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A note on higher-order Bernoulli polynomials

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Abstract

Let $\mathbb{P}_n = \{p(x) \in \mathbf{Q}[x] \mid \deg p(x) \le n\}$ be the $(n+1)$-dimensional vector space over $\mathbf{Q}$. From the property of the basis $B_0^{(r)}, B_1^{(r)}, \ldots, B_n^{(r)}$ for the space $\mathbb{P}_n$, we derive some interesting identities of higher-order Bernoulli polynomials.

1 Introduction

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$. For a fixed $r \in \mathbf{Z}_+$, the nth Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = e^{B^{(r)}(x)t} = \sum_{n=0}^{\infty} \frac{B_n^{(r)}(x)t^n}{n!} \quad \text{(see [1–11])} \tag{1}$$

with the usual convention about replacing $(B^{(r)}(x))^n$ by $B_n^{(r)}(x)$. In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the nth Bernoulli numbers of order r.

From (1), we note that

$$\begin{aligned} B_n^{(r)}(x) &= \sum_{k=0}^{n} \binom{n}{k} B_k^{(r)} x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(r)} x^k \\ &= \sum_{n_1+\cdots+n_r+n_{r+1}=n} \binom{n}{n_1, \ldots, n_r, n_{r+1}} B_{n_1} \cdots B_{n_r} x^{n_{r+1}}. \end{aligned} \tag{2}$$

Thus, by (2) we get the Euler-type sums of products of Bernoulli numbers as follows:

$$B_n^{(r)} = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} B_{n_1} B_{n_2} \cdots B_{n_r} \quad \text{(see [11–17])}. \tag{3}$$

By (2) and (3), we see that $B_n^{(r)}(x)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}$.

From (2), we note that

$$\left(B_n^{(r)}(x)\right)' = \frac{d}{dx} B_n^{(r)}(x) = n B_{n-1}^{(r)}(x) \quad \text{(see [11–17])} \tag{4}$$

and

$$B_n^{(r)}(x+1) - B_n^{(r)}(x) = n B_{n-1}^{(r-1)}(x). \tag{5}$$

Let Ω denote the space of real-valued differential functions on $(-\infty,\infty)=\mathbf{R}$. Now, we define three linear operators I, $\triangle$, D on Ω as follows:

$$If(x)=\int_x^{x+1} f(t)\,dt, \qquad \triangle f(x)=f(x+1)-f(x), \qquad Df(x)=f'(x). \tag{6}$$

Then we see that (i) $DI=ID=\triangle$, (ii) $\triangle I=I\triangle$, (iii) $\triangle D=D\triangle$.

Let $\mathbb{P}_n=\{p(x)\in\mathbf{Q}(x)|\deg p(x)\le n\}$ be the $(n+1)$-dimensional vector space over $\mathbf{Q}$. Probably, $\{1,x,\ldots,x^n\}$ is the most natural basis for this space. But $\{B_0^{(r)}(x),B_1^{(r)}(x),B_2^{(r)}(x),\ldots,B_n^{(r)}(x)\}$ is also a good basis for the space $\mathbb{P}_n$ for our purpose of arithmetical and combinatorial applications.

Let $p(x)\in\mathbb{P}_n$. Then $p(x)$ can be generated by $B_0^{(r)}(x),B_1^{(r)}(x),B_2^{(r)}(x),\ldots,B_n^{(r)}(x)$ as follows:

$$p(x)=\sum_{k=0}^{n} a_k B_k^{(r)}(x).$$

In this paper, we develop methods for uniquely determining a_k from the information of $p(x)$. From those methods, we derive some interesting identities of higher-order Bernoulli polynomials.

2 Higher-order Bernoulli polynomials

For $r=0$, by (1), we get $B_n^{(0)}=x^n$ $(n\in\mathbf{Z}_+)$. Let $p(x)\in\mathbb{P}_n$.

For a fixed $r\in\mathbf{Z}_+$, $p(x)$ can be generated by $B_0^{(r)}(x),B_1^{(r)}(x),B_2^{(r)}(x),\ldots,B_n^{(r)}(x)$ as follows:

$$p(x)=\sum_{k=0}^{n} a_k B_k^{(r)}(x). \tag{7}$$

From (6) and (7), we can derive the following identities:

$$\begin{aligned} IB_n^{(r)}(x) &= \int_x^{x+1} B_n^{(r)}(x)\,dx \\ &= \frac{1}{n+1}\bigl(B_{n+1}^{(r)}(x+1)-B_{n+1}^{(r)}(x)\bigr). \end{aligned} \tag{8}$$

By (5) and (8), we get

$$IB_n^{(r)}(x)=\frac{n+1}{n+1}B_n^{(r-1)}(x)=B_n^{(r-1)}(x). \tag{9}$$

It is easy to show that

$$\triangle B_n^{(r)}(x)=B_n^{(r)}(x+1)-B_n^{(r)}(x)=nB_{n-1}^{(r-1)}(x), \tag{10}$$

and

$$DB_n^{(r)}(x)=nB_{n-1}^{(r)}(x). \tag{11}$$

By (7) and (9), we get

$$I^r p(x) = \sum_{k=0}^{n} a_k B_k^{(0)}(x) = \sum_{k=0}^{n} a_k x^k. \tag{12}$$

From (6) and (12), we note that

$$D^k I^r p(x) = \sum_{l=k}^{n} a_l \frac{l!}{(l-k)!} x^{l-k}. \tag{13}$$

Thus, by (13) we get

$$D^k I^r p(0) = k! a_k. \tag{14}$$

Hence, from (14) we have

$$a_k = \frac{D^k I^r p(0)}{k!}. \tag{15}$$

Case 1. Let $r > n$. Then $r > k$ for all $k = 0, 1, 2, \dots, n$.
By (15), we get

$$\begin{aligned} a_k &= \frac{1}{k!} D^k I^k I^{r-k} p(0) = \frac{1}{k!} (DI)^k I^{r-k} p(0) \\ &= \frac{1}{k!} \triangle^k I^{r-k} p(0) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} I^{r-k} p(j). \end{aligned} \tag{16}$$

Case 2. Assume that $r \leq n$.
(i) For $0 \leq k \leq r$, by (15) we get

$$a_k = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} I^{r-k} p(j). \tag{17}$$

(ii) For $r \leq k \leq n$, by (15) we see that

$$\begin{aligned} a_k &= \frac{1}{k!} D^{k-r} D^r I^r p(0) = \frac{1}{k!} D^{k-r} (DI)^r p(0) = \frac{1}{k!} D^{k-r} \triangle^r p(0) \\ &= \frac{1}{k!} \triangle^r D^{k-r} p(0) = \frac{1}{k!} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} D^{k-r} p(j). \end{aligned} \tag{18}$$

Therefore, by (7), (16), (17) and (18), we obtain the following theorem.

Theorem 1
(a) *For $r > n$, we have*

$$p(x) = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \frac{1}{k!} (-1)^{k-j} \binom{k}{j} I^{r-k} p(j) \right) B_k^{(r)}(x).$$

(b) *For $r \leq n$, we have*

$$p(x) = \sum_{k=0}^{r-1}\left(\sum_{j=0}^{k}\frac{1}{k!}(-1)^{k-j}\binom{k}{j}I^{r-k}p(j)\right)B_k^{(r)}(x) + \sum_{k=r}^{n}\left(\sum_{j=0}^{r}\frac{1}{k!}(-1)^{r-j}D^{k-r}p(j)\right)B_k^{(r)}(x).$$

Let us take $p(x) = x^n \in \mathbb{P}_n$. Then x^n can be expressed as a linear combination of $B_0^{(r)}, B_1^{(r)}, \dots, B_n^{(r)}$. For $r > n$, we have

$$I^{r-k}x^n = \frac{n!}{(n+r-k)!}\sum_{l=0}^{r-k}(-1)^{r-k-l}\binom{r-k}{l}(x+l)^{n+r-k}. \tag{19}$$

Therefore, by Theorem 1 and (19), we obtain the following corollary.

Corollary 2 *For $n, r \in \mathbf{Z}_+$ with $r > n$, we have*

$$x^n = \sum_{k=0}^{n}\left\{\sum_{j=0}^{k}\sum_{l=0}^{r-k}(-1)^{r-j-l}\frac{n!\binom{k}{j}\binom{r-k}{l}}{k!(n+r-k)!}(j+l)^{n+r-k}\right\}B_k^{(r)}(x).$$

Let us assume that $r, n \in \mathbf{Z}_+$ with $r \leq n$. Observe that

$$D^{k-r}x^n = n(n-1)\cdots(n-k+r+1)x^{n-k+r} = \frac{n!}{(n-k+r)!}x^{n-k+r}. \tag{20}$$

Thus, by Theorem 1 and (20), we obtain the following corollary.

Corollary 3 *For $n, r \in \mathbf{Z}_+$ with $r \leq n$, we have*

$$x^n = \sum_{k=0}^{r-1}\left\{\sum_{j=0}^{k}\sum_{l=0}^{r-k}(-1)^{r-j-l}\frac{n!\binom{k}{j}\binom{r-k}{l}}{k!(n+r-k)!}(j+l)^{n+r-k}\right\}B_k^{(r)}(x) + \sum_{k=r}^{n}\left\{\sum_{j=0}^{r}(-1)^{r-j}\frac{n!\binom{r}{j}}{k!(n+r-k)!}j^{n+r-k}\right\}B_k^{(r)}(x).$$

Let us take $p(x) = B_n^{(s)}(x) \in \mathbb{P}_n$ ($s \in \mathbf{Z}_+$). Then $p(x)$ can be generated by $\{B_0^{(r)}(x), B_1^{(r)}(x), \dots, B_n^{(r)}(x)\}$ as follows:

$$B_n^{(s)}(x) = \sum_{k=0}^{n}a_k B_k^{(r)}(x). \tag{21}$$

For $r > n$, we have

$$I^{r-k}B_n^{(s)}(x) = B_n^{(s-r+k)}(x). \tag{22}$$

Thus, by Theorem 1 and (22), we obtain the following theorem.

Theorem 4 *For $r, n, s \in \mathbf{Z}_+$ with $r > n$, we have*

$$B_n^{(s)}(x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} (-1)^{k-j} \frac{1}{k!} \binom{k}{j} B_n^{(s+k-r)}(j) \right\} B_k^{(r)}(x).$$

In particular, for $r = s$, we have

$$\begin{aligned} B_n^{(r)}(x) &= 0B_0^{(r)}(x) + 0B_1^{(r)}(x) + \cdots + 0B_{n-1}^{(r)}(x) + 1B_n^{(r)}(x) \\ &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} (-1)^{k-j} \frac{1}{k!} \binom{k}{j} B_n^{(k)}(j) \right\} B_k^{(r)}(x). \end{aligned} \tag{23}$$

By comparing coefficients on the both sides of (23), we get

$$\sum_{j=0}^{k} (-1)^{k-j} \frac{1}{k!} \binom{k}{j} B_n^{(k)}(j) = \delta_{kn}, \quad \text{for } 0 \le k \le n. \tag{24}$$

Therefore, by (24), we obtain the following corollary.

Corollary 5

(a) *For $n, k \in \mathbf{Z}_+$ with $0 \le k \le n-1$, we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n^{(k)}(j) = 0.$$

(b) *In particular, $k = n$, we get*

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_n^{(n)}(j) = n!.$$

Let us assume that $r \le n$ in (21). Then we have

$$D^{k-r} B_n^{(s)}(x) = n(n-1)\cdots(n-k+r+1) B_{n+r-k}^{(s)}(x) = \frac{n!}{(n-k+r!)} B_{n+r-k}^{(s)}(x). \tag{25}$$

Therefore, by Theorem 1, (21) and (25), we obtain the following theorem.

Theorem 6 *For $r, n \in \mathbf{Z}_+$ with $r \le n$, we have*

$$\begin{aligned} B_n^{(s)}(x) &= \sum_{k=0}^{n-1} \left\{ \sum_{j=0}^{k} (-1)^{k-j} \frac{1}{k!} \binom{k}{j} B_n^{(s+k-r)}(j) \right\} B_k^{(r)}(x) \\ &\quad + \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} (-1)^{r-j} \frac{n! \binom{r}{j}}{k!(n+r-k)!} B_{n+r-k}^{(s)}(j) \right\} B_k^{(r)}(x). \end{aligned}$$

Let $p(x) = E_n^{(s)}(x)$ ($s \in \mathbf{Z}_+$) be Euler polynomials of order s. Then $E_n^{(s)}$ can be expressed as a linear combination of $B_0^{(r)}(x), B_1^{(r)}(x), \ldots, B_n^{(r)}(x)$.

Assume that $r, n \in \mathbf{Z}_+$ with $r > n$.

By (6), we get

$$I^{r-k}E_n^{(s)}(x) = \frac{1}{(n+1)\cdots(n+r-k)}\sum_{l=0}^{r-k}(-1)^{r-k-l}\binom{r-k}{l}E_{n+r-k}^{(s)}(x+l)$$
$$= \frac{n!}{(n+r-k)!}\sum_{l=0}^{r-k}(-1)^{r-k-l}\binom{r-k}{l}E_{n+r-k}^{(s)}(x+l). \tag{26}$$

Therefore, by Theorem 1 and (26), we obtain the following theorem.

Theorem 7 *For $r, n \in \mathbf{Z}_+$ with $r > n$, we have*

$$E_n^{(s)}(x) = \sum_{k=0}^{n}\left\{\sum_{j=0}^{k}\sum_{l=0}^{r-k}(-1)^{r-j-l}\frac{n!\binom{k}{j}\binom{r-k}{l}}{k!(n+r-k)!}E_{n+r-k}^{(s)}(j+l)\right\}B_k^{(r)}(x).$$

For $r, n \in \mathbf{Z}_+$ with $r \leq n$, we have

$$D^{k-r}E_n^{(s)}(x) = n(n-1)\cdots(n-k+r+1)E_{n-k+r}^{(s)}(x). \tag{27}$$

By Theorem 1 and (27), we obtain the following theorem.

Theorem 8 *For $r, n \in \mathbf{Z}_+$ with $r \leq n$, we have*

$$E_n^{(s)}(x) = \sum_{k=0}^{r-1}\left\{\sum_{j=0}^{k}\sum_{l=0}^{r-k}(-1)^{r-j-l}\frac{n!\binom{k}{j}\binom{r-k}{l}}{k!(n+r-k)!}E_{n+r-k}^{(s)}(j+l)\right\}B_k^{(r)}(x)$$
$$+ \sum_{k=r}^{n}\left\{\sum_{j=0}^{r}(-1)^{r-j}\frac{n!\binom{r}{k}}{k!(n+r-k)!}E_{n+r-k}^{(s)}(j)\right\}B_k^{(r)}(x).$$

Remarks (a) For $r \leq 0$, by (40) we get

$$I^r x^n = \frac{n!}{(n+r)!}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(x+j)^{n+r} = \frac{1}{\binom{n+r}{r}}\frac{1}{r!}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}(x+j)^{n+r}.$$

Thus, for $x = 0$, we have

$$I^r x^n|_{x=0} = \frac{n!}{(n+r)!}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}j^{n+r} = \frac{1}{\binom{n+r}{r}}\frac{1}{r!}\triangle^r 0^{n+r} = \frac{S(n+r,r)}{\binom{n+r}{r}}, \tag{28}$$

where $S(n,r)$ is the Stirling number of the second kind.

(b) Assume

$$\sum_{k=0}^{n}\alpha_k x^k = \sum_{k=0}^{n}a_k B_k^{(r)}(x) \quad (r \geq 0). \tag{29}$$

Applying I^t on both sides ($t \geq 0$), we get

$$\sum_{k=0}^{n} a_k B_k^{(r-t)}(x) = \sum_{k=0}^{n} \alpha_k I^t x^k = \sum_{k=0}^{n} \frac{\alpha_k}{\binom{\alpha+t}{t}} \frac{1}{t!} \sum_{j=0}^{t} (-1)^{(t-j)} \binom{t}{j} (x+j)^{k+t}. \tag{30}$$

From (28) and (30), we have

$$\sum_{k=0}^{n} a_k B_k^{(r-t)} = \sum_{k=0}^{n} \frac{\alpha_k}{\binom{\alpha+t}{t}} S(k+t,t).$$

Remark Let us define two operators d, $\tilde{d}$ as follows:

$$d = e^{-D} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} D^n, \qquad \tilde{d} = e^{D} = \sum_{n=0}^{\infty} \frac{D^n}{n!}. \tag{31}$$

From (31), we note that

$$\begin{aligned} \tilde{d}x^n &= \sum_{l=0}^{n} \binom{n}{l} x^{n-l} = (x+1)^n, \\ dx^n &= \sum_{l=0}^{n} \binom{n}{l} (-1)^l x^{n-l} = (x-1)^n. \end{aligned} \tag{32}$$

Thus, by (31) and (32), we get

$$\tilde{d}B_n^{(r)}(x) = B_n^{(r)}(x+1), \qquad dB_n^{(r)}(x) = B_n^{(r)}(x-1), \tag{33}$$

and

$$\tilde{d}E_n^{(r)}(x) = E_n^{(r)}(x+1), \qquad dE_n^{(r)}(x) = E_n^{(r)}(x-1). \tag{34}$$

3 Further remarks

For any $r_0, r_1, \ldots, r_n \in \mathbf{Z}_+$, $\{B_0^{(r_0)}(x), B_1^{(r_1)}(x), \ldots, B_0^{(r_n)}(x)\}$ forms a basis for $\mathbb{P}_n$. Let $r = \max\{r_i | i = 0,1,2,\ldots,n\}$. Let $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expressed as a linear combination of $B_0^{(r_0)}(x), B_1^{(r_1)}(x), \ldots, B_n^{(r_n)}(x)$ as follows:

$$p(x) = a_0 B_0^{(r_0)}(x) + a_1 B_1^{(r_1)}(x) + \cdots + a_n B_n^{(r_n)}(x) = \sum_{l=0}^{n} a_l B_l^{(r_l)}(x). \tag{35}$$

Thus, by (6) and (35), we get

$$\begin{aligned} I^r p(x) &= \sum_{l=0}^{n} a_l I^r B_l^{(r_l)}(x) \\ &= \sum_{l=0}^{n} a_l I^{r-r_l} I^{r_l} B_l^{(r_l)}(x) = \sum_{l=0}^{n} a_l I^{r-r_l} B_l^{(0)}(x) \\ &= \sum_{l=0}^{n} a_l I^{r-r_l} x^l. \end{aligned} \tag{36}$$

Now, for each $k = 0, 1, 2, \dots, n$, by (36) we get

$$\begin{aligned} D^k I^r p(x) &= \sum_{l=0}^{n} a_l D^k I^{r-r_l} x^l = \sum_{l=0}^{n} a_l I^{r-r_l}\left(D^k x^l\right) \\ &= \sum_{l=k}^{n} a_l I^{r-r_l}\left(\frac{l!}{(l-k)!} x^{l-k}\right) = \sum_{l=k}^{n} \frac{a_l l!}{(l-k)!} I^{r-r_l} x^{l-k}. \end{aligned} \tag{37}$$

Let us take $x = 0$ in (37). Then, by (28) and (37), we get

$$\begin{aligned} D^k I^r p(0) &= \sum_{l=k}^{n} \frac{l! a_l}{(l-k)!} \times \frac{S(l-k+r-r_l, r-r_l)}{\binom{l-k+r-r_l}{r-r_l}} \\ &= \sum_{l=k}^{n} \frac{a_l (r-r_l)! l!}{(l-k)!} S(l-k+r-r_l, r-r_l). \end{aligned} \tag{38}$$

Case 1. For $r > n$, we have

$$\begin{aligned} D^k I^r p(0) &= D^k I^k I^{r-k} p(0) = (DI)^k I^{r-k} p(0) = \triangle^k I^{r-k} p(0) \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} I^{r-k} p(j). \end{aligned} \tag{39}$$

Case 2. Let $r \le n$.

(i) For $0 \le k < r$, we have

$$D^k I^r p(0) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} I^{r-k} p(j). \tag{40}$$

(ii) For $r \le k \le n$, we have

$$\begin{aligned} D^k I^r p(0) &= D^{k-r} D^r I^r p(0) = D^{k-r} (DI)^r p(0) = D^{k-r} \triangle^r p(0) \\ &= \triangle^r D^{k-r} p(0) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D^{k-r} p(j). \end{aligned} \tag{41}$$

Thus, by (38), (39), (40) and (41), we can determine $a_0, a_1, a_2, \dots, a_n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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References

1. Abramowitz, M, Stegun, IA: Handbook of Mathematical Functions. Natl. Bur. of Standards, Washington (1964)
2. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math - Jang'jun Math. Soc. **22**(3), 399-406 (2012)
3. Bayad, A, Kim, T: Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials. Russ. J. Math. Phys. **18**(2), 133-143 (2011)
4. Gould, HW: Explicit formulas for Bernoulli numbers. Am. Math. Mon. **79**, 44-51 (1972)
5. Kim, T: Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials. J. Differ. Equ. Appl. **14**(12), 1267-1277 (2008)
6. Kim, DS, Kim, T, Kim, YH, Dolgy, DV: A note on Eulerian polynomials associated with Bernoulli and Euler numbers and polynomials. Adv. Stud. Contemp. Math - Jang'jun Math. Soc. **22**(3), 379-389 (2012)
7. Kim, DS, Kim, T: Euler basis, identities, and their applications. Int. J. Math. Math. Sci. **2012**, Article ID 343981 (2012). doi:10.1155/2012/343981
8. Kim, DS, Kim, T: Bernoulli basis and the product of several Bernoulli polynomials. Int. J. Math. Math. Sci. **2012**, Article ID 463659 (2012). doi:10.1155/2012/463659
9. Kim, T: Sums of products of q-Bernoulli numbers. Arch. Math. **76**(3), 190-195 (2001)
10. Kim, T, Adiga, C: Sums of products of generalized Bernoulli numbers. Int. Math. J. **5**(1), 1-7 (2004)
11. Kim, T: Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**(1), 93-96 (2009)
12. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory **132**(12), 2854-2865 (2012)
13. Kim, T: Some formulae for the q-Bernstein polynomials and q-deformed binomial distributions. J. Comput. Anal. Appl. **14**(5), 917-933 (2012)
14. Petojević, A: New sums of products of Bernoulli numbers. Integral Transforms Spec. Funct. **19**(1-2), 105-114 (2008)
15. Rim, S-H, Jeong, J: On the modified q-Euler numbers of higher order with weight. Adv. Stud. Contemp. Math - Jang'jun Math. Soc. **22**(1), 93-98 (2012)
16. Ryoo, CS: Some relations between twisted q-Euler numbers and Bernstein polynomials. Adv. Stud. Contemp. Math - Jang'jun Math. Soc. **21**(2), 217-223 (2011)
17. Kim, T: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$. Rocky Mt. J. Math. **41**(1), 239-247 (2011)

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