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Remarks on contractive mappings via Ω-distance

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Abstract

Very recently, some authors discovered that some fixed point results in the context of a *G*-metric space can be derived from the fixed point results in the context of a quasi-metric space and hence the usual metric space. In this article, we investigate some fixed point results in the framework of a *G*-metric space via Ω-distance that cannot be obtained by the usual fixed point results in the literature. We also add an application to illustrate our results.

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1 Introduction and preliminaries

Very recently, Jleli and Samet [1] and Samet *et al.* [2] proved that some fixed point results in the setting of *G*-metric spaces, introduced by Sims and Mustafa [3], are consequences of the well-known fixed point theorem in the context of the usual metric space. Indeed, authors in [1, 2] noticed that $G(x, y, y) = q(x, y)$ is a quasi-metric and obtained that the results are just a characterization of existence results in the framework of a quasi-metric. On the other hand, a *G*-metric was introduced as a generalization of the (usual) metric. Basically, *G*-metrics claim the geometry of three points instead of two points. Consequently, Jleli and Samet [1] and Samet *et al.* [2] concluded that if the expression in the fixed point theorem can be reduced to two points, then it can be written as a consequence of the related existence result in the literature.

Recently, Saadati *et al.* [4] introduced the concept of Ω-distance on a complete *G*-metric space as a generalized notion of ω-distance due to Kada *et al.* [5]. In these papers, the authors investigate the existence/uniqueness of a fixed point of certain operators in this setting. In this paper, we revise some published papers (see, *e.g.*, [6, 7]) and improve the statements in a way that cannot be manipulated by the techniques used in [1, 2] (see also [8–10]).

We first recall some necessary definitions and basic results on the topics in the literature.

Definition 1 ([3]) Let X be a non-empty set. A function $G : X \times X \times X \to [0, \infty)$ is called a *G*-metric if the following conditions are satisfied:

  (i) $G(x, y, z) = 0$ if $x = y = z$ (coincidence),
  (ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,
  (iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
  (iv) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

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(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2 ([3]) Suppose that (X, G) is a G-metric space.

(1) A sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$.

(2) A sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_m, x_n, x) < \varepsilon$.

Definition 3 ([4]) Let (X, G) be a G-metric space. Then a function $\Omega : X \times X \times X \longrightarrow [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,

(b) $\Omega(x, y, \cdot), \Omega(x, \cdot, y) : X \to [0, \infty)$ are lower semi-continuous for any $x, y \in X$,

(c) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

Example 4 ([4]) Suppose that (X, d) is a metric space. Let $G : X^3 \longrightarrow [0, \infty)$ be defined as follows:

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then one can easily show that $\Omega = G$ is an Ω-distance on X.

Example 5 ([4]) Let $X = \mathbb{R}$ and (X, G) be a G-metric, where

$$G(x, y, z) = \frac{1}{3}\left(|x - y| + |y - z| + |x - z|\right)$$

for all $x, y, z \in X$. If we define $\Omega : \mathbb{R}^3 \longrightarrow [0, \infty)$ as follows:

$$\Omega(x, y, z) = \frac{1}{3}\left(|z - x| + |x - y|\right)$$

for all $x, y, z \in X$, then it is an Ω-distance on $\mathbb{R}$.

We refer, *e.g.*, to [4, 11] for more details and examples on the topic.

Lemma 6 [4] *Suppose that (X, G) is a G-metric space and Ω is an Ω-distance on X. Let $\{x_n\}, \{y_n\}$ be sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and $x, y, z, a \in X$. Then*

(a) *if $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$, and hence $y = z$;*

(b) *if $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$, then $G(y_n, y_m, z) \to 0$, and hence $y_n \to z$;*

(c) *if $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a G-Cauchy sequence;*

(d) *if $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a G-Cauchy sequence.*

Definition 7 ([4]) Suppose that (X, G) is a G-metric space and Ω is an Ω-distance on X. (X, G) is called Ω-bounded if there is a constant $C > 0$ with $\Omega(x, y, z) \leq C$ for all $x, y, z \in X$.

Definition 8 Let $(X, \leq)$ be a partially ordered set. A self-mapping $T : X \to X$ is said to be non-decreasing if, for $x, y \in X$,

$$x \leq y \quad \Longrightarrow \quad T(x) \leq T(y).$$

The tripled $(X, G, \leq)$ is called a partially ordered G-metric space if $(X, \leq)$ is a partially ordered set endowed with a G-metric on X; see also [12, 13].

2 Fixed point theorems on partially ordered *G*-metric spaces

We start this section with the following classes of mappings:

$$\Phi = \big\{\phi | \phi : [0, \infty) \to [0, \infty) \text{ continuous, non-decreasing}\big\} \quad \text{and}$$

$$\Psi = \big\{\psi | \psi : [0, \infty) \to [0, \infty) \text{ continuous, non-decreasing}\big\}$$

with $\phi^{-1}(\{0\}) = \psi^{-1}(\{0\}) = \{0\}$.

Definition 9 Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. A self-mapping $T : X \to X$ is said to be a generalized weak-contraction mapping if it satisfies the following condition:

$$\psi\big(\Omega\big(Tx, T^2x, Ty\big)\big) \leq \psi\big(\Omega(x, Tx, y)\big) - \phi\big(\Omega(x, Tx, y)\big) \quad \text{for all } x, y \in X, \text{ with } x \leq y,$$

where $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 10 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, and let Ω be an Ω-distance on X. Suppose that a non-decreasing self-mapping $T : X \to X$ is a generalized weak-contraction mapping, that is,*

$$\psi\big(\Omega\big(Tx, T^2x, Ty\big)\big) \leq \psi\big(\Omega(x, Tx, y)\big) - \phi\big(\Omega(x, Tx, y)\big) \quad \text{for all } x, y \in X, \text{ with } x \leq Tx,$$

with $\psi \in \Psi$ and $\phi \in \Phi$. Suppose also that $\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \leq Tx\} > 0$ for every $y \in X$ with $y \neq Ty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point, say $u \in X$. Moreover, $\Omega(u, u, u) = 0$.

Proof If $x_0 = Tx_0$, then the proof is finished. Suppose that $x_0 \neq Tx_0$. Since $x_0 \leq Tx_0$ and T is non-decreasing, we obtain

$$x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^{n+1}x_0 \leq \cdots.$$

Now, if for some $n \in \mathbb{N}$, $\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = 0$, then

$$\psi\big(\Omega\big(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0\big)\big) \leq \psi\big(\Omega\big(T^nx_0, T^{n+1}x_0, T^{n+1}x_0\big)\big)$$
$$- \phi\big(\Omega\big(T^nx_0, T^{n+1}x_0, T^{n+1}x_0\big)\big),$$

then $\Omega(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0) = 0$. Due to [(a), Definition 3], we have $\Omega(T^nx_0, T^{n+2}x_0, T^{n+2}x_0) = 0$. On the other hand, by [(c), Definition 3], we easily derive that $G(T^nx_0, T^{n+2}x_0, T^{n+2}x_0) = 0$, which completes the proof.

Consequently, throughout the proof, we suppose that $\Omega(T^n x_0, T^{n+1} x_0, T^{n+1} x_0) > 0$ for all $n \in \mathbb{N}$. Hence, we have

$$\psi\big(\Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big)\big) \leq \psi\big(\Omega\big(T^{n-1} x_0, T^n x_0, T^n x_0\big)\big)$$
$$- \phi\big(\Omega\big(T^{n-1} x_0, T^n x_0, T^n x_0\big)\big), \tag{2.1}$$

which yields that

$$\psi\big(\Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big)\big) \leq \psi\big(\Omega\big(T^{n-1} x_0, T^n x_0, T^n x_0\big)\big).$$

As a result, we conclude that $\{\Omega(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)\}$ is non-increasing. Thus, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} \Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big) = r.$$

We shall show that $r = 0$. Suppose, on the contrary, that $r > 0$. Then we have $\phi(r) > 0$. Letting $n \to \infty$ on (2.1), we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

a contraction. Hence, we have

$$\lim_{n \to \infty} \Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big) = 0. \tag{2.2}$$

Recursively, we obtain that

$$\lim_{n \to \infty} \Omega\big(T^n x_0, T^{n+1} x_0, T^{n+t} x_0\big) = 0 \tag{2.3}$$

for every $t \in \mathbb{N}$.

Let $l \geq m \geq n$ with $m = n + k$ and $l = m + t$ ($k, t \in \mathbb{N}$). By the triangle inequality, we derive that

$$\Omega\big(T^n x_0, T^m x_0, T^l x_0\big) \leq \Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big) + \Omega\big(T^{n+1} x_0, T^m x_0, T^l x_0\big)$$
$$\leq \Omega\big(T^n x_0, T^{n+1} x_0, T^{n+1} x_0\big) + \Omega\big(T^{n+1} x_0, T^{n+2} x_0, T^{n+2} x_0\big)$$
$$+ \cdots + \Omega\big(T^{m-1} x_0, T^m x_0, T^l x_0\big).$$

Letting $n \to \infty$ in the inequality above, by keeping the limits (2.2) and (2.3), we obtain

$$\lim_{n,m,l \to \infty} \Omega\big(T^n x_0, T^m x_0, T^l x_0\big) = 0.$$

Therefore, $\{T^n x_0\}$ is a G-Cauchy sequence. Since X is G-complete, $\{T^n x_0\}$ converges to a point $u \in X$. Now, for $\varepsilon > 0$ and by the lower semi-continuity of Ω,

$$\Omega\big(T^n x_0, T^m x_0, u\big) \leq \liminf_{p \to \infty} \Omega\big(T^n x_0, T^m x_0, T^p x_0\big) \leq \varepsilon, \quad m \geq n,$$

and

$$\Omega\big(T^n x_0, u, T^l x_0\big) \leq \liminf_{p \to \infty} \Omega\big(T^n x_0, T^p x_0, T^l x_0\big) \leq \varepsilon, \quad l \geq n.$$

Assume that $u \neq Tu$. Since $T^n x_0 \leq T^{n+1} x_0$,

$$0 < \inf\big\{\Omega\big(T^n x_0, u, T^n x_0\big) + \Omega\big(T^n x_0, u, T^{n+1} x_0\big) + \Omega\big(T^n x_0, T^{n+1} x_0, u\big) : n \in \mathbb{N}\big\} \leq 3\varepsilon,$$

a contraction. Hence, we have $u = Tu$.

We shall show that u is the unique fixed point of T. Suppose, on the contrary, that v is another fixed point of T. So, we have

$$
\begin{aligned}
\psi\big(\Omega(u, u, v)\big) &= \psi\big(\Omega(Tu, T^2 u, Tv)\big) \\
&\leq \psi\big(\Omega(u, Tu, v)\big) - \phi\big(\Omega(u, Tu, v)\big) \\
&= \psi\big(\Omega(u, u, v)\big) - \phi\big(\Omega(u, u, v)\big) \\
&< \psi\big(\Omega(u, u, v)\big),
\end{aligned}
$$

a contraction. Thus, the fixed point u is unique. Now, since $u = Tu$, we have

$$
\begin{aligned}
\psi\big(\Omega(u, u, u)\big) &= \psi\big(\Omega(Tu, T^2 u, Tu)\big) \\
&\leq \psi\big(\Omega(u, Tu, u)\big) - \phi\big(\Omega(u, Tu, u)\big) \\
&= \psi\big(\Omega(u, u, u)\big) - \phi\big(\Omega(u, u, u)\big).
\end{aligned}
$$

So, $\Omega(u, u, u) = 0$. $\qquad\qquad\square$

Definition 11 Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. A self-mapping $T : X \to X$ is said to be a weak-contraction mapping if it satisfies the following condition:

$$\Omega\big(Tx, T^2 x, Ty\big) \leq \Omega(x, Tx, y) - \phi\big(\Omega(x, Tx, y)\big) \quad \text{for all } x, y \in X, \text{ with } x \leq y,$$

where $\phi \in \Phi$.

Corollary 12 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, and let Ω be an Ω-distance on X. Suppose that a non-decreasing self-mapping $T : X \to X$ is a weak-contraction mapping, that is,*

$$\Omega\big(Tx, T^2 x, Ty\big) \leq \Omega(x, Tx, y) - \phi\big(\Omega(x, Tx, y)\big) \quad \text{for all } x, y \in X, \text{ with } x \leq Tx,$$

where $\phi \in \Phi$. Suppose also that $\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \leq Tx\} > 0$ for every $y \in X$ with $y \neq Ty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point, say $u \in X$. Moreover, $\Omega(u, u, u) = 0$.

If we take $\phi(t) = kt$, where $k \in [0, 1)$, we derive Theorem 2.2 [4] as the following corollary.

Corollary 13 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, and let Ω be an Ω-distance on X. Suppose that there exists $k \in [0, 1)$ such that*

$$\Omega(Tx, T^2x, Ty) \leq k\Omega(x, Tx, y) \quad \text{for all } x, y \in X, \text{ with } x \leq Tx.$$

Suppose also that $\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \leq Tx\} > 0$ *for every $y \in X$ with $y \neq Ty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point, say $u \in X$. Moreover, $\Omega(u, u, u) = 0$.*

Definition 14 Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. A self-mapping $T : X \to X$ is said to be a Ćirić-type contraction mapping if it satisfies that there exists $0 \leq k < 1$ such that

$$\Omega(Tx, T^2x, Ty) \leq kM(x, x, y),$$

where

$$M(x, x, y) = \max\left\{\Omega(x, Tx, Tx), \Omega(y, Ty, Ty), \frac{1}{2}\Omega(x, Ty, Ty)\right\}$$

for all $x, y \in X$ with $x \leq y$.

Theorem 15 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, and let Ω be an Ω-distance on X. Suppose that a non-decreasing self-mapping $T : X \longrightarrow X$ is a Ćirić-type contraction mapping.*
   (i) *For every $x \in X$ and $y \in X$ with $y \neq T(y)$,*
      $\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \leq T(x)\} > 0$,
   (ii) *There exists $x_0 \in X$ such that $x_0 \leq T(x_0)$,*
then T has a fixed point u in X and $\Omega(u, u, u) = 0$.

Proof By assumption (ii), there exists $x_0 \in X$ such that $x_0 \leq T(x_0)$. We fix $x_1 \in X$ such that $x_1 = T(x_0)$. Since T is a non-decreasing mapping, $Tx_0 \leq Tx_1$. There exists $x_2 \in X$ such that $Tx_1 = x_2$. Recursively, we construct the sequence $\{x_n\}$ in the following way:

$$x_{n+1} = Tx_n \leq Tx_{n+1} = x_{n+2} \quad \text{for all } n \geq 0.$$

Since T is a Ćirić-type contraction mapping, by replacing $x = x_n$ and $y = x_{n+1}$, we get that

$$\Omega(x_{n+1}, x_{n+2}, x_{n+2}) = \Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq kM(x_n, x_n, x_{n+1}), \tag{2.4}$$

where

$$M(x_n, x_n, x_{n+1}) = \max\left\{\Omega(x_n, Tx_n, Tx_n), \Omega(x_{n+1}, Tx_{n+1}, Tx_{n+1}),\right.$$

$$\left.\frac{1}{2}\Omega(x_n, Tx_{n+1}, Tx_{n+1})\right\}$$

$$= \max\left\{\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n+1}, x_{n+2}, x_{n+2}),\right.$$

$$\frac{1}{2}\Omega(x_n, x_{n+2}, x_{n+2})\Big\}$$

$$\leq \max\Big\{\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n+1}, x_{n+2}, x_{n+2}),$$

$$\frac{1}{2}\big[\Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2})\big]\Big\}$$

$$= \max\big\{\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n+1}, x_{n+2}, x_{n+2})\big\}.$$

Notice that if $M(x_n, x_n, x_{n+1}) \leq \Omega(x_{n+1}, x_{n+2}, x_{n+2})$, then (2.4) yields a contradiction since $k < 1$.

Thus, $M(x_n, x_n, x_{n+1}) \leq \Omega(x_n, x_{n+1}, x_{n+1})$ and inequality (2.4) and $k < 1$ turn into

$$\Omega(x_{n+1}, x_{n+2}, x_{n+2}) \leq k\Omega(x_n, x_{n+1}, x_{n+1}). \tag{2.5}$$

Upon the discussion above, we conclude that the sequence $\{\Omega(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = r.$$

We shall show that $r = 0$. By a standard calculation, using inequality (2.5) and keeping $k < 1$ in mind, we obtain $\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$. We claim that the sequence $\{x_n\}$ is G-Cauchy. Let $l \geq m \geq n$ with $m = n + k$ and $l = m + t$ ($k, t \in \mathbb{N}$). By the triangle inequality, we derive that

$$\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_m, x_l)$$

$$\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l). \tag{2.6}$$

On the other hand, we have

$$\Omega(x_{m-1}, x_m, x_{m+t}) \leq kM(x_{m-2}, x_{m-2}, x_{m+t-1})$$

$$= k \max\Big\{\Omega(x_{m-2}, x_{m-1}, x_{m-1}), \Omega(x_{m+t-1}, x_{m+t}, x_{m+t}),$$

$$\frac{1}{2}\Omega(x_{m-2}, x_{m+t}, x_{m+t})\Big\}$$

$$\leq k \max\Big\{\Omega(x_{m-2}, x_{m-1}, x_{m-1}), \Omega(x_{m+t-1}, x_{m+t}, x_{m+t}),$$

$$\frac{1}{2}\big[\Omega(x_{m-2}, x_{m-1}, x_{m-1}) + \Omega(x_{m-1}, x_m, x_m)$$

$$+ \cdots + \Omega(x_{m+t-1}, x_{m+t}, x_{m+t})\big]\Big\}. \tag{2.7}$$

By combining expressions (2.6) and (2.7), we find that

$$\Omega(x_n, x_m, x_l)$$

$$\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-2}, x_{m-1}, x_{m-1})$$

$$+ k \max\left\{\Omega(x_{m-2}, x_{m-1}, x_{m-1}), \Omega(x_{m+t-1}, x_{m+t}, x_{m+t}), \frac{1}{2}\left[\Omega(x_{m-2}, x_{m-1}, x_{m-1})\right.\right.$$

$$\left.\left. + \Omega(x_{m-1}, x_m, x_m) + \cdots + \Omega(x_{m+t-1}, x_{m+t}, x_{m+t})\right]\right\}. \tag{2.8}$$

Taking $n \to \infty$ in (2.8), we conclude that

$$\lim_{n,m,l \to \infty} \Omega(x_n, x_m, x_l) = 0,$$

and hence $\{x_n\}$ is a G-Cauchy sequence due to expression (c) of Lemma 6. Since X is G-complete, $\{x_n\}$ converges to a point $u \in X$. Thus, for $\varepsilon > 0$ and by the lower semi-continuity of Ω, we have

$$\Omega(x_n, x_m, u) \leq \liminf_{p \to \infty} \Omega(x_n, x_m, x_p) \leq \varepsilon, \quad m \geq n,$$

and

$$\Omega(x_n, u, x_l) \leq \liminf_{p \to \infty} \Omega(x_n, x_p, x_l) \leq \varepsilon, \quad l \geq n.$$

Assume that $u \neq Tu$. Since $x_{n+1} \leq x_{n+2}$,

$$0 < \inf\left\{\Omega(x_{n+1}, u, x_{n+1}) + \Omega(x_{n+1}, u, x_{n+2}) + \Omega(x_{n+1}, x_{n+2}, u) : n \in \mathbb{N}\right\} \leq 3\varepsilon$$

for every $\varepsilon > 0$, that is a contraction. Therefore, we have $u = Tu$ and $\Omega(u, u, u) = 0$. $\qquad\square$

Definition 16 Let $(X, \leq)$ be a partially ordered space and $f, g : X \to X$. We say that g is an f-monotone mapping if

$$x, y \in X, \quad f(x) \leq f(y) \implies g(x) \leq g(y).$$

Theorem 17 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, and let Ω be an Ω-distance on X such that X is Ω-bounded. Let $f : X \longrightarrow X$ and $g : f(X) \longrightarrow X$ commute, f be non-decreasing and g be an f-monotone mapping such that*:

(a) *$gf(X) \subseteq f^2(X)$;*

(b) *$\Omega(gfx, gy, g^2x) \leq kM(x, x, y)$, where*
   $M(x, x, y) = \max\{\Omega(f^2x, fy, fgx), \Omega(fy, fy, gy), \Omega(f^2x, f^2x, fgx)\}$ for all $x, y \in X$ with
   $f(x) \leq f(y)$ and $0 \leq k < 1$;

(c) *for every $x \in X$ and $z \in X$ with $f^2z \neq gfz$,*

$$\inf\left\{\Omega(x, z, x) + \Omega(x, x, z) + \Omega(f^2x, gx, gfx) : f^2x \leq gfx\right\} > 0;$$

(d) *there exists $x_0 \in f(X)$ such that $f(x_0) \leq g(x_0)$;*
then f and g have a unique common fixed point u in X and $\Omega(u, u, u) = 0$.

Proof Let $x_0 \in f(X)$ such that $f(x_0) \leq g(x_0)$. By part (a), we can choose $x_1 \in f(X)$ such that $f(x_1) = g(x_0)$. Again from part (a), we can choose $x_2 \in f(X)$ such that $f(x_2) = g(x_1)$.

Continuing this process, we can construct sequences $\{x_n\}$ in $f(X)$ and $\{z_n\}$ in $f^2(X)$ such that

$$y_n = gx_n = fx_{n+1}, \tag{2.9}$$

and

$$z_n = gy_{n-1} = gfx_n = fgx_n = fy_n. \tag{2.10}$$

Since $f(x_0) \leq g(x_0)$ and $f(x_1) = g(x_0)$, we have $f(x_0) \leq f(x_1)$. Then by Definition 16, $g(x_0) \leq g(x_1)$. Continuing, we obtain

$$gx_n \leq gx_{n+1}, \quad \forall n \geq 0. \tag{2.11}$$

So, by (2.9) and (2.11), for all $t \geq 1$, $fx_n \leq fx_{n+t}$. Now, for all $s \geq 0$,

$$
\begin{aligned}
\Omega(z_n, z_{n+s}, z_{n+1}) &= \Omega\big(gfx_n, gx_{n+s-1}, g^2x_n\big) \\
&\leq k \max\big\{\Omega\big(f^2x_n, fy_{n+s-1}, fgx_n\big), \Omega(fy_{n+s-1}, fy_{n+s-1}, gy_{n+s-1}), \\
&\quad\ \Omega\big(f^2x_n, f^2x_n, fgx_n\big)\big\} \\
&= k \max\big\{\Omega(z_{n-1}, z_{n+s-1}, z_n), \Omega(z_{n+s-1}, z_{n+s-1}, z_{n+s}), \\
&\quad\ \Omega(z_{n-1}, z_{n-1}, z_n)\big\}.
\end{aligned}
$$

Then, for $s = 0$,

$$\Omega(z_n, z_n, z_{n+1}) \leq k\Omega(z_{n-1}, z_{n-1}, z_n).$$

For $s = 1$,

$$\Omega(z_n, z_{n+1}, z_{n+1}) \leq k^{1+1} \max\big\{\Omega(z_{n-1}, z_n, z_n), \Omega(z_{n-1}, z_{n-1}, z_n)\big\}.$$

For $s = 2$,

$$\Omega(z_n, z_{n+2}, z_{n+1}) \leq k^{1+2} \max\big\{\Omega(z_{n-1}, z_{n+1}, z_n), \Omega(z_{n-1}, z_{n-1}, z_n)\big\}$$

and

$$
\begin{aligned}
\Omega(z_{n-1}, z_{n-1}, z_n) &\leq k \max\big\{\Omega(z_{n-2}, z_{n-2}, z_{n-1}), \Omega(z_{n-2}, z_{n-2}, z_{n-1}), \Omega(z_{n-2}, z_{n-2}, z_{n-1})\big\} \\
&= k\Omega(z_{n-2}, z_{n-2}, z_{n-1}) \\
&\quad \vdots \\
&\leq k^{n-1}\Omega(z_0, z_0, z_1).
\end{aligned}
$$

Therefore, for all $n \geq 1$ and $s \geq 0$,

$$\Omega(z_n, z_{n+s}, z_{n+1}) \leq k^{n+s} \max\big\{\Omega(z_{n-1}, z_{n+s-1}, z_n), \Omega(z_0, z_0, z_1)\big\}. \tag{2.12}$$

Notice that if $\Omega(z_n, z_{n+s}, z_{n+1}) \leq k^{n+s} \Omega(z_0, z_0, z_1)$, so for all $s \geq 0$, $\lim_{n\to\infty} \Omega(z_n, z_{n+s}, z_{n+1}) = 0$. If $\Omega(z_n, z_{n+s}, z_{n+1}) \leq k^{n+s} \Omega(z_{n-1}, z_{n+s-1}, z_n)$, so $\{\Omega(z_{n-1}, z_{n+s-1}, z_n)\}$ is non-increasing and bounded below. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \Omega(z_{n-1}, z_{n+s-1}, z_n) = r.$$

We shall show that $r = 0$. By a standard calculation, using inequality (2.12) and keeping $k < 1$ in mind, we obtain $\lim_{n\to\infty} \Omega(z_{n-1}, z_{n+s-1}, z_n) = 0$. Now, for any $l \geq m \geq n$ with $m = n + k$ and $l = m + t$ $(k, t \in \mathbb{N})$, we have

$$\Omega(z_n, z_m, z_l) \leq \Omega(z_n, z_{n+1}, z_{n+1}) + \Omega(z_{n+1}, z_m, z_l)$$

$$\leq \Omega(z_n, z_{n+1}, z_{n+1}) + \Omega(z_{n+1}, z_{n+2}, z_{n+2}) + \cdots + \Omega(z_{m-1}, z_m, z_l)$$

$$\leq \Omega(z_n, z_{n+1}, z_{n+1}) + \Omega(z_{n+1}, z_{n+2}, z_{n+2}) + \cdots + \Omega(z_{m-1}, z_m, z_m)$$

$$+ \Omega(z_m, z_{m+1}, z_{m+1}) + \cdots + \Omega(z_{m+t-1}, z_m, z_{m+t}).$$

So,

$$\lim_{n,m,l\to\infty} \Omega(z_n, z_m, z_l) = 0,$$

and consequently, by Part (3) of Lemma 6, $\{z_n\}$ is a G-Cauchy sequence. Since X is G-complete, $\{z_n\}$ converges to a point $z \in X$. Thus, for $\varepsilon > 0$ and by the lower semi-continuity of Ω, we have

$$\Omega(z_n, z_m, z) \leq \liminf_{p\to\infty} \Omega(z_n, z_m, z_p) \leq \varepsilon, \quad m \geq n,$$

and

$$\Omega(z_n, z, z_l) \leq \liminf_{p\to\infty} \Omega(z_n, z_p, z_l) \leq \varepsilon, \quad l \geq n.$$

Assume that $f^2 z \neq gfz$. Since f is non-decreasing, we obtain

$$z_n = f^2 x_{n+1} = f(fx_{n+1}) \leq f(fx_{n+2}) = gfx_{n+1} = z_{n+1},$$

then $z_n \leq z_{n+1}$. Also, for all $n \geq 1$,

$$\Omega\big(f^2 z_n, g z_n, gf z_n\big) = \Omega\big(gf z_{n-1}, g z_n, g^2 z_{n-1}\big)$$

$$\leq k \max\big\{\Omega\big(f^2 z_{n-1}, f z_n, fg z_{n-1}\big), \Omega(f z_n, f z_n, g z_n),$$

$$\Omega\big(f^2 z_{n-1}, f^2 z_{n-1}, fg z_{n-1}\big)\big\}$$

$$= k \max\big\{\Omega\big(gf z_{n-2}, g z_{n-1}, g^2 z_{n-2}\big), \Omega(f z_n, f z_n, g z_n),$$

$$\Omega\big(gf z_{n-2}, gf z_{n-2}, g^2 z_{n-2}\big)\big\}$$

$$\leq k^3 \max\big\{\Omega\big(f^2 z_{n-2}, f z_{n-1}, fg z_{n-2}\big), \Omega(f z_{n-1}, f z_{n-1}, g z_{n-1}),$$

$$\Omega\big(f^2 z_{n-2}, f^2 z_{n-2}, fg z_{n-2}\big), \Omega(f z_n, f z_n, g z_n),$$

$$\Omega\big(f^2 z_{n-2}, f^2 z_{n-2}, fg z_{n-2}\big), \Omega\big(f^2 z_{n-2}, f^2 z_{n-2}, gf z_{n-2}\big),$$
$$\Omega\big(f^2 z_{n-2}, f^2 z_{n-2}, fg z_{n-2}\big)\big\}$$
$$= k^3 \max\big\{\Omega\big(f^2 z_{n-2}, f z_{n-1}, fg z_{n-2}\big), \Omega(f z_{n-1}, f z_{n-1}, g z_{n-1}),$$
$$\Omega\big(f^2 z_{n-2}, f^2 z_{n-2}, fg z_{n-2}\big), \Omega(f z_n, f z_n, g z_n)\big\}$$
$$\vdots$$
$$\leq k^{2n+1} \max\big\{\Omega\big(f^2 z_1, g z_1, gf z_1\big), \Omega\big(f^2 z_1, f^2 z_1, fg z_1\big),$$
$$\Omega(f z_i, f z_i, g z_i), 0 \leq i \leq n\big\}$$
$$\leq k^{2n+1} C,$$

where $C = \max\{\Omega(f^2 z_1, g z_1, gf z_1), \Omega(f^2 z_1, f^2 z_1, fg z_1), \Omega(f z_i, f z_i, g z_i), 0 \leq i \leq n\}$, and consequently $\lim_{n \to \infty} \Omega(f^2 z_n, g z_n, gf z_n) = 0$. Therefore,

$$0 < \inf\big\{\Omega(z_n, z, z_n) + \Omega(z_n, z_n, z) + \Omega\big(f^2 z_n, g z_n, gf z_n\big) : n \in \mathbb{N}\big\} \leq 3\varepsilon$$

for every $\varepsilon > 0$, that is a contraction. So, we have $f^2 z = gf z$. Then, by (b),

$$\Omega\big(gf^2 z, g(gf z), g^2 f z\big) \leq k \max\big\{\Omega\big(f^2 f z, f(gf z), fg(f z)\big), \Omega\big(f(gf z), f(gf z), g(gf z)\big),$$
$$\Omega\big(f^2(f z), f^2(f z), fg(f z)\big)\big\}.$$

So, $\Omega(gf^2 z, g(gf z), g^2 f z) = 0$. Since X is Ω-bounded, $\Omega(gf^2 z, g(gf z), g^2 f z) = 0 < M$. Similarly, $\Omega(gf^2 z, gf z, g^2 f z) \leq k \Omega(f^2 z, f^2 z, f^2 z) < M$. By part (c) of Definition 3, $G(gf^2 z, gf z, g^2 f z) = 0$. Then $g^2 f z = gf z$, which implies that $gf z$ is a fixed point for g. Now,

$$f(gf z) = gf^2 z = g^2 f z = gf z.$$

Then $u = gf z$ is a common fixed point of f and g.

Uniqueness. Assume that there exists $v \in X$ such that $f v = g v = v$. Hence, we have

$$\Omega(v, v, v) \leq k \Omega(v, v, v),$$

and so $\Omega(v, v, v) = \Omega(u, u, u) = 0$. Also, $\Omega(v, u, v) = 0$. Then, by Part (c) of Definition 3, $u = v$ and $\Omega(u, u, u) = 0$. □

The following corollary is a generalization of Theorem 2.1 [14].

Denote by Λ the set of all functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(i) λ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$,

(ii) for every $\varepsilon > 0$, we have $\int_0^\varepsilon \lambda(s)\, ds > 0$,

(iii) $\|\lambda\| < 1$, where $\|\lambda\|$ denotes the norm of λ.

Now, we have the following corollary.

Corollary 18 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, let Ω be an Ω-distance on X, and let $T : X \to X$ be a non-decreasing self-mapping. Suppose that $\psi \in \Psi$*

and $\phi \in \Phi$ such that

$$\int_0^{\psi(\Omega(Tx,T^2x,Ty))} \lambda(s)\,ds \leq \int_0^{\psi(\Omega(x,Tx,y))} \lambda(s)\,ds - \int_0^{\phi(\Omega(x,Tx,y))} \lambda(s)\,ds, \tag{2.13}$$

for all $x \leq Tx$, $y \in X$, where $\lambda \in \Lambda$. Also, for every $x \in X$,

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\} > 0$$

for every $y \in X$ with $y \neq Ty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point.

Proof Define $\gamma : [0,+\infty) \to [0,+\infty)$ by $\gamma(t) = \int_0^t \lambda(s)\,ds$, then from inequality (2.13), we have

$$\gamma\big(\psi\big(\Omega\big(Tx,T^2x,Ty\big)\big)\big) \leq \gamma\big(\psi\big(\Omega(x,Tx,y)\big)\big) - \gamma\big(\phi\big(\Omega(x,Tx,y)\big)\big),$$

which can be written as

$$\psi_1\big(\Omega\big(Tx,T^2x,Ty\big)\big) \leq \psi_1\big(\Omega(x,Tx,y)\big) - \phi_1\big(\Omega(x,Tx,y)\big),$$

where $\psi_1 = \gamma \circ \psi$ and $\phi_1 = \gamma \circ \phi$. Since the functions ψ_1 and ϕ_1 satisfy the properties of ψ and ϕ, by Theorem 10, T has a unique fixed point. $\qquad\square$

Corollary 19 *Let $(X, G, \leq)$ be a partially ordered complete G-metric space, let Ω be an Ω-distance on X, and let $T : X \to X$ be a non-decreasing self-mapping. Suppose that there exists $0 \leq k < 1$ such that*

$$\int_0^{\psi(\Omega(Tx,T^2x,Ty))} k\lambda(s)\,ds \leq \int_0^{M(x,x,y)} \lambda(s)\,ds \tag{2.14}$$

for all $x \leq Tx$, $y \in X$, where

$$M(x,x,y) = \max\left\{\Omega(x,Tx,Tx), \Omega(y,Ty,Ty), \frac{1}{2}\Omega(x,Ty,Ty)\right\}$$

and $\lambda \in \Lambda$. Also, for every $x \in X$,

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\} > 0$$

for every $y \in X$ with $y \neq Ty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point.

3 Application

In this section, we give an existence theorem for a solution of the following integral equations:

$$x(t) = \int_0^1 K\big(t,s,x(s)\big)\,ds + g(t), \quad t \in [0,1]. \tag{3.1}$$

Let $X = C([0,1])$ be the set of all continuous functions defined on $[0,1]$. Define $G: X \times X \times X \to \mathbb{R}$ by

$$G(x,y,z) = \|x - y\| + \|y - z\| + \|z - x\|,$$

where $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Then (X, G) is a complete G-metric space. Let $\Omega = G$. Then Ω is an Ω-distance on X. Define an ordered relation $\leq$ on X by

$$x \leq y \quad \text{iff} \quad x(t) \leq y(t), \quad \forall t \in [0,1].$$

Then $(X, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 20 *Suppose the following hypotheses hold*:

(1) $K : [0,1] \times [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0,1] \to \mathbb{R}$ *are continuous mappings*,

(2) K *is non-decreasing in its first coordinate and* g *is non-decreasing*,

(3) *There exists a continuous function* $G : [0,1] \times [0,1] \to [0, +\infty)$ *such that*

$$\bigl| K(t,s,u) - K(t,s,v) \bigr| \leq G(t,s)|u - v|$$

for every comparable $u, v \in \mathbb{R}^+$ *and* $s, t \in [0,1]$ *with* $\sup_{t \in [0,1]} \int_0^1 G(t,s)\, ds \leq \frac{1}{2}$,

(4) *There exist continuous, non-decreasing functions* $\phi, \psi : [0,\infty) \to (0,\infty)$ *with* $\psi^{-1}(\{0\}) = \phi^{-1}(\{0\}) = \{0\}$ *and* $\psi(r) \leq \psi(2r) - \phi(2r)$ *for all* $r \in [0,\infty)$.

Then the integral equation has a solution in $C([0,1])$.

Proof Define $Tx(t) = \int_0^1 K(t,s,x(s))\, ds + g(t)$. By hypothesis (2), we have that T is non-decreasing.

Now, if

$$\inf\bigl\{ \Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx \bigr\} = 0$$

for every $y \in X$ with $y \neq Ty$, then for each $n \in \mathbb{N}$, there exists $x_n \in C([0,1])$ with $x_n \leq Tx_n$ such that

$$\Omega(x_n, y, x_n) + \Omega(x_n, y, Tx_n) + \Omega(x_n, Tx_n, y) \leq \frac{1}{n}.$$

Then we have

$$\Omega(x_n, y, Tx_n) = \sup_{t \in [0,1]} |x_n - y| + \sup_{t \in [0,1]} |y - Tx_n| + \sup_{t \in [0,1]} |Tx_n - x_n| \leq \frac{1}{n}.$$

Thus,

$$\lim_{n \to \infty} x_n(t) = y(t), \qquad \lim_{n \to \infty} Tx_n(t) = y(t).$$

By the continuity of K, we have

$$y(t) = \lim_{n \to \infty} Tx_n(t) = \int_0^1 K\Bigl(t,s, \lim_{n \to \infty} x_n(s)\Bigr) ds + g(t)$$

$$= \int_0^1 K\bigl(t,s,y(s)\bigr) ds + g(t) = Ty(t),$$

which is a contradiction. Therefore,

$$\inf\big\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\big\} > 0.$$

Now, for $x, y \in X$ with $x \leq Tx$, we have

$$
\psi\big(\Omega\big(Tx, T^2x, Ty\big)\big) = \psi\bigg(\sup_{t\in[0,1]} \big|Tx(t) - T^2x(t)\big| + \sup_{t\in[0,1]} \big|T^2x(t) - Ty(t)\big|
$$
$$
+ \sup_{t\in[0,1]} \big|Ty(t) - Tx(t)\big|\bigg)
$$
$$
\leq \psi\bigg(\sup_{t\in[0,1]} \int_0^1 \big|K\big(t,s,x(s)\big) - K\big(t,s,Tx(s)\big)\big|\,ds
$$
$$
+ \sup_{t\in[0,1]} \int_0^1 \big|K\big(t,s,Tx(s)\big) - K\big(t,s,y(s)\big)\big|\,ds
$$
$$
+ \sup_{t\in[0,1]} \int_0^1 \big|K\big(t,s,y(s)\big) - K\big(t,s,x(s)\big)\big|\,ds\bigg)
$$
$$
\leq \psi\bigg(\sup_{t\in[0,1]} \bigg(\int_0^1 G(t,s)\big|x(s) - Tx(s)\big|\,ds\bigg)
$$
$$
+ \sup_{t\in[0,1]} \bigg(\int_0^1 G(t,s)\big|Tx(s) - y(s)\big|\,ds\bigg)
$$
$$
+ \sup_{t\in[0,1]} \bigg(\int_0^1 G(t,s)\big|y(s) - x(s)\big|\,ds\bigg)\bigg)
$$
$$
\leq \psi\bigg(\sup_{t\in[0,1]} \big(\big|x(t) - Tx(t)\big|\big) \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds
$$
$$
+ \sup_{t\in[0,1]} \big(\big|Tx(t) - y(t)\big|\big) \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds
$$
$$
+ \sup_{t\in[0,1]} \big(\big|y(t) - x(t)\big|\big) \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds\bigg)
$$
$$
\leq \psi\bigg(\frac{1}{2} \sup_{t\in[0,1]} \big(\big|x(t) - Tx(t)\big|\big) + \frac{1}{2} \sup_{t\in[0,1]} \big(\big|Tx(t) - y(t)\big|\big)
$$
$$
+ \frac{1}{2} \sup_{t\in[0,1]} \big(\big|y(t) - x(t)\big|\big)\bigg)
$$
$$
\leq \psi\bigg(\frac{1}{2}\Omega(x,Tx,y)\bigg) \leq \psi\big(\Omega(x,Tx,y)\big) - \phi\big(\Omega(x,Tx,y)\big).
$$

Thus, by Theorem 10, there exists a solution $u \in C[0,1]$ of integral equation (3.1). $\qquad\square$

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