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An elementary operator and generalized Weyl's theorem

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Abstract
A Hilbert space operator T belongs to class A if $|T^2| - |T|^2 \geq 0$. Let d_{AB} denote either δ_{AB} or $\triangle_{AB}$, where δ_{AB} and $\triangle_{AB}$ denote the generalized derivation and the elementary operator on a Banach space $B(\mathcal{H})$ defined by $\delta_{AB}X = AX - XB$ and $\triangle_{AB}X = AXB - X$ respectively. If A and B^* are class A operators, we show that d_{AB} is polaroid and generalized Weyl's theorem holds for $f(d_{AB})$, generalized a-Weyl's theorem holds for $f((d_{AB})^*)$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$, where $H(\sigma(d_{AB}))$ denotes the set of all analytic functions in a neighborhood of $\sigma(d_{AB})$.
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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{C}$ be the set of complex numbers. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators and the ideal of compact operators acting on $\mathcal{H}$ respectively. For operators $A, B \in B(\mathcal{H})$, let $\delta_{AB} \in B(B(\mathcal{H}))$ denote the generalized derivation on a Banach space $B(\mathcal{H})$ defined by $\delta_{AB}X = AX - XB$; let $\triangle_{AB} \in B(B(\mathcal{H}))$ denote the elementary operator on a Banach space $B(\mathcal{H})$ defined by $\triangle_{AB} = AXB - X$. Let d_{AB} denote either δ_{AB} or $\triangle_{AB}$. d_{AB} has been studied by a number of authors [1–4]. Also let $\alpha(T) = \dim\ker T$, $\beta(T) = \dim\ker T^*$, and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum and approximate point spectrum of T. An operator $T \in B(\mathcal{H})$ is called upper (resp. lower) semi-Fredholm if $\operatorname{ran} T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). In the sequel, let $SF_+(\mathcal{H})$ denote the set of all upper semi-Fredholm operators. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. Let $\sigma_e(T)$, $\sigma_w(T)$ and $\sigma_b(T)$ denote the essential spectrum, the Weyl spectrum and the Browder spectrum of $T \in B(\mathcal{H})$. Let $\operatorname{iso}\mathcal{K}$ denote the isolated points of $\mathcal{K} \subseteq \mathbb{C}$. We write $\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$, $\pi_{00}^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$, and $p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$. It is evident that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \bigcup \operatorname{acc}\sigma(T)$ and $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$, where $\operatorname{acc}\sigma(T) = \sigma(T) \setminus \operatorname{iso}\sigma(T)$.

We say that Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and that Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

By definition, $\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(\mathcal{H})\}$ is the essential approximate point spectrum of T, and $\sigma_{ab}(T) = \bigcap\{\sigma_a(T+K) : K \in K(\mathcal{H})$ and $KT = TK\}$ is the Browder approximate point spectrum of T.

We say that a-Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that a-Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

For a bounded linear operator T and a nonnegative integer n, define T_n to be the restriction of T to $\operatorname{ran} T^n$ viewed as a map from $\operatorname{ran} T^n$ into $\operatorname{ran} T^n$ (in particular $T_0 = T$). If for some integer n, the range space $\operatorname{ran} T^n$ is closed and T_n is a Fredholm operator, then T is called a B-Fredholm operator. If T is a B-Fredholm operator of index zero, then T is called a B-Weyl operator. The B-Fredholm spectrum $\sigma_{\mathrm{BF}}(T)$ and B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T are defined by $\sigma_{\mathrm{BF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Fredholm operator}\}$ and $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}$. An operator $T \in B(\mathcal{H})$ satisfies generalized Weyl's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \pi_0(T),$$

where $\pi_0(T)$ is the set of all isolated eigenvalues of T, and $T \in B(\mathcal{H})$ satisfies generalized Browder's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus p_0(T),$$

where $p_0(T)$ is the set of all poles of the resolvent of T.

Let $\mathrm{SBF}_+(\mathcal{H})$ be the class of all the upper semi-B-Fredholm operators and $\mathrm{SBF}_+^-(\mathcal{H})$ be the class of all $T \in \mathrm{SBF}_+(\mathcal{H})$ such that $\mathrm{ind}(T) \leq 0$. Let

$$\sigma_{\mathrm{SBF}_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SBF}_+^-(\mathcal{H})\right\}$$

be called the semi-B-essential approximate point spectrum of T. We say that $T \in B(\mathcal{H})$ satisfies generalized a-Weyl's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_a(T) \setminus \pi_0^a(T),$$

where $\pi_0^a(T)$ is the set of all eigenvalues of T which are isolated points of $\sigma_a(T)$.

The following implications are known to hold:

$$\begin{aligned}&\text{generalized } a\text{-Weyl's theorem}\\&\quad\Longrightarrow\quad \text{generalized Weyl's theorem}\quad\Longrightarrow\quad \text{Weyl's theorem}\\&\quad\Longrightarrow\quad \text{Browder's theorem;}\\&\text{generalized } a\text{-Weyl's theorem}\\&\quad\Longrightarrow\quad a\text{-Weyl's theorem}\quad\Longrightarrow\quad a\text{-Browder's theorem}\\&\quad\Longrightarrow\quad \text{Browder's theorem.}\end{aligned}$$

In this paper, we shall study the generalized Weyl's theorem for the elementary operator and the generalized derivation with class A operators as entries. Recall that $T \in B(\mathcal{H})$ is called p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [6]; when $p = 1$, T is called hyponormal. And T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathscr{H}$ [7, 8]. In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), Furuta, Ito and Yamazaki [9] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T, and they showed that class A is a subclass of paranormals and contains p-hyponormal and log-hyponormal operators.

Definition 1.1 An operator $T \in B(\mathcal{H})$ is said to have the single valued extension property (SVEP) at $\lambda \in \mathbb{C}$ if for every open neighborhood $\mathcal{G}$ of λ, the only function $f \in H(\mathcal{G})$ such that $(T-\mu)f(\mu) = 0$ on G is $0 \in H(\mathcal{G})$, where $H(\mathcal{G})$ means the space of all analytic functions on G. When T has SVEP at each $\lambda \in \mathbb{C}$, say that T has SVEP.

The single valued extension property dates back to the early days of local spectral theory; see the recent monograph of Laursen and Neumann [10] or Aiena [11]. In addition to the definition of SVEP, there are notions of property (β), property (δ) and condition (C). The interested reader is referred to [10] for more details.

2 The main results

For $T \in B(\mathcal{H})$, let L_T and R_T denote the operators of left and right multiplication by T respectively.

Chō and Yamazaki proved that class A operators have property β in [12] Theorem 3.1; unfortunately, there are some mistakes in the proof of this theorem; see details in [13]. So, Theorem 3.1 in [12] is still an open problem.

Lemma 2.1 *Let A and B^* be class A operators satisfying property (β), then d_{AB} has SVEP.*

Proof By assumption and [10] Theorem 2.5.5, A satisfies property (β) and B satisfies property (δ). Hence, both L_A and R_B satisfy condition (C) by [10] Corollary 3.6.11. Clearly, L_A and R_B commute. By Theorem 3.6.3 and Note 3.6.19 on p.283 of [10], $L_A - R_B$ and L_AR_B have SVEP, which implies that d_{AB} has SVEP. □

It is well known that the isolated points of the spectrum of a class A (indeed, paranormal) operator T are poles of the resolvent of the operator (hence, eigenvalues of the operator), the restriction of T to an invariant subspace is again of class A (resp., paranormal), and that if T has countable spectrum then T is normal. (We shall use this information freely in the following without any further reference.)

Recall, [14], that $\sigma(\delta_{AB}) = \{\lambda : \lambda \in \sigma(A) - \sigma(B)\}$ and $\sigma(\triangle_{AB}) = \{\lambda : \lambda \in \sigma(A)\sigma(B) - 1\}$. If $\lambda \in \operatorname{iso}\sigma(d_{AB})$, then we have one of the following two cases:

(1) $\lambda \neq -1$ if $d_{AB} = \triangle_{AB}$. Then there exist finite sequences $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$, where $\alpha_i \in \operatorname{iso}\sigma(A)$ and $\beta_i \in \operatorname{iso}\sigma(B)$ respectively, such that $\lambda = \alpha_i - \beta_i$ if $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$ and $\lambda = \alpha_i\beta_i - 1$ if $\lambda \in \operatorname{iso}\sigma(\triangle_{AB})$, for all $1 \le i \le m$.

(2) $\lambda = -1$ and $d_{AB} = \triangle_{AB}$. Then either $0 \in \operatorname{iso}\sigma(A)$ and $0 \in \operatorname{iso}\sigma(B)$ or $0 \in \operatorname{iso}\sigma(A)$ and $0 \notin \sigma(B)$ or $0 \in \operatorname{iso}\sigma(B)$ and $0 \notin \sigma(A)$.

Theorem 2.2 *Let A and B^* be class A operators, then $H_0(d_{AB} - \lambda) = \ker(d_{AB} - \lambda)$ for all $\lambda \in \operatorname{iso}\sigma(d_{AB})$.*

Proof We consider the case $d_{AB} = \delta_{AB}$ and $d_{AB} = \triangle_{AB}$ respectively.

(1) We consider the case $d_{AB} = \delta_{AB}$. The idea comes from [1]. If $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$, then there exist finite sets $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\{\beta_1, \beta_2, \ldots, \beta_m\}$, where $\alpha_i \in \operatorname{iso}\sigma(A)$ and $\beta_i \in \operatorname{iso}\sigma(B)$ such that $\lambda = \alpha_i - \beta_i$ for all $1 \le i \le m$. Let

$$M_1 = \bigvee_{i=1}^m \ker(A - \alpha_i), \qquad M_2 = \mathcal{H} \ominus M_1$$

and

$$N_1 = \bigvee_{i=1}^m \ker\bigl(B^* - \overline{\beta_i}\bigr), \qquad N_2 = \mathcal{H} \ominus N_1.$$

Then A and B have representations $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ on $M_1 \oplus M_2$ and $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$ on $N_1 \oplus N_2$ respectively, where A_{11} and B_{11} are normal, $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$, $\sigma(B) = \sigma(B_{11}) \cup \sigma(B_{22})$ and $\lambda \notin \sigma(\delta_{A_{ii}B_{jj}})$ for all $1 \le i, j \le 2$ other than $i = j = 1$. Consider an $X \in H_0(\delta_{AB} - \lambda)$. Letting $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$, we have

$$(\delta_{AB} - \lambda)^n X = \begin{pmatrix} * & * \\ * & (\delta_{A_{22}B_{22}} - \lambda)^n X_{22} \end{pmatrix}$$

for some yet to be determined entries $*$.

Since

$$\lim_{n \longrightarrow \infty} \bigl\| (\delta_{AB} - \lambda)^n X \bigr\|^{\frac{1}{n}} = 0,$$

we have that

$$\lim_{n \longrightarrow \infty} \bigl\| (\delta_{A_{22}B_{22}} - \lambda)^n X_{22} \bigr\|^{\frac{1}{n}} = 0.$$

Since $\lambda \notin \sigma(\delta_{A_{22}B_{22}})$, we have that $\delta_{A_{22}B_{22}} - \lambda$ is invertible. Hence, $X_{22} = 0$. So, we have that

$$(\delta_{AB} - \lambda)^n X = \begin{pmatrix} * & (\delta_{A_{11}B_{22}} - \lambda)^n X_{12} \\ (\delta_{A_{22}B_{11}} - \lambda)^n X_{21} & 0 \end{pmatrix}.$$

Since

$$\lim_{n\longrightarrow\infty}\left\|(\delta_{A_{ii}B_{jj}}-\lambda)^n X_{ij}\right\|^{\frac{1}{n}}=0$$

for all $1\le i,j\le 2$ other than $i=j=1$ and since $\lambda\notin\sigma(\delta_{A_{11}B_{22}})$ and $\lambda\notin\sigma(\delta_{A_{22}B_{11}})$, we have that $X_{12}=X_{21}=0$. Hence,

$$(\delta_{AB}-\lambda)^n X=\begin{pmatrix}(\delta_{A_{11}B_{11}}-\lambda)^n X_{11} & 0\\ 0 & 0\end{pmatrix}.$$

Since A_{11} and B_{11} are normal,

$$\lim_{n\longrightarrow\infty}\left\|(\delta_{A_{11}B_{11}}-\lambda)^n X_{11}\right\|^{\frac{1}{n}}=0$$

if and only if

$$(\delta_{A_{11}B_{11}}-\lambda)X_{11}=0$$

by [15] Lemma 2. Hence, we have $H_0(\delta_{AB}-\lambda)\subseteq\ker(\delta_{AB}-\lambda)$. Since $\ker(\delta_{AB}-\lambda)\subseteq H_0(\delta_{AB}-\lambda)$ is always true, we have

$$H_0(\delta_{AB}-\lambda)=\ker(\delta_{AB}-\lambda).$$

(2) We consider the case $d_{AB}=\triangle_{AB}$. When $\lambda\neq-1$, the proof is similar to the proof of the first part. We omit the proof. When $\lambda=-1$, then either $0\in\operatorname{iso}\sigma(A)$ and $0\in\operatorname{iso}\sigma(B)$ or $0\in\operatorname{iso}\sigma(A)$ and $0\notin\sigma(B)$ or $0\in\operatorname{iso}\sigma(B)$ and $0\notin\sigma(A)$. If 0 is both in $\operatorname{iso}\sigma(A)$ and $\operatorname{iso}\sigma(B)$, then let $M_1=\ker(A)$, $M_2=\mathcal{H}\ominus M_1$ and $N_1=\ker(B^*)$, $N_2=\mathcal{H}\ominus N_1$.

Then we have $A=\left(\begin{smallmatrix}0 & C_1\\ 0 & A_2\end{smallmatrix}\right)$ on $\mathcal{H}=M_1\oplus M_2$ and $B=\left(\begin{smallmatrix}0 & 0\\ C_2 & B_2\end{smallmatrix}\right)$ on $\mathcal{H}=N_1\oplus N_2$ for some operators C_1, A_2 and C_2, B_2 respectively. Here both A_2 and B_2 are invertible. So, we have that $\triangle_{A_2B_2}-\lambda=L_{A_2}R_{B_2}$ is invertible. Let $X:N_1\oplus N_2\longrightarrow M_1\oplus M_2$ have the matrix representation $X=[X_{ij}]_{i,j=1}^2$. If $X\in H_0(\triangle_{AB}-\lambda)=H_0(L_AR_B)$, it follows that $X_{22}=0$ as in the proof of the first part. Hence, $L_AR_BX=0$ for every $X\in H_0(\triangle_{AB}-\lambda)=H_0(L_AR_B)$. So, we have $H_0(\triangle_{AB}-\lambda)\subseteq\ker(\triangle_{AB}-\lambda)$. Since $\ker(\triangle_{AB}-\lambda)\subseteq H_0(\triangle_{AB}-\lambda)$ is always true, we have $H_0(\triangle_{AB}-\lambda)=\ker(\triangle_{AB}-\lambda)$. The proofs of the other remaining cases are similar, we consider $0\in\operatorname{iso}\sigma(A)$ and $0\notin\sigma(B)$. Here $\triangle_{AB}-\lambda=L_AR_B$. In the following, we shall prove that $H_0(L_AR_B)=H_0(L_A)$. If $X\in H_0(L_AR_B)$, then

$$\lim_{n\longrightarrow\infty}\left\|(L_A)^n X\right\|^{\frac{1}{n}}=\lim_{n\longrightarrow\infty}\left\|(L_AR_B)^n XB^{-n}\right\|^{\frac{1}{n}}\le\left\|B^{-1}\right\|\lim_{n\longrightarrow\infty}\left\|(L_AR_B)^n X\right\|^{\frac{1}{n}}=0.$$

On the other hand, if $X\in H_0(L_A)$, then

$$\lim_{n\longrightarrow\infty}\left\|(L_AR_B)^n X\right\|^{\frac{1}{n}}\le\|B\|\lim_{n\longrightarrow\infty}\left\|(L_A)^n X\right\|^{\frac{1}{n}}=0.$$

Hence, $H_0(L_AR_B)=H_0(L_A)$. Next, we shall prove that $H_0(L_A)=\ker(L_A)$. Let $M_1=\ker A$, $M_2=\mathcal{H}\ominus M_1$. We have $A=\left(\begin{smallmatrix}0 & C_1\\ 0 & A_2\end{smallmatrix}\right)$ on $\mathcal{H}=M_1\oplus M_2$, where A_2 is invertible. Let X have the following matrix representation: $X=[X_{ij}]_{i,j=1}^2$ on $\mathcal{H}=M_1\oplus M_2$. If $X\in H_0(L_A)$, as in the proof above, we have that $X_{21} = X_{22} = 0$. So, we have that $L_A X = 0$. Hence, $H_0(L_A) \subseteq \ker(L_A)$. Since $\ker(L_A) \subseteq H_0(L_A)$ is always true, we have that $H_0(L_A) = \ker(L_A)$. Since B is invertible, we have $\ker(L_A R_B) = \ker(L_A)$. Therefore, we have that

$$H_0(L_A R_B) = H_0(L_A) = \ker(L_A) = \ker(L_A R_B),$$

hence

$$H_0(L_A R_B) = \ker(L_A R_B).$$

That is,

$$H_0(\triangle_{AB} - \lambda) = \ker(\triangle_{AB} - \lambda).$$

This completes the proof. □

An operator $T \in B(\mathcal{H})$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid then it is isoloid.

Lemma 2.3 *Let A and B^* be class A operators, then d_{AB} and d_{AB}^* are polaroid. In particular, d_{AB} and d_{AB}^* are isoloid.*

Proof We only need to prove that d_{AB} is polaroid. Let $\mathcal{X} = B(\mathcal{H})$ and $\lambda \in \operatorname{iso}\sigma(d_{AB})$. Then we have that $H_0(d_{AB} - \lambda) = \ker(d_{AB} - \lambda)$ by Theorem 2.2. Hence,

$$\mathcal{X} = H_0(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda) = \ker(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda).$$

So, we have

$$(d_{AB} - \lambda)\mathcal{X} = 0 \oplus (d_{AB} - \lambda)\big(K(d_{AB} - \lambda)\big) = K(d_{AB} - \lambda).$$

Therefore,

$$\mathcal{X} = \ker(d_{AB} - \lambda) \oplus (d_{AB} - \lambda)\mathcal{X}.$$

Thus, isolated points of $\sigma(d_{AB})$ are simple poles of the resolvent of d_{AB}. Hence, d_{AB} is polaroid. So, we have that d_{AB} and d_{AB}^* are polaroid. Since polaroid operators are always isoloid, we have that d_{AB} and d_{AB}^* are isoloid. □

Theorem 2.4 *Let A and B^* be class A operators satisfying property (β). Then generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

Proof Suppose that A and B^* are class A operators. By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d_{AB} is polaroid. So, we have that generalized Weyl's theorem holds for d_{AB} by [16, Theorem 3.10(ii)]. Since d_{AB} has SVEP and d_{AB} is isoloid, we have that generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ by [17, Theorem 2.2]. □

Corollary 2.5 *Let A and B^* be class A operators satisfying property (β). Then Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

A bounded linear operator $T \in B(\mathcal{H})$ is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Note that every a-isoloid operator is isoloid and the converse is not true in general.

Lemma 2.6 *Let A and B^* be class A operators satisfying property (β), then d^*_{AB} is a-isoloid.*

Proof Let λ be an isolated point of $\sigma_a(d^*_{AB})$. Suppose that A and B^* are class A operators satisfying property (β). By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d^*_{AB} is isoloid. Hence, $\sigma_a(d^*_{AB}) = \sigma(d^*_{AB})$ by [18, Corollary 7]. We have that λ is an isolated point of $\sigma(d^*_{AB})$. Since d^*_{AB} is isoloid, we have that λ is an eigenvalue of d^*_{AB}. Hence, d^*_{AB} is a-isoloid. □

Theorem 2.7 *Let A and B^* be class A operators satisfying property (β). Then generalized a-Weyl's theorem holds for $f(d^*_{AB})$ for every $f \in H(\sigma(d_{AB}))$, and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

Proof Suppose that A and B^* are class A operators satisfying property (β). By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d^*_{AB} is polaroid. By Corollary 2.5, Weyl's theorem holds for d_{AB}. Hence, d^*_{AB} satisfies Weyl's theorem by [19, Proposition 2.1]. Since d_{AB} has SVEP and d^*_{AB} is polaroid, generalized a-Weyl's theorem holds for d^*_{AB} by [16, Theorem 3.10]. T is a-isoloid by Lemma 2.6, hence generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$ by [17, Theorem 2.4]. □

Corollary 2.8 *Let A and B^* be class A operators satisfying property (β). Then a-Weyl's theorem holds for $f(d^*_{AB})$ for every $f \in H(\sigma(d_{AB}))$, and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

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