic function of finite order. By Lemma 3.1, we have

$$T(r, F(z + c)) = nT(r, f(z + c)) = nT(r, f) + S(r, f). \tag{4.1}$$

From (4.1), we see that $F(z + c)$ and $\Delta_c F$ are meromorphic functions of finite order, and obviously $S(r, F(z)) = S(r, F(z + c)) = S(r, f)$.

Set $\omega = e^{\frac{2\pi i}{n}}$. Then by the second main theorem, we have

$$\begin{aligned} m\left(r, \frac{1}{F(z) - 1}\right) &= m\left(r, \frac{1}{(f(z) - \omega^0)(f(z) - \omega^1) \cdots (f(z) - \omega^{n-1})}\right) \\ &\le \sum_{j=0}^{n-1} m\left(r, \frac{1}{f(z) - \omega^j}\right) \\ &\le 2T(r, f) - m(r, f) - \left(2N(r, f) - N(r, f') + N\left(r, \frac{1}{f'}\right)\right) + S(r, f) \\ &\le T(r, f) + \left(N(r, f') - N(r, f) - N\left(r, \frac{1}{f'}\right)\right) + S(r, f) \\ &\le T(r, f) + \overline{N}(r, f) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\le 2T(r, f) + S(r, f). \end{aligned} \tag{4.2}$$

Since $F(z)$ and $\Delta_c F$ share 1, ∞ CM, we have

$$\frac{F(z)-1}{\Delta_c F-1}=e^{h(z)}, \tag{4.3}$$

where $h(z)$ is a polynomial.

By (4.2) and (4.3), we get

$$\begin{aligned} T(r,e^{h(z)}) &= T(r,e^{-h(z)})+O(1)=m(r,e^{-h(z)})+O(1)\\ &= m\left(r,\frac{\Delta_c F-1}{F(z)-1}\right)+O(1)\\ &\le m\left(r,\frac{\Delta_c F}{F(z)-1}\right)+m\left(r,\frac{1}{F(z)-1}\right)+O(1)\\ &\le S(r,F(z))+2T(r,f)+S(r,f)=2T(r,f)+S(r,f). \end{aligned} \tag{4.4}$$

Thus, $S(r,e^{h(z)})=o(T(r,f))$.

Now we rewrite (4.3) as

$$F(z+c)-(e^{h(z)}+1)e^{-h(z)}F(z)+e^{-h(z)}\equiv 1. \tag{4.5}$$

Set $F_1(z)=F(z+c)$, $F_2(z)=-(e^{h(z)}+1)e^{-h(z)}F(z)$ and $F_3(z)=e^{-h(z)}$. Then

$$F_1(z)+F_2(z)+F_3(z)\equiv 1$$

and

$$T(r)=\max_{1\le j\le 3}\{T(r,F_j)\}\ge T(r,F(z+c))=nT(r,f)+S(r,f),$$
$$S(r)=o(T(r)).$$

We easily get

$$\begin{aligned} &\overline{N}(r,F_2)=\overline{N}(r,f)\le T(r,f)+S(r,f),\\ &\overline{N}(r,F_3)=0,\quad N_2\left(r,\frac{1}{F_3}\right)=0. \end{aligned} \tag{4.6}$$

From Lemma 3.1, we get

$$\begin{aligned} \overline{N}(r,F_1)=\overline{N}(r,f(z+c)) &\le T(r,f(z+c))+S(r,f(z+c))\\ &= T(r,f)+S(r,f), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} N_2\left(r,\frac{1}{F_1}\right) &= 2\overline{N}\left(r,\frac{1}{F_1}\right)=2\overline{N}\left(r,\frac{1}{f(z+c)}\right)\le 2T\left(r,\frac{1}{f(z+c)}\right)\\ &= 2T(r,f)+S(r,f). \end{aligned} \tag{4.8}$$

By (4.4), we see that

$$\begin{aligned} N_2\left(r,\frac{1}{F_2}\right) &\le 2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{e^{h(z)}+1}\right)\\ &\le 2\overline{N}\left(r,\frac{1}{f}\right)+T(r,e^{h(z)})+S(r,e^{h(z)})+O(1)\\ &\le 4T(r,f)+S(r,f). \end{aligned} \tag{4.9}$$

By (4.6)-(4.9), we obtain

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} \overline{N}(r, F_j) \le 8T(r, f) + S(r, f) \le \frac{8}{n}T(r) + S(r).$$

Noting that $n \ge 9$, we get from Lemma 2.2 that $F_2(z) \equiv 1$ or $F_3(z) \equiv 1$.
If $F_3(z) \equiv 1$, we have $e^{h(z)} \equiv 1$. By (4.3), the conclusion holds.
If $F_2(z) \equiv 1$, we have $F(z) \equiv -\frac{e^{h(z)}}{e^{h(z)}+1}$. Then by (44), we see that

$$nT(r, f) = T(r, F) = T\left(r, -\frac{e^{h(z)}}{e^{h(z)}+1}\right) = T(r, e^{h(z)}) + O(1) \le 2T(r, f) + S(r, f),$$

which is a contradiction since $n \ge 9$.

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Competing interests
The authors declare that they have no competing interests.

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