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The convergence of the modified Mann and Ishikawa iterations in Banach spaces

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Abstract
In this paper, under the new condition we show that the convergence of the modified Mann and Ishikawa iterations is equivalent for uniformly L-Lipschitz asymptotically pseudocontractive mappings in real Banach spaces. Our results extend and improve the corresponding results of Zeng (Acta Math. Sin. 47:219-228, 2004).
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1 Introduction and preliminaries

Throughout the paper, we assume that E is an arbitrary real Banach space, D is a nonempty closed convex subset of E, $T : D \to D$ is a self-mapping and $F(T)$ is the fixed point set of T, *i.e.*, $F(T) = \{x \in D : Tx = x\}$. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}, \quad \forall x \in E, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. The single-valued normalized duality mapping is denoted by j.

Definition 1.1 (see [1]) (1) A mapping T is said to be *uniformly L-Lipschitz* if there exists a constant $L > 0$ such that, for all $x, y \in D$,

$$\left\|T^n x - T^n y\right\| \le L\|x - y\|, \quad \forall n \ge 1. \tag{1.2}$$

(2) The mapping T is said to be *asymptotically nonexpansive* with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x, y \in D$,

$$\left\|T^n x - T^n y\right\| \le k_n\|x - y\|, \quad \forall n \ge 1. \tag{1.3}$$

(3) The mapping T is said to be *asymptotically pseudocontractive* with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x, y \in D$, there exists $j(x - y) \in J(x - y)$ such that

$$\left\langle T^n x - T^n y, j(x - y)\right\rangle \le k_n\|x - y\|^2, \quad \forall n \ge 1. \tag{1.4}$$

Obviously, an asymptotically nonexpansive mapping is both asymptotically pseudocontractive and uniformly L-Lipschitz, but the converse is not true in general. For more details on uniformly L-Lipschitz asymptotically nonexpansive and asymptotically pseudocontractive mappings, see [2–6] and [7–11].

Definition 1.2 (see [1]) For any $u_1, x_1 \in D$, the sequences $\{u_n\}$ and $\{x_n\}$ in D defined by

$$u_{n+1} = (1 - a_n)u_n + a_n T^n u_n, \quad \forall n \geq 1, \tag{1.5}$$

and

$$\begin{cases} y_n = (1 - b_n)x_n + b_n T^n x_n, \\ x_{n+1} = (1 - a_n)x_n + a_n T^n y_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

are called the *modified Mann* and *Ishikawa iterations*, respectively, where $\{a_n\}$, $\{b_n\}$ are two real sequences in $[0,1]$ satisfying some conditions. For more details on the Mann and Ishikawa iterations, see [4, 12] and [11].

In 2001, Chidume and Mutangadura [13] constructed an example for every nontrivial Mann iteration failing to converge while Ishikawa iteration converges. Therefore, there exist some differences between convergence of two kinds of the iterative sequences. Since then, many authors have shown that the Mann (modified Mann) and Ishikawa (modified Ishikawa) iterations (with errors) converge strongly to fixed points of pseudocontractive mappings and others under appropriate conditions.

Especially, Chang [1] proved the following.

Theorem 1.3 [1, Theorem 2.1] *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $L \geq 1$. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences in $[0,1]$ satisfying the following conditions*:

(a) $a_n, b_n \to 0$ *as* $n \to \infty$;

(b) $\sum_{n=0}^{\infty} a_n = \infty$.

For any $x_0 \in D$, let $\{x_n\}$ be the modified Ishikawa iteration defined by (1.2). *If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that*

$$\bigl\langle T^n x_{n+1} - q, j(x_{n+1} - q)\bigr\rangle \leq k_n \|x_{n+1} - q\|^2 - \Phi\bigl(\|x_{n+1} - q\|\bigr), \quad \forall n \geq 0,$$

where $j(x_{n+1} - q) \in J(x_{n+1} - q)$, then $\{x_n\}$ converges strongly to a fixed point q of T.

Theorem 1.4 [1, Theorem 2.3] *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $L \geq 1$. Let $\{a_n\}$ be the real sequence in $[0,1]$ satisfying the following conditions*:

(a) $a_n \to 0$ *as* $n \to \infty$;

(b) $\sum_{n=0}^{\infty} a_n = \infty$.

For any $u_0 \in D$, let $\{u_n\}$ be the modified Mann iteration defined by (1.1). *If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ satisfying the condition* (2.1) *of* [1, Theorem 2.1], *then $\{u_n\}$ converges strongly to a fixed point q of T.*

Motivated by Theorems 1.3 and 1.4, Zeng [14] gave another interesting results as follows.

Theorem 1.5 [14, Theorem 2.1] *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,+\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $L \geq 1$. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences in $[0,1]$ satisfying the following conditions*:

(a) $a_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=0}^{\infty} a_n = \infty$;
(b) $\sum_{n=0}^{\infty} a_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} a_n(k_n - 1) < \infty$;
(c) $\sum_{n=0}^{\infty} a_n b_n < \infty$.

For arbitrary $x_0 \in D$, let $\{x_n\}$ be the modified Ishikawa iteration defined by (1.2). *If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that*

$$\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle \leq k_n \|x_{n+1} - q\|^2 - \Phi(\|x_{n+1} - q\|), \quad \forall n \geq 0,$$

where $j(x_{n+1} - q) \in J(x_{n+1} - q)$, then $\{x_n\}$ converges strongly to the fixed point q of T.

Theorem 1.6 [14, Theorem 2.3] *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,+\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $L \geq 1$. Let $\{a_n\}$ be a real sequence in $[0,1]$ satisfying the following conditions*:

(a) $a_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=0}^{\infty} a_n = \infty$;
(b) $\sum_{n=0}^{\infty} a_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} a_n(k_n - 1) < \infty$.

For arbitrary $u_0 \in D$, let $\{u_n\}$ be the modified Mann iteration defined by (1.1). *If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ satisfying the condition* (2.1) *of* [1, Theorem 2.1], *then $\{u_n\}$ converges strongly to the fixed point q of T.*

It is worth mentioning that the result of Chang [1] is different from that of Zeng [14]. This can be seen from the following example.

Example 1.7 Set

$$a_n = \begin{cases} 0, & n = 2i, \\ \frac{1}{n}, & n = 2i-1, \end{cases} \qquad b_n = \begin{cases} \frac{1}{2}, & n = 2i, \\ \frac{1}{n}, & n = 2i-1, \end{cases} \qquad k_n = 1 + \frac{1}{n}, \quad \forall i \geq 1, n \geq 1.$$

Then $a_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n^2 < \infty$, $\sum_{n=1}^{\infty} a_n b_n < \infty$, $\sum_{n=1}^{\infty} a_n(k_n - 1) < \infty$, but $b_n \to 0$ as $n \to \infty$ does not hold. On the other hand, let

$$a_n = \begin{cases} 0, & n = 2i, \\ \frac{1}{\sqrt{n}}, & n = 2i-1, \end{cases} \qquad b_n = \begin{cases} 0, & n = 2i, \\ \frac{1}{\sqrt{n}}, & n = 2i-1, \end{cases} \qquad k_n = 1 + \frac{1}{\sqrt{n}}, \quad \forall i \geq 1, n \geq 1.$$

Then $a_n, b_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$, but $\sum_{n=1}^{\infty} a_n^2 = \infty$, $\sum_{n=1}^{\infty} a_n b_n = \infty$ and $\sum_{n=1}^{\infty} a_n(k_n - 1) = \infty$.

The aim of this paper is to extend and improve Theorem 1.5 and Theorem 1.6.
For this, we need to use the following lemmas.

Lemma 1.8 [1] *Let E be a real Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then, for all $x, y \in E$ and $j(x+y) \in J(x+y)$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{1.7}$$

Lemma 1.9 [14] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq 0. \tag{1.8}$$

If $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2 Main results

Now, we give the main results in this paper.

Theorem 2.1 *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $L \geq 1$. Let $\{a_n\}$ be a real sequence in $[0,1]$ satisfying the following conditions*:

(a) $a_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(b) $\sum_{n=1}^{\infty} a_n^2 < \infty$ *and* $\sum_{n=1}^{\infty} a_n(k_n - 1) < \infty$.

For arbitrary $u_1 \in D$, let $\{u_n\}$ be the modified Mann iteration defined by (1.5). *If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that*

$$\langle T^n u_{n+1} - q, j(u_{n+1} - q)\rangle \leq k_n \|u_{n+1} - q\|^2 - \frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2}, \quad \forall n \geq 1,$$

where $j(u_{n+1} - q) \in J(u_{n+1} - q)$, then $\{u_n\}$ converges strongly to a fixed point q of T.

Proof Applying (1.5) and Lemma 1.8, we have

$$\begin{aligned}
\|u_{n+1} - q\|^2 &= \big\|(1-a_n)(u_n - q) + a_n\big(T^n u_n - q\big)\big\|^2 \\
&\leq (1-a_n)^2\|u_n - q\|^2 + 2a_n\langle T^n u_n - q, j(u_{n+1} - q)\rangle \\
&= (1-a_n)^2\|u_n - q\|^2 + 2a_n\langle T^n u_n - T^n u_{n+1}, j(u_{n+1} - q)\rangle \\
&\quad + 2a_n\langle T^n u_{n+1} - T^n q, j(u_{n+1} - q)\rangle \\
&\leq (1-a_n)^2\|u_n - q\|^2 + 2a_n L\|u_n - u_{n+1}\| \cdot \|u_{n+1} - q\| \\
&\quad + 2a_n\left[k_n\|u_{n+1} - q\|^2 - \frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2}\right].
\end{aligned} \tag{2.1}$$

Observe that

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|a_n(T^n u_n - u_n)\| \\ &\le a_n \|T^n u_n - T^n q + q - u_n\| \\ &\le a_n(1+L)\|u_n - q\|. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\begin{aligned} \|u_{n+1} - q\|^2 \le{} & (1-a_n)^2\|u_n - q\|^2 + a_n^2 L(1+L)\big(\|u_n - q\|^2 + \|u_{n+1} - q\|^2\big) \\ & + 2a_n\left[k_n\|u_{n+1} - q\|^2 - \frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2}\right]. \end{aligned} \tag{2.3}$$

Since $a_n \to 0$ and $k_n \to 1$ as $n \to \infty$, without loss of generality, we assume that

$$\frac{1}{2} < 1 - a^2 L(1+L) - 2a_n k_n < 1, \quad \forall n \ge 1.$$

Then (2.3) implies that

$$\begin{aligned} & \|u_{n+1} - q\|^2 \\ & \quad \le \frac{(1-a_n)^2 + a_n^2 L(1+L)}{1 - a^2 L(1+L) - 2a_n k_n}\|u_n - q\|^2 \\ & \qquad - \frac{2a_n}{1 - a^2 L(1+L) - 2a_n k_n} \cdot \frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} \\ & \quad \le \left\{1 + \frac{2a_n(k_n - 1) + a_n^2[1 + 2L(1+L)]}{1 - a^2 L(1+L) - 2a_n k_n}\right\}\|u_n - q\|^2 - \frac{2a_n\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} \\ & \quad \le \big\{1 + 4a_n(k_n - 1) + 2a_n^2\big[1 + 2L(1+L)\big]\big\}\|u_n - q\|^2 - \frac{2a_n\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} \\ & \quad \le \big\{1 + 4a_n(k_n - 1) + 2a_n^2\big[1 + 2L(1+L)\big]\big\}\|u_n - q\|^2. \end{aligned} \tag{2.4}$$

Since $\sum_{n=1}^{\infty}\{4a_n(k_n - 1) + 2a_n^2[1 + 2L(1+L)]\} < \infty$, by Lemma 1.9, $\lim_{n\to\infty}\|u_n - q\|$ exists. Denote $M = \sup_{n\ge 1}\{\|u_n - q\|\}$.

On the other hand, from (2.4), we have

$$\begin{aligned} & \|u_{n+1} - q\|^2 \\ & \quad \le \big\{1 + 4a_n(k_n - 1) + 2a_n^2\big[1 + 2L(1+L)\big]\big\}\|u_n - q\|^2 \\ & \qquad - \frac{2a_n\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} \\ & \quad \le \|u_n - q\|^2 + \big\{4a_n(k_n - 1) + 2a_n^2\big[1 + 2L(1+L)\big]\big\}M^2 \\ & \qquad - \frac{2a_n\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2}. \end{aligned} \tag{2.5}$$

Let $\inf_{n\ge 1}\frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} = \delta$. Then $\delta = 0$. Assume $\delta > 0$. Then we have

$$\frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} \ge \delta, \quad \forall n \ge 1.$$

It follows from (2.5) that

$$2a_n\delta \leq \|u_n - q\|^2 - \|u_{n+1} - q\|^2 + \bigl[4a_n(k_n - 1) + 2a_n^2\bigl(1 + 2L(1 + L)\bigr)\bigr]M^2, \tag{2.6}$$

which implies that

$$2\delta \sum_{n=1}^{\infty} a_n \leq \|u_1 - q\|^2 + \sum_{n=1}^{\infty}\bigl[4a_n(k_n - 1) + 2a_n^2\bigl(1 + 2L(1 + L)\bigr)\bigr]M^2 < \infty, \tag{2.7}$$

which is a contradiction, and so $\delta = 0$. Thus, there exists a subsequence

$$\left\{\frac{\Phi(\|u_{n_i+1} - q\|)}{1 + \Phi(\|u_{n_i+1} - q\|) + \|u_{n_i+1} - q\|^2}\right\}$$

of

$$\left\{\frac{\Phi(\|u_{n+1} - q\|)}{1 + \Phi(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2}\right\}$$

such that

$$\lim_{i\to\infty} \frac{\Phi(\|u_{n_i+1} - q\|)}{1 + \Phi(\|u_{n_i+1} - q\|) + \|u_{n_i+1} - q\|^2} = 0.$$

Since $0 \leq \|u_n - q\| \leq M$, it follows that

$$0 \leq \frac{\Phi(\|u_{n_i+1} - q\|)}{1 + \Phi(M) + M^2} \leq \frac{\Phi(\|u_{n_i+1} - q\|)}{1 + \Phi(\|u_{n_i+1} - q\|) + \|u_{n_i+1} - q\|^2}.$$

Thus, $\lim_{i\to\infty} \Phi(\|u_{n_i+1} - q\|) = 0$. By the strictly increasing continuous function Φ, we obtain that $\lim_{i\to\infty} \|u_{n_i+1} - q\| = 0$ and so $\lim_{n\to\infty} \|u_n - q\| = 0$. This completes the proof. □

Theorem 2.2 *Let E be a real Banach space and D be a nonempty closed convex subset of E. Let $T : D \to D$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Suppose that $\{a_n\}$ and $\{b_n\}$ are two real sequences in $[0, 1]$ satisfying the following conditions*:

(a) $a_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(b) $\sum_{n=1}^{\infty} a_n^2 < \infty$ *and* $\sum_{n=1}^{\infty} a_n(k_n - 1) < \infty$;
(c) $\sum_{n=1}^{\infty} a_n b_n < \infty$.

For any $u_1, x_1 \in D$, let $\{u_n\}$ and $\{x_n\}$ be the modified Mann and Ishikawa iterations defined by (1.5) *and* (1.6), *respectively. If $F(T) \neq \emptyset$, $q \in F(T)$ and there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that*

$$\begin{aligned}&\bigl\langle T^n x_{n+1} - T^n u_{n+1}, j(x_{n+1} - u_{n+1})\bigr\rangle \\ &\quad\leq k_n\|x_{n+1} - u_{n+1}\|^2 - \frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2},\end{aligned}$$

where $j(x_{n+1} - u_{n+1}) \in J(x_{n+1} - u_{n+1})$. Then the following two assertions are equivalent:

(1) $\{u_n\}$ *converges strongly to the fixed point* q *of* T.
(2) $\{x_n\}$ *converges strongly to the fixed point* q *of* T.

Proof If the iteration (1.6) converges to a fixed point q, then, by putting $b_n = 0$, we can get the convergence of the iteration (1.5).

Conversely, we only need to prove that the iteration (1.5) $\Rightarrow$ the iteration (1.6), *i.e.*, $\|u_n - q\| \to 0$ as $n \to \infty \Rightarrow \|x_n - q\| \to 0$ as $n \to \infty$. Here, without loss of generality, let $\|u_n - q\| \leq 1$. Then $\|T^n u_n - u_n\| \leq (1 + L)$.

Applying the iterations (1.5), (1.6) and Lemma 1.8, we have

$$\begin{aligned} \|x_{n+1} - u_{n+1}\|^2 &= \big\|(1 - a_n)(x_n - u_n) + a_n\big(T^n y_n - T^n u_n\big)\big\|^2 \\ &\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\big\langle T^n y_n - T^n u_n, j(x_{n+1} - u_{n+1})\big\rangle \\ &= (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\big\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - u_{n+1})\big\rangle \\ &\quad + 2a_n\big\langle T^n x_{n+1} - T^n u_{n+1}, j(x_{n+1} - u_{n+1})\big\rangle \\ &\quad + 2a_n\big\langle T^n u_{n+1} - T^n u_n, j(x_{n+1} - u_{n+1})\big\rangle \\ &\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n L\|y_n - x_{n+1}\| \cdot \|x_{n+1} - u_{n+1}\| \\ &\quad + 2a_n\left[k_n\|x_{n+1} - u_{n+1}\|^2 - \frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2}\right] \\ &\quad + 2a_n L\|u_{n+1} - u_n\| \cdot \|x_{n+1} - u_{n+1}\|. \end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned} \|u_{n+1} - u_n\| &= \big\|a_n\big(T^n u_n - u_n\big)\big\| \\ &\leq a_n(1 + L), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \|y_n - u_n\| &= \big\|(1 - b_n)(x_n - u_n) + b_n\big(T^n x_n - u_n\big)\big\| \\ &\leq (1 - b_n)\|x_n - u_n\| + b_n\big\|T^n x_n - T^n u_n\big\| + b_n\big\|T^n u_n - u_n\big\| \\ &\leq (1 - b_n + b_n L)\|x_n - u_n\| + b_n(1 + L) \\ &\leq (1 + b_n L)\|x_n - u_n\| + b_n(1 + L), \end{aligned} \tag{2.10}$$

$$\begin{aligned} \|y_n - x_{n+1}\| &= \big\|b_n\big(T^n x_n - x_n\big) + a_n\big(x_n - T^n y_n\big)\big\| \\ &\leq b_n\big[\big\|T^n x_n - T^n u_n\big\| + \big\|T^n u_n - u_n\big\| + \|u_n - x_n\|\big] \\ &\quad + a_n\big[\|x_n - u_n\| + \big\|u_n - T^n u_n\big\| + \big\|T^n u_n - T^n y_n\big\|\big] \\ &\leq b_n\big[(L + 1)\|x_n - u_n\| + (L + 1)\big] \\ &\quad + a_n\big[\|x_n - u_n\| + (L + 1) + L\|u_n - y_n\|\big] \\ &\leq b_n\big[(L + 1)\|x_n - u_n\| + (L + 1)\big] \\ &\quad + a_n\big[\|x_n - u_n\| + (L + 1) + L\big[(1 + b_n L)\|x_n - u_n\| + b_n(1 + L)\big]\big] \\ &= A_n\|x_n - u_n\| + B_n, \end{aligned} \tag{2.11}$$

where $A_n = b_n(L+1) + a_n[1 + L(1 + b_nL)] \to 0$ and $B_n = b_n(L+1) + a_n(L+1)(1 + Lb_n) \to 0$ as $n \to \infty$. Substituting (2.9), (2.11) into (2.8), we obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\quad\leq (1-a_n)^2\|x_n - u_n\|^2 + 2a_nL\big(A_n\|x_n - u_n\| + B_n\big)\|x_{n+1} - u_{n+1}\| + 2a_n^2L(1+L) \\
&\qquad \cdot \|x_{n+1} - u_{n+1}\| + 2a_n\left[k_n\|x_{n+1} - u_{n+1}\|^2 - \frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2}\right] \\
&\quad\leq (1-a_n)^2\|x_n - u_n\|^2 + a_nLA_n\|x_n - u_n\|^2 + a_nLA_n\|x_{n+1} - u_{n+1}\|^2 \\
&\qquad + a_nLB_n + a_nLB_n\|x_{n+1} - u_{n+1}\|^2 + a_n^2L(1+L)\|x_{n+1} - u_{n+1}\|^2 + a_n^2L(1+L) \\
&\qquad + 2a_n\left[k_n\|x_{n+1} - u_{n+1}\|^2 - \frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2}\right] \\
&\quad= \big[(1-a_n)^2 + a_nLA_n\big]\|x_n - u_n\|^2 + \big[a_nLA_n + a_nLB_n + a_n^2L(1+L) \\
&\qquad + 2a_nk_n\big]\|x_{n+1} - u_{n+1}\|^2 + a_nLB_n + a_n^2L(1+L) \\
&\qquad - 2a_n\frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2}.
\end{aligned} \tag{2.12}$$

Since $a_n, b_n, A_n, B_n, k_n - 1 \to 0$ as $n \to \infty$, without loss of generality, we assume that

$$\frac{1}{2} < 1 - a_nLA_n - a_nLB_n - a_n^2L(1+L) - 2a_nk_n < 1, \quad \forall n \geq 1.$$

Then (2.12) implies that

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\quad\leq \frac{(1-a_n)^2 + a_nLA_n}{1 - a_nLA_n - a_nLB_n - a_n^2L(1+L) - 2a_nk_n}\|x_n - u_n\|^2 \\
&\qquad + \frac{a_nLB_n + a_n^2L(1+L)}{1 - a_nLA_n - a_nLB_n - a_n^2L(1+L) - 2a_nk_n} \\
&\qquad - \frac{2a_n}{1 - a_nLA_n - a_nLB_n - a_n^2L(1+L) - 2a_nk_n} \\
&\qquad \cdot \frac{\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2} \\
&\quad\leq \|x_n - u_n\|^2 + \frac{2a_nLA_n + a_nLB_n + a_n^2(1 + L + L^2) + 2a_n(k_n - 1)}{1 - a_nLA_n - a_nLB_n - a_n^2L(1+L) - 2a_nk_n}\|x_n - u_n\|^2 \\
&\qquad + 2a_nLB_n + 2a_n^2L(1+L) - \frac{2a_n\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2} \\
&\quad\leq \big[1 + 4a_nLA_n + 2a_nLB_n + 2a_n^2\big(1 + L + L^2\big) + 4a_n(k_n - 1)\big]\|x_n - u_n\|^2 \\
&\qquad + 2a_nLB_n + 2a_n^2L(1+L) - \frac{2a_n\Phi(\|x_{n+1} - u_{n+1}\|)}{1 + \Phi(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2} \\
&\quad\leq \big[1 + 4a_nLA_n + 2a_nLB_n + 2a_n^2\big(1 + L + L^2\big) + 4a_n(k_n - 1)\big]\|x_n - u_n\|^2 \\
&\qquad + 2a_nLB_n + 2a_n^2L(1+L).
\end{aligned} \tag{2.13}$$

Since

$$\sum_{n=1}^{\infty}\bigl[4a_nLA_n + 2a_nLB_n + 2a_n^2\bigl(1+L+L^2\bigr) + 4a_n(k_n-1)\bigr] < \infty$$

and $\sum_{n=1}^{\infty}[2a_nLB_n + 2a_n^2L(1+L)] < \infty$, by Lemma 1.9, $\lim_{n\to\infty}\|x_n - u_n\|$ exists. Denote $M_0 = \sup_{n\geq 1}\{\|x_n - u_n\|\}$.

On the other hand, from (2.13), it follows that

$$\begin{aligned}
&\|x_{n+1}-u_{n+1}\|^2 \\
&\quad\leq \bigl[1 + 4a_nLA_n + 2a_nLB_n + 2a_n^2\bigl(1+L+L^2\bigr) + 4a_n(k_n-1)\bigr]\|x_n - u_n\|^2 \\
&\qquad + 2a_nLB_n + 2a_n^2L(1+L) - \frac{2a_n\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\Phi(\|x_{n+1}-u_{n+1}\|) + \|x_{n+1}-u_{n+1}\|^2} \\
&\quad\leq \|x_n-u_n\|^2 + \bigl[4a_nLA_n + 2a_nLB_n + 2a_n^2\bigl(1+L+L^2\bigr) + 4a_n(k_n-1)\bigr]M^2 \\
&\qquad + 2a_nLB_n + 2a_n^2L(1+L) - \frac{2a_n\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\Phi(\|x_{n+1}-u_{n+1}\|) + \|x_{n+1}-u_{n+1}\|^2}.
\end{aligned}\tag{2.14}$$

Let $\inf_{n\geq 1}\frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\Phi(\|x_{n+1}-u_{n+1}\|)+\|x_{n+1}-u_{n+1}\|^2} = \delta$. Then $\delta = 0$. Assume $\delta > 0$. Then we have

$$\frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\Phi(\|x_{n+1}-u_{n+1}\|) + \|x_{n+1}-u_{n+1}\|^2} \geq \delta, \quad \forall n \geq 1.$$

It follows from (2.14) that

$$\begin{aligned}
2a_n\delta &\leq \|x_n-u_n\|^2 - \|x_{n+1}-u_{n+1}\|^2 \\
&\quad + \bigl[4a_nLA_n + 2a_nLB_n + 2a_n^2\bigl(1+L+L^2\bigr) + 4a_n(k_n-1)\bigr]M^2 \\
&\quad + 2a_nLB_n + 2a_n^2L(1+L),
\end{aligned}\tag{2.15}$$

which implies that

$$\begin{aligned}
2\delta\sum_{n=1}^{\infty}a_n &\leq \|x_1-u_1\|^2 + \sum_{n=1}^{\infty}\bigl[4a_nLA_n + 2a_nLB_n + 2a_n^2(1+L) + 4a_n(k_n-1)\bigr]M^2 \\
&\quad + \sum_{n=1}^{\infty}\bigl[2a_nLB_n + 2a_n^2L(1+L)\bigr] < \infty,
\end{aligned}\tag{2.16}$$

which is a contradiction, and so $\delta = 0$. Thus, there exists a subsequence

$$\left\{\frac{\Phi(\|x_{n_i+1}-u_{n_i+1}\|)}{1+\Phi(\|x_{n_i+1}-u_{n_i+1}\|) + \|x_{n_i+1}-u_{n_i+1}\|^2}\right\}$$

of

$$\left\{\frac{\Phi(\|x_{n+1}-u_{n_i+1}\|)}{1+\Phi(\|x_{n+1}-u_{n_i+1}\|) + \|x_{n+1}-u_{n+1}\|^2}\right\}$$

such that

$$\lim_{i\to\infty}\frac{\Phi(\|x_{n_i+1}-u_{n_i+1}\|)}{1+\Phi(\|x_{n_i+1}-u_{n_i+1}\|)+\|x_{n_i+1}-u_{n_i+1}\|^2}=0.$$

Since $0\le\|x_n-u_n\|\le M$, it follows that

$$0\le\frac{\Phi(\|x_{n_i+1}-u_{n_i+1}\|)}{1+\Phi(M)+M^2}\le\frac{\Phi(\|x_{n_i+1}-u_{n_i+1}\|)}{1+\Phi(\|x_{n_i+1}-u_{n_i+1}\|)+\|x_{n_i+1}-u_{n_i+1}\|^2}.$$

Thus, $\lim_{i\to\infty}\Phi(\|x_{n_i+1}-u_{n_i+1}\|)=0$. By the strictly increasing continuous function Φ, we obtain that $\lim_{i\to\infty}\|x_{n_i+1}-u_{n_i+1}\|=0$ and so $\lim_{n\to\infty}\|x_n-u_n\|=0$. Using the inequality $\|x_n-q\|\le\|x_n-u_n\|+\|u_n-q\|\to 0$ as $n\to\infty$, we know that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Theorem 2.3 *Let E be a real Banach space and D be a nonempty closed convex subset of E. Let $T:D\to D$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with a sequence $\{k_n\}\subset[1,+\infty)$ such that $\lim_{n\to\infty}k_n=1$. Suppose that $\{a_n\}$ and $\{b_n\}$ are two real sequences in $[0,1]$ satisfying the following conditions*:

(a) $a_n\to 0$ *as* $n\to\infty$ *and* $\sum_{n=1}^{\infty}a_n=\infty$;
(b) $\sum_{n=1}^{\infty}a_n^2<\infty$ *and* $\sum_{n=1}^{\infty}a_n(k_n-1)<\infty$;
(c) $\sum_{n=1}^{\infty}a_nb_n<\infty$.

For any $x_1\in D$, let $\{x_n\}$ be the modified Ishikawa iteration defined in (1.6). *If $F(T)\neq\emptyset$, $q\in F(T)$ and there exists a strictly increasing continuous function $\Phi:[0,+\infty)\to[0,+\infty)$ with $\Phi(0)=0$ such that*

$$\langle T^nx_{n+1}-q,j(x_{n+1}-q)\rangle\le k_n\|x_{n+1}-q\|^2-\frac{\Phi(\|x_{n+1}-q\|)}{1+\Phi(\|x_{n+1}-q\|)+\|x_{n+1}-q\|^2},\quad\forall n\ge 1,$$

where $j(x_{n+1}-q)\in J(x_{n+1}-q)$. Then $\{x_n\}$ converges strongly to the fixed point q of T.

Proof By Theorem 2.1 and Theorem 2.2, we obtain the proof of Theorem 2.3. □

Remark 2.4 Since the condition $\langle T^nx-q,j(x-q)\rangle\le k_n\|x-q\|^2-\frac{\Phi(\|x-q\|)}{1+\Phi(\|x-q\|)+\|x-q\|^2}$ is weaker than $\langle T^nx-q,j(x-q)\rangle\le k_n\|x-q\|^2-\Phi(\|x-q\|)$, Theorem 2.1 and Theorem 2.3 generalize the corresponding results of Zeng [14]. Further, our proof methods are different from those of Zeng [14].

For the sake of convenience, we give the following definitions.

Definition 2.5 A mapping $T:D\to E$ is said to be *weak generalized asymptotically φ-hemi-contractive* with a sequence $\{k_n\}\subset[1,+\infty)$ such that $\lim_{n\to\infty}k_n=1$ if there exists a strictly increasing continuous function $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(0)=0$ such that, for any $x\in D$ and $y\in F(T)$, there exists $j(x-y)\in J(x-y)$ such that

$$\langle T^nx-T^ny,j(x-y)\rangle\le k_n\|x-y\|^2-\frac{\varphi(\|x-y\|)}{1+\varphi(\|x-y\|)+\|x-y\|^2},\quad\forall n\ge 1.\tag{2.17}$$

If the condition (2.17) is replaced by the following inequality:

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2 - \varphi(\|x-y\|), \quad \forall n \ge 1, \tag{2.18}$$

then T is called a *generalized asymptotically φ-hemi-contractive mapping*. Clearly, if T is a generalized asymptotically asymptotically φ-hemi-contractive, then T must be a weak generalized asymptotically asymptotically φ-hemi-contractive mapping. However, the converse is not true in general. This can be seen from the following examples.

Example 2.6 Let $E = R$ be the set of real numbers with the usual norm $|\cdot|$ and $D = [0, +\infty)$. Define a mapping $T : D \to D$ by

$$Tx = \frac{2x^3}{1+2x^2}, \quad \forall x \in D.$$

Then T is a monotonically increasing function with a fixed point $q = 0 \in D$. Define two functions $\Phi, \varphi : [0, +\infty) \to [0, +\infty)$ by $\Phi(t) = \frac{t^2}{1+2t^2}$ and $\varphi(t) = t^2$, respectively. Then Φ and φ are two strictly increasing continuous functions with $\Phi(0) = \varphi(0) = 0$. For all $x \in D$ and $q \in F(T)$, let $k_n = 1$. Then we obtain that

$$\left|T^n x - T^n q\right| \le |Tx| \le |x-q| = k_n |x-q|, \tag{2.19}$$

$$\begin{aligned}\langle T^n x - T^n q, j(x-q)\rangle &= \langle T^n x, j(x-0)\rangle \le \langle Tx, j(x-0)\rangle \\ &= \left\langle \frac{2x^3}{1+2x^2}, x\right\rangle = \frac{2x^4}{1+2x^2} \\ &= |x-q|^2 - \frac{|x-q|^2}{1+2|x-q|^2} \\ &= k_n|x-q|^2 - \Phi(|x-q|) \\ &= k_n|x-q|^2 - \frac{\varphi(|x-q|)}{1+\varphi(|x-q|)+|x-q|^2}. \end{aligned} \tag{2.20}$$

Then T is a generalized asymptotically Φ-hemi-contraction and a weak generalized asymptotically φ-hemi-contraction.

Example 2.7 Let $E = R$ be the set of real numbers with the usual norm and $R^+ = [0, +\infty)$. Define a mapping $T : R^+ \to R$ by

$$Tx = \frac{x + x^3 + x^{5/2} - x^{1/2}}{1 + x^{3/2} + x^2}, \quad \forall x \in R^+.$$

Then T has a fixed point $q = 0 \in R^+$. Define a function $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = t^{3/2}$. Then φ is a strictly increasing continuous function with $\varphi(0) = 0$. For all $x \in R^+$ and $q \in F(T)$, let $n = 1$ and $k_n = 1$. Then we have

$$\begin{aligned}\langle Tx - Tq, j(x-q)\rangle &= \left\langle \frac{x + x^3 + x^{5/2} - x^{1/2}}{1 + x^{3/2} + x^2} - 0, j(x-0)\right\rangle \\ &= \left\langle \frac{x + x^3 + x^{5/2} - x^{1/2}}{1 + x^{3/2} + x^2}, x\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^2 + x^4 + x^{7/2} - x^{3/2}}{1 + x^{3/2} + x^2} \\
&= x^2 - \frac{x^{3/2}}{1 + x^{3/2} + x^2} \\
&= |x-q|^2 - \frac{|x-q|^{3/2}}{1 + |x-q|^{3/2} + |x-q|^2} \\
&= |x-q|^2 - \frac{\varphi(|x-q|)}{1 + \varphi(|x-q|) + |x-q|^2}.
\end{aligned} \tag{2.21}
$$

Then T is a weak generalized asymptotically φ-hemi-contraction, but not a generalized asymptotically Φ-hemi-contraction with $n = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper and read and approved the final manuscript.

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