A generalised frequency detuning method for multi-degree-of-freedom oscillators with nonlinear stiffness

S.A. Neild and D.J. Wagg
Department of Mechanical Engineering,
University of Bristol, Bristol, BS8 1TR

Citation: Nonlinear Dynamics, **73**:649-663, 2013.
doi:10.1007/s11071-013-0818-7

Abstract

In this paper we derive a frequency detuning method for multi-degree-of-freedom oscillators with nonlinear stiffness. This approach includes a matrix of detuning parameters which are used to model the amplitude dependent variation in resonant frequencies for the system. As a result, we compare three different approximations for modelling the affect of the nonlinear stiffness on the linearized frequency of the system. In each case the response of the primary resonances can be captured with the same level of accuracy. However harmonic and subharmonic responses away from the primary response are captured with significant differences in accuracy. The detuning analysis is carried out using a normal form technique, and the analytical results are compared with numerical simulations of the response. Two examples are considered, the second of which is a two degree-of-freedom oscillator with cubic stiffnesses.

1 Introduction

In this paper a generalised frequency detuning method is derived for multi-degree-of-freedom oscillators with nonlinear stiffness. We assume that the oscillator is forced sinusoidally and has viscous damping. This type of oscillator system will have a series of primary resonant frequencies and, because of the nonlinear terms, a series of secondary resonances at harmonic, subharmonic and potentially also combination frequencies. In this study we are concerned with finding an approximate solution which captures all these responses with the highest level of accuracy.

We will consider three different definitions of linearized frequency of the

system, using either (i) tangent stiffness, (ii) nonlinear resonant frequency, or (iii) frequency detuning. For unforced systems, resonant and detuning are the same, but for forced systems (with light damping), these two cases will normally be slightly different. What we show in the examples is that in each case the response of the primary resonance can be captured with the same level of accuracy. However harmonic and subharmonic responses away from the primary response show considerable variations in accuracy. For the examples considered here the nonlinear resonant frequency approach gives the most accurate results, and we derive a generalised method based on a normal form approach which can be applied to both single- and multi-degree-of-freedom oscillators.

In terms of practical motivation, resonances between primary and/or secondary resonant frequencies are important for a wide range of physical applications — see for example [1, 2, 3, 4, 5, 6, 7]. Typically models of the these type of systems are in the form of weakly nonlinear oscillators. Analytical approximations, such as harmonic balance, averaging, multiple scales and other perturbation techniques are often used to study these type of oscillator systems [8, 9, 10]. In the case where there is more than a single-degree-of-freedom, analysis of this type becomes significantly more complex, because for each primary resonance, there can be multiple secondary resonances [8, 11]. For example, in the case of musical instruments, an integer (or near integer) relationship between primary and secondary resonances is deliberately exploited to give an instrument its characteristic sound quality [1].

The analytical method we use here a version of normal forms [12, 13, 14, 15, 16]. Normal forms is usually applied to first-order nonlinear oscillator equations and an assessment of their accuracy is given in [17]. Here we consider a recently developed formulation that can be applied directly to second-order nonlinear oscillators directly, termed *second-order normal forms* [18, 19]. The second-order normal form technique has the useful property that the nonlinear transform removes non-resonant terms for each mode, rather than for each state, as is the case using the first-order formulation. These non-resonant terms are those in the equation of motion that result in harmonics of the primary resonance, which is either the natural frequency (in the case of the unforced system) or of the dominant response frequency (in the case of forced systems). Transforming these terms out of the equations of motion, for the nth mode, allows the use of a trial solution of the form $U_n \cos(\omega_{rn} t - \phi_n)$ to solve the equation exactly, thereby removing the need for a harmonic balance type approximation. By generalising this approach to include a detuning matrix, we can obtain excellent approximate solutions for the behaviour of both the primary and secondary resonances.

In Section 2 the generalised detuning method is derived based on the second-order normal form technique from [18], and the effect of the selection of linearised frequency has on the predicted response is analysed. Then in Section 3, the first example we give is of a single degree-of-freedom Duffing oscillator, which is used to shown that the detuning approximation is equivalent to linearising the system using the resonant frequency. In Section 4 a two mode system is introduced and it is shown that linearising the system using the resonant frequencies gives better results than using the detuning approximation in calculating the non-resonant

response of the system. To assess the accuracy of the detuning approximation we introduce a method of finding the second-order normal form to accuracy ε^2 and consider the small assumptions made in the derivation.

2 The generalised detuning method

Consider the N degree-of-freedom system in terms of the $N \times 1$ displacement vector $\mathbf{x}$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{K}_{nl}(\mathbf{x}) = \mathbf{P}_x\mathbf{r}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $N \times N$ mass, damping and stiffness matrices respectively. The nonlinear stiffness terms are represented by the $N \times 1$ vector $\mathbf{K}_{nl}$. Sinusoidal forcing is included as $\mathbf{P}_x\mathbf{r}$, where $\mathbf{P}_x$ is a $N \times 2$ forcing amplitude matrix and $\mathbf{r} = \{r_p \;\; r_m\}^T$ is a 2×1 forcing vector with $r_p = \mathrm{e}^{\mathrm{i}\Omega t}$ and $r_m = \mathrm{e}^{-\mathrm{i}\Omega t}$. The subscripts p and m indicate the sign of the complex exponential term, plus and minus respectively.

In the following analysis both the nonlinear and damping terms are assumed to be small. As a result, these terms can be grouped together to give

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{N}_x(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{P}_x\mathbf{r}, \tag{2}$$

where $\mathbf{N}_x = \mathbf{K}_{nl}(\mathbf{x}) + \mathbf{C}\dot{\mathbf{x}}$.

In this approach three transformations are applied to (2). Firstly a linear modal transformation, based on the unforced linear equivalent system, is applied: $\mathbf{x} \to \mathbf{q}$. This results in coupled modal equations of motion. The second and third transformations, a forcing transformation: $\mathbf{q} \to \mathbf{v}$ and a nonlinear near-identity transformation: $\mathbf{v} \to \mathbf{u}$ respectively, remove the non-resonant terms from each of these modal equations of motion. These transformations allow the resulting dynamic equation in $\mathbf{u}$ to be solved exactly using a trial solution for the nth mode of $u_n = U_n \cos(\omega_{rn}t - \phi_n)$. Information regarding the modal response at other frequencies is contained within the transformation equation relating $\mathbf{u}$ to $\mathbf{q}$.

Before considering each of these transformations, it is helpful to define some frequency terms that are used in the following discussion:

ω_{rn} the response frequency of the nth mode.

Ω the external forcing frequency.

ω_{0n} the undamped nonlinear resonant frequency of the nth mode.

$\omega_{\gamma n}$ a linearised approximation to the the undamped nonlinear resonant frequency of the nth mode.

ω_{nn} the undamped natural frequency of the nth mode of the linearised system (i.e. when all nonlinear terms are set to zero).

ω_{an} the natural frequency of the nth mode used in the normal form calculation (see section 2.3).

Considering a one degree-of-freedom system, if there is near-resonant forcing then we can write $\omega_r = \Omega$. Alternatively if there is no forcing or damping then we have $\omega_r = \omega_0$. Note that the subscript n is dropped when discussing a one degree-of-freedom system.

2.1 Linear modal transformation: $\mathbf{x} \to \mathbf{q}$

First the undamped linear terms are decoupled using a linear modal transform. Consider the unforced linear form of the equation of motion, $\ddot{\mathbf{x}} + \mathbf{M}^{-1}\mathbf{K}\mathbf{x} = 0$. Eigenvector analysis can be used to find the mode shape (i.e. eigenvector) matrix $\mathbf{\Phi}$ and the corresponding natural frequency diagonal matrix $\mathbf{\Lambda}$ (via eigenvalues), in which the nth diagonal element is ω_{nn}^2 – the square of the nth linear undamped natrual frequency.

By applying the transform $\mathbf{x} = \mathbf{\Phi}\mathbf{q}$, where $\mathbf{q}$ are the modal co-ordinates, and noting that by definition $\mathbf{M}^{-1}\mathbf{K}\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Lambda}$, (2) may be written as

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} + \mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{P}_q\mathbf{r}, \tag{3}$$

where

$$\begin{aligned} \mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) &= (\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{N}_x(\mathbf{\Phi}\mathbf{q}, \mathbf{\Phi}\dot{\mathbf{q}}) \\ \mathbf{P}_q &= (\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{P}_x. \end{aligned} \tag{4}$$

Here the original equations of motion were for discrete locations, as are typical when using FE or spring-mass models. If, instead, the Galerkin technique is applied to partial differential equations, the resulting equations of motion are in the form of (3), so this first transformation is unnecessary.

The diagonal matrix $\mathbf{\Lambda}$ consists of the (squared) natural frequencies of the linear system (when $\mathbf{N}_q$ is set to zero). However these are not normally the best estimate of the frequencies of the nonlinear system. This raises the question: what is the effect of linearising the system using different approximations to natural frequencies, such as the resonant response frequencies? To assess this we introduce a diagonal matrix of linearised natural frequencies for the nonlinear system, $\mathbf{\Lambda}_\gamma$ with $\mathbf{\Lambda}_\gamma = \mathbf{\Lambda} + \Gamma$ and with the nth diagonal element in $\mathbf{\Lambda}_\gamma$ and $\mathbf{\Gamma}$ being $\omega_{\gamma n}^2$ and γ_n respectively, where γ_n will be used as a general frequency shift parameter. With this the equation for the system dynamics becomes

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}_\gamma\mathbf{q} + (\mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{\Gamma}\mathbf{q}) = \mathbf{P}_q\mathbf{r}, \qquad \mathbf{\Lambda}_\gamma = \mathbf{\Lambda} + \Gamma \tag{5}$$

Now the small nonlinear term is given by $\mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{\Gamma}\mathbf{q}$, note that $\mathbf{\Gamma}\mathbf{q}$ is small as the new frequencies are close to the linear natural frequencies.

2.2 Force transformation: $\mathbf{q} \to \mathbf{v}$

The second transform is also linear and, for each mode, removes any non-resonant forcing terms in the modal equation of motion and places them in the $\mathbf{q} \to \mathbf{v}$ transform

$$\mathbf{q} = \mathbf{v} + \mathbf{e}\mathbf{r}, \tag{6}$$

where $\mathbf{e}$ has size $N \times 2$. Substituting this transformation into the modal equation of motion, (5), gives

$$\ddot{\mathbf{v}} + \mathbf{\Lambda}_\gamma \mathbf{v} + (\mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) - \mathbf{\Gamma}\mathbf{v}) = \mathbf{P}_v \mathbf{r}, \tag{7}$$

where the relationships between the pre- and post-transformed nonlinear and forcing terms are

$$\mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) = \mathbf{N}_q(\mathbf{v} + \mathbf{e}\mathbf{r}, \dot{\mathbf{v}} + \mathbf{e}\mathbf{W}\mathbf{r}). \tag{8}$$

$$\tilde{\mathbf{e}} + \mathbf{P}_v = \mathbf{P}_q, \quad \text{with} \quad \tilde{\mathbf{e}} = \mathbf{e}\mathbf{W}\mathbf{W} + \mathbf{\Lambda}\mathbf{e}, \tag{9}$$

respectively. Here $\mathbf{W}$ is a 2×2 diagonal matrix with the first and second diagonal values being $\mathrm{i}\Omega$ and $-\mathrm{i}\Omega$ respectively. Also the nth row ($n = 1, 2 \ldots N$) and kth column ($k = 1, 2$) of $\tilde{\mathbf{e}}$ may be written in terms of the corresponding element in $\mathbf{e}$ using

$$e_{n,k} = \tilde{e}_{n,k}/(\omega_{nn}^2 - \Omega^2). \tag{10}$$

Note that this relationship and hence the transform is unaffected by the introduction of Γ.

For each element in turn, one of two options can now be selected to satisfy (9) and hence define the transform matrix $\mathbf{e}$. Considering the (n, k)th element, if the forcing is close to the natural frequency (i.e. $\Omega \approx \omega_{nn}$) then the forcing term is kept in the equation of motion by writing

$$\text{Option F1:} \quad e_{n,k} = 0, \quad P_{v,n,k} = P_{q,n,k}, \tag{11}$$

where the n, k subscripts indicates the (n, k)th element. If, however, the (n, k)th element corresponds to a forcing term that is not approximately equal to resonance then it is transformed out of the equation of motion by writing

$$\text{Option F2:} \quad e_{n,k} = P_{q,n,k}/(\omega_{nn}^2 - \Omega^2), \quad P_{v,n,k} = 0, \tag{12}$$

using (10).

2.3 Nonlinear near-identity transformation: $\mathbf{v} \to \mathbf{u}$

The third transformation removes the non-resonant terms from the equations of motion and places them in a nonlinear transformation. This results in (7) being transformed into a form that can be solved using a single frequency trial solution for each mode, thereby eliminating the need for a harmonic balance type approximation.

To keep track of the relative size of the different terms, small terms are marked with ε which may be seen as a book-keeping aid [20]. As the nonlinear and damping terms are small, $\mathbf{N}_v$ can be expressed as a power series of ε starting with an ε^1 term. In addition the term arising from choice of linearised natural frequencies, $\mathbf{\Gamma}\mathbf{v}$, has been taken to be small, order ε, giving

$$\ddot{\mathbf{v}} + \mathbf{\Lambda}_\gamma \mathbf{v} + (\mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) - \mathbf{\Gamma}\mathbf{v}) = \mathbf{P}_v \mathbf{r} \tag{13}$$

$$\text{with } \mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) = \varepsilon \mathbf{n}_{v1} + \varepsilon^2 \mathbf{n}_{v2} + \ldots, \quad \mathbf{\Gamma}\mathbf{v} = -\varepsilon \mathbf{n}_{v\gamma}(\mathbf{v}),$$

The near-identity nonlinear transform that is applied to this equation is

$$\mathbf{v} = \mathbf{u} + \mathbf{h}(\mathbf{u}, \dot{\mathbf{u}}) + \varepsilon \mathbf{h}_\gamma, \quad \text{with } \mathbf{h}(\mathbf{u}, \dot{\mathbf{u}}) = \varepsilon \mathbf{h}_1 + \varepsilon^2 \mathbf{h}_2 + \dots. \tag{14}$$

Note that there is no ε^0 term as the transform is near-identity and therefore $\mathbf{h}$ is small. This transform results in the equations of motion in terms of $\mathbf{u}$

$$\ddot{\mathbf{u}} + \mathbf{\Lambda}_\gamma \mathbf{u} + \mathbf{N}_u(\mathbf{u}, \dot{\mathbf{u}}) + \varepsilon \mathbf{n}_{u\gamma} = \mathbf{P}_u \mathbf{r}, \quad \text{with } \mathbf{N}_u(\mathbf{u}, \dot{\mathbf{u}}) = \varepsilon \mathbf{n}_{u1} + \varepsilon^2 \mathbf{n}_{u2} + \dots. \tag{15}$$

Again the nonlinear terms have been expressed as a power series of ε starting with ε^1 to reflect the assumption that they are small. Note also that the terms $\mathbf{h}_\gamma(\mathbf{u})$ and $\mathbf{n}_{u\gamma}(\mathbf{u})$ has been introduced to account for additional terms due to the $\mathbf{n}_{v\gamma}(\mathbf{v})$ term in (13).

The state vector $\mathbf{v}$ can be eliminated from (13) using (14) and then $\ddot{\mathbf{u}}$ can be eliminated using (15), to produce

$$\begin{aligned} \mathbf{P}_u \mathbf{r} - \varepsilon[\mathbf{n}_{u1}(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{n}_{u\gamma}(\mathbf{u})] + \varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{h}_\gamma(\mathbf{u})) + \\ \varepsilon \mathbf{\Lambda}_\gamma[\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}})) + \mathbf{h}_\gamma(\mathbf{u})] + \varepsilon[\mathbf{n}_{v1}(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{n}_{v\gamma}(\mathbf{u})] = \mathbf{P}_v \mathbf{r} + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{16}$$

where a Taylor series expansion has been applied to the terms $\mathbf{n}_{v1}$ and $\mathbf{n}_{v\gamma}$. Equating the zero and first-order powers of ε produces

$$\varepsilon^0 : \qquad \mathbf{P}_u \mathbf{r} = \mathbf{P}_v \mathbf{r}, \tag{17}$$

$$\varepsilon^1 : \quad \mathbf{n}_{u1}(\mathbf{u}, \dot{\mathbf{u}}) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}})) = \mathbf{\Lambda}_\gamma \mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{n}_{v1}(\mathbf{u}, \dot{\mathbf{u}}), \tag{18}$$

$$\varepsilon^1 : \qquad \mathbf{n}_{u\gamma}(\mathbf{u}) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{h}_\gamma(\mathbf{u})) = \mathbf{\Lambda}_\gamma \mathbf{h}_\gamma(\mathbf{u}) + \mathbf{n}_{v\gamma}(\mathbf{u}). \tag{19}$$

Here the ε^1 equation has been split into two parts, the second dealing with the $\mathbf{n}_{v\gamma}$ terms and mirrored terms in the transform and resulting dynamic equation. The ε^2 relationship is derived in the Appendix.

The ε^0 equation is satisfied by setting $\mathbf{P}_u = \mathbf{P}_v$. To satisfy the ε^1 equation, (18), the form of the response needs to be considered. Since the near-identity transform removes non-resonant nonlinear terms from the equations of motion, the response for each state $u_1, u_2 \dots u_N$ is at a single response frequency, $\omega_{r1}, \omega_{r2}, \dots \omega_{rN}$. The differential equation in $\mathbf{u}$ is second-order, so trial solutions for the states must consist of both positive and negative complex exponential terms giving $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_m$, with the nth row of this vector expression being written as

$$u_n = u_{np} + u_{nm} : \quad u_{np} = (U_n \mathrm{e}^{-\mathrm{i}\phi_n}/2)\mathrm{e}^{\mathrm{i}\omega_{rn}t}, \quad u_{nm} = (U_n \mathrm{e}^{\mathrm{i}\phi_n}/2)\mathrm{e}^{-\mathrm{i}\omega_{rn}t}. \tag{20}$$

This results in the form of solution $u_n = U_n \cos(\omega_{rn}t - \phi_n)$ and therefore U_n is taken to be real to ensure a real response to the real excitation. The time derivatives of $\mathbf{u}$ may now be written as $\dot{\mathbf{u}} = \mathbf{\Upsilon}(\mathbf{u}_p - \mathbf{u}_m)$ and $\ddot{\mathbf{u}} = \mathbf{\Upsilon}^2(\mathbf{u}_p + \mathbf{u}_m)$, where nth diagonal element of diagonal matrix $\mathbf{\Upsilon}$ is $\mathrm{i}\omega_{rn}$.

It is at this stage that a detuning approximation is normally applied (see [[18]], for more details). The approximation is based on the fact that the response of u_n will be close to the nth natural frequency, i.e $\omega_{rn} \approx \omega_n$ (since only the resonant forcing terms are present $\mathbf{u}$). Therefore as the nth diagonal elements of matrices $\mathbf{\Lambda}_\gamma$ and $\mathbf{\Upsilon}^2$ are ω_γ^2 and $-\omega_{rn}^2$ respectively, it can be seen that these matrices are similar (but opposite sign). Hence we can write $\mathbf{\Lambda}_\gamma = -\mathbf{\Upsilon}^2 + \mathcal{O}(\varepsilon^1)$ such that $\mathbf{\Lambda}_\gamma$ can be replaced by $-\mathbf{\Upsilon}^2$ in the order ε^1 equation, (18). It is this detuning approximation that we will discuss in this paper.

Here we will be more general and replace $\mathbf{\Lambda}_\gamma$ with $\mathbf{A}^2$ in (18) giving

$$\mathbf{n}_{u1}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{r}) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{r})\right) = \mathbf{A}^2\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{r}) + \mathbf{n}_{v1}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{r}), \tag{21}$$

where $\mathbf{A}^2 = \mathbf{\Lambda}_\gamma$ if no detuning is applied or $\mathbf{A}^2 = -\mathbf{\Upsilon}^2$ if the detuning approximation is used. We define ω_{an}^2 as the nth element in the diagonal matrix $\mathbf{A}^2$.

To proceed, a vector $\mathbf{u}^*$ (of length L) is specified. It contains all the combinations of u_{np}, u_{nm} $(1 \leq n \leq N)$, r_p and r_m terms that are present in $\mathbf{n}_{v1}(\mathbf{u}, \dot{\mathbf{u}})$, allowing the expressions

$$\begin{aligned} \mathbf{n}_{v1}(\mathbf{u}, \dot{\mathbf{u}}) =& \mathbf{n}_{v1}^* \mathbf{u}^*(\mathbf{u_p}, \mathbf{u_m}), \\ \mathbf{n}_{u1}(\mathbf{u}, \dot{\mathbf{u}}) =& \mathbf{n}_{u1}^* \mathbf{u}^*(\mathbf{u_p}, \mathbf{u_m}), \\ \mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}) =& \mathbf{h}_1^* \mathbf{u}^*(\mathbf{u_p}, \mathbf{u_m}), \end{aligned} \tag{22}$$

to be defined, where $\mathbf{n}_{v\gamma 1}^*$, $\mathbf{n}_{u1}^*$ and $\mathbf{h}_1^*$ are coefficient matrices of size $N \times L$ and $\mathbf{n}_{u1}^*$ and $\mathbf{h}_1^*$ are as yet unknown. To simplify (21), the general form of the ℓ^{th} element in vector $\mathbf{u}^*$ is written as

$$u_\ell^* = r_p^{m_{\ell p}} r_m^{m_{\ell m}} \prod_{n=1}^{N} \left\{ u_{np}^{s_{\ell np}} u_{nm}^{s_{\ell nm}} \right\}, \tag{23}$$

where m and s are powers for each of the terms. Substituting this expression along with (22) into (21) results in the relationship

$$\mathbf{n}_{u1}^* = \mathbf{n}_{v1}^* - \tilde{\mathbf{h}}_1^*. \tag{24}$$

In this equation $\tilde{\mathbf{h}}_1^*$ is size $N \times L$ and the element in the nth row and ℓth column of $\tilde{\mathbf{h}}_1^*$ is related to the same element in $\mathbf{h}_1^*$ via

$$\tilde{h}_{1,n,\ell}^* = \left(\left[(m_{\ell p} - m_{\ell m})\Omega + \sum_{n=1}^{N} \left\{ (s_{\ell np} - s_{\ell nm})\omega_{rn} \right\} \right]^2 - \omega_{an}^2 \right) h_{1,n,\ell}^* = \beta_{1,n,\ell} h_{1,n,\ell}^*, \tag{25}$$

where we define $\beta_{1,n,\ell}$ which relates $h_{1,n,\ell}^*$ to $\tilde{h}_{1,n,\ell}^*$. Please see [18, 19, 21] for more details of the derivation. Here we use the already defined ω_a parameter such that, either $\omega_{an} = \omega_{\gamma n}$ if no detuning is applied or $\omega_{an} = \omega_{rn}$ for the detuning case.

Now $\mathbf{n}^*_{u1}$ and $\mathbf{h}^*_1$ can be selected by considering the size of the $\beta_{1,n,\ell}$ terms. There are two options to satisfy (24). Considering each term in turn, where possible the term in $\mathbf{n}^*_{u1}$ is set to zero:

$$\text{Option N1 (non-resonant terms):} \quad n^*_{u1,n,\ell} = 0, \quad h^*_{1,n,\ell} = n^*_{v1,n,\ell}/\beta_{1,n,\ell}, \tag{26}$$

in which, for example, $n^*_{u1,n,\ell}$ is the (n,ℓ) element in $\mathbf{n}^*_{u1}$. However, in the cases where the term in $\mathbf{u}^*$ is *near-resonant*, $\beta_{1,n,\ell}$ is small and hence $h_{1,n,\ell}$ would be large if this option is selected. To avoid breaking the near-identity constraint, these near-resonant terms are kept in the equation of motion by setting

$$\text{Option N2 (near-resonant terms):} \quad n^*_{u1,n,\ell} = n^*_{v1,n,\ell}, \quad h^*_{1,n,\ell} = 0, \tag{27}$$

and as a result these terms are unaffected by the transform. Note that the selection of the resonant and non-resonant term is not effected by the detuning approximation or the selection of Γ as these are all small effects.

Now considering the ε^1 relationship between the terms relating directly to γ, (19), we can see that the form of the equation is identical to (18). Therefore, the method for selecting $\mathbf{h}_\gamma$ and $\mathbf{n}_{u\gamma}$ is identical to that for selecting $\mathbf{h}_1$ and $\mathbf{n}_{u1}$. As the form of $\mathbf{n}_{v\gamma}$ is known, we can write $\mathbf{n}^*_{v1}$ and $\mathbf{u}^*$, where $\mathbf{n}_{v\gamma} = \mathbf{n}^*_{v1}\mathbf{u}^*$ (as in (22)) to give

$$\mathbf{n}^*_{v1} = \begin{bmatrix} -\hat{\gamma}_1 & -\hat{\gamma}_1 & 0 & 0 \\ 0 & 0 & -\hat{\gamma}_2 & -\hat{\gamma}_2 \end{bmatrix}, \quad \mathbf{u}^* = \begin{bmatrix} u_{1p} & u_{1m} & u_{2p} & u_{2m} \end{bmatrix}^T, \tag{28}$$

for the case where the system has two degrees-of-freedom. By inspection of these expressions and (25), it can be seen that all the non-zero terms in $\mathbf{n}^*_{v1}$ equate to resonant terms. This means that we can write $\mathbf{n}_{u\gamma} = \mathbf{n}_{v\gamma}$ and $\mathbf{h}_\gamma = 0$ and simplify the near-identity nonlinear transform and transformed dynamics to

$$\mathbf{v} = \mathbf{u} + \mathbf{h}(\mathbf{u}, \dot{\mathbf{u}}), \tag{29}$$

$$\ddot{\mathbf{u}} + \mathbf{\Lambda}\mathbf{u} + \mathbf{N}_u(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{P}_u\mathbf{r}. \tag{30}$$

It is important to note that the dynamic equation is identical to that for the case where $\mathbf{\Gamma} = 0$ and the transform equation has the same form, however the individual non-zero terms in $\mathbf{h}$ are different due to ω_{an} being used in (25) and hence (26).

By adopting this method, the equation of motion for $\mathbf{u}$ can be solved exactly using the trial solution in the form $u_n = U_n \cos(\omega_{rn}t - \phi_n)$ for the nth mode. As the dynamic equation in $\mathbf{u}$ is unchanged by detuning or the use of Γ, the predicted resonant response is independent of these effects. Information regarding the response of each mode at other frequencies is contained within the transform equation $\mathbf{v} = \mathbf{u} + \mathbf{h}(\mathbf{u}, \dot{\mathbf{u}})$. Here the detuning approximation and the use of Γ have the same effect, namely they alter ω_{an} which appears in (25). Hence they both alter the magnitude of the non resonant response terms. Importantly, these two effect result in the same change to the form of

the equations. Hence the detuning approximation may be seen to be equivalent to linearisation around the response frequencies ω_{rn}, i.e. detuning is the same as writing $\omega_{\gamma n} = \omega_{rn}$.

To examine the detuning approximation the normal form technique will first be applied to a single degree-of-freedom unforced system and then in Section 4 to a two degree-of-freedom forced system. In the following discussion three options for the normal form will be considered:

- *tangent approach:* taking the linearised approximation to the undamped nonlinear resonant frequency, $\omega_{\gamma n}$, to be the natural frequency of the linear frequency without applying detuning, such that $\omega_{an} = \omega_{nn}$.
- *detuning approach:* applying detuning such that $\omega_{an} = \omega_{rn}$ – this is the method reported in [18].
- *resonant approach:* taking the linearised approximation to the undamped nonlinear resonant frequency to be the undamped nonlinear resonant frequency, ω_{0n}, without applying detuning, such that $\omega_{an} = \omega_{0n}$.

The choice of approach and hence the selection of ω_{an} only effects the value of β, (25). In turn, as already discussed, this only changes the magnitude of the near-resonant terms calculated using (26). As a result all three approaches can be implemented by considering the normal form transformation without the introduction of $\mathbf{\Gamma}$, i.e considering (3) rather than (5), along with the appropriate selection of ω_{an} in the equation for β, (25).

3 The Unforced Oscillator

Consider the unforced single degree-of-freedom system

$$\ddot{x} + \omega_n^2 x + X(x) = 0, \tag{31}$$

where X is a small, order ε, nonlinearity, and ω_n is the natural frequency for the case where there is no nonlinearity present, i.e. when $X(x) = 0$. Following the approach described above, first we introduce a small parameter γ by writing

$$\ddot{x} + \omega_\gamma^2 x + (X - \gamma x) = 0, \text{with } \omega_\gamma^2 = \omega_n^2 + \gamma. \tag{32}$$

Now the linearised natural frequency is ω_γ and the nonlinear term (which now contains a small linear component) is $X - \gamma x$.

To examine the linearised stiffness for this oscillator first consider a single-degree-of-freedom mass-spring system with mass m and nonlinear restoring force $kx + \alpha x^3$, Figure 1 shows the restoring force-displacement relationship. The dashed line is the force-displacement relationship for the linearised tangent stiffness, k, which defines $\omega_n = \sqrt{k/m}$ as in the *tangent* approach. It can be seen that the discrepancy between the solid and dashed lines increases with increasing magnitude of x indicating the increasing contribution due to the nonlinear

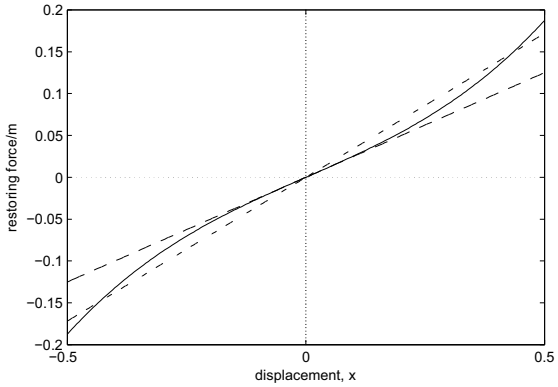

Figure 1: Force-displacement relationship for the nonlinear spring, in which $\alpha = 1$ and $\omega_n = 0.5$. The solid line is $F = kx + \alpha x^3$ and the linearised natural frequency lines are for the cases $\omega_\gamma = \omega_n$ (dashed) and $\omega_\gamma = \omega_0$, the resonant frequency (dotted) for the case where the harmonic oscillations are taken to be over a range $x = \pm A$ with $A = 0.5$.

term αx^3. The dotted line is the relationship for an alternative linearised system, based on the backbone stiffness $\ddot{x}+\omega_\gamma^2 x = 0$ where $\omega_\gamma = \omega_0$, i.e. the system is linearised about its resonant frequency as in the *resonant* approach (this is achieved to order ε^1 by setting $\gamma = 3\alpha A^2/4$ where oscillations are over the range $x = \pm A$). It can be seen that in this case the discrepancy, and therefore the nonlinear contribution, between the linear and nonlinear systems is smaller.

3.1 Applying the normal form technique

As discussed at the end of section 2, we do not need to analyse (32), we can analyse (31) instead provided we calculate β, (25), using the appropriate ω_a for the approach, either *tangent*, *detuning* or *resonant*, being considered. Considering (31), since the system has one degree-of-freedom and is unforced, the first two transforms are unity transforms, $x = q = v$, giving

$$\ddot{v} + \Lambda v + N_v(v) = 0, \quad \Lambda = \omega_n^2, \quad N_v(v) = \alpha V, \tag{33}$$

where $V = X(v)$. Note that where matrices are scalar the bold notation is dropped. The third transform, the near-identity transform, uses (14) to convert this equation into a dynamic equation of the form given in (15). To evaluate h and N_u, the nonlinear term $N_v(v)$ must be expressed as a power series in ε, (13). Taking the nonlinearity to be order ε^1, we write $N_v(v) = \varepsilon n_{v1}(v)$ where $n_{v1}(v) = \hat{\alpha}V$ with $\alpha = \varepsilon\hat{\alpha}$. Now $n_{v1}(v)$ is written in terms of $u = u_p + u_m$, giving $n_{v1}(u) = n_{v1}(u_p + u_m) = \hat{\alpha}V(u_p + u_m)$ and expressed in matrix form $n_{v1}(u) = \mathbf{n}_{v1}^* \mathbf{u}^*(u_p, u_m)$, (22).

Now consider a cubic stiffness nonlinearity $X_x(x) = x^3$ (the unforced Duffing Oscillator). Following the procedure discussed in the previous section, the nonlinear term can be written as $n_{v1}(u_p + u_m) = \hat{\alpha}(u_p + u_m)^3$. Using (22) gives

$$\mathbf{u}^* = \left[\begin{array}{cccc} u_p^3 & u_p^2 u_m & u_p u_m^2 & u_m^3 \end{array} \right]^T, \mathbf{n}_{v1}^* = \left[\begin{array}{cccc} \hat{\alpha} & 3\hat{\alpha} & 3\hat{\alpha} & \hat{\alpha} \end{array} \right]. \tag{34}$$

Using (25) gives

$$\beta_{1,e} = \left[\begin{array}{cccc} 9\omega_r^2 - \omega_a^2 & \omega_r^2 - \omega_a^2 & \omega_r^2 - \omega_a^2 & 9\omega_r^2 - \omega_a^2 \end{array} \right], \tag{35}$$

from which the second and third terms can be identified as resonant for the case as $\omega_a \approx \omega_r$ for all three approaches.

Using (26) and (27) gives

$$n_{u1} = 3\hat{\alpha}(u_p^2 u_m + u_p u_m^2), \tag{36}$$

$$h_1 = \frac{\hat{\alpha}}{9\omega_r^2 - \omega_a^2}(u_p^3 + u_m^3). \tag{37}$$

Then using (29) and (30) results in the resonant dynamic and transform equations

$$\ddot{u} + \left(\omega_n^2 + \frac{3\alpha}{4}U^2\right) u = 0, \tag{38}$$

$$x = v = U\cos(\omega_r t - \phi) + \frac{\alpha U^3}{4(9\omega_r^2 - \omega_a^2)}\cos(3[\omega_r t - \phi]), \tag{39}$$

respectively, where the substitution $u = U\cos(\omega_r t - \phi)$, (20), has been used. From the dynamic equation in u, (38), the response frequency can be written to order ε^1 as

$$\omega_r = \sqrt{\omega_n^2 + 3\alpha U^2/4}. \tag{40}$$

Considering the transformation equation, the ratio of the amplitude of the 3rd harmonic to the fundamental response amplitude, M, may be written as

$$M = \frac{\alpha U^2}{4(9\omega_r^2 - \omega_a^2)} = \frac{R}{4(9 - \omega_a^2/\omega_r^2)}, \tag{41}$$

where $R = \alpha U^2/\omega_r^2$. Noting that $\omega_a^2/\omega_r^2 = 1$ for the *detuning* and *resonant* approaches and that, using (40), $\omega_a^2/\omega_r^2 = 1 - 3R/4$ for the *tangent* approach, this can be rewritten for the three approaches as

$$\textit{tangent}\text{: } M = \frac{R}{32} - \frac{3R^2}{1024}, \quad \textit{detuning, resonant}\text{: } M = \frac{R}{32}. \tag{42}$$

Here, for the *tangent* approach, a Taylor series expansion has been used in write $(1 + 3R/32)^{-1} = 1 - 3R/32 + \mathcal{O}\{R^2\}$. The expressions in (42) are equivalent at order ε^1, since R contains α and hence is order ε^1. Note that for the unforced, undamped system the response frequency ω_r is the same as the undamped

nonlinear resonant frequency ω_0, hence the *detuning* and *resonant* approaches are the same.

To assess which is a more accurate approximation, we consider the normal form prediction to ε^2-order accuracy solution – see the Appendix for the development of the ε^2-order refinement to the normal form and its application to the unforced Duffing equation. For both the *detuning/resonant* and the *tangent* approaches the ε^2-accurate normal form technique predicts

$$\textit{tangent, detuning, resonant: } M = \frac{R}{32} + \frac{3R^2}{1024} \quad \text{to order } \varepsilon^2. \tag{43}$$

In this expression $R = \alpha U^2/\omega^2_{r,\varepsilon^1}$, where ω_{r,ε^1} is ω_r to order ε^1, such that this expression is directly comparable to the equivalent order ε^1 expressions given in (42). The response frequency at order ε^2, ω_{r,ε^2}, was found to be

$$\omega^2_{r,\varepsilon^2} = \omega^2_{r,\varepsilon^1}\left(1 + \frac{3R^2}{128}\right) \tag{44}$$

for both the *detuning/resonant* and the *tangent* approaches. Note that these expressions are consistent to order ε^2 with a Fourier based solution presented in [22].

From this analysis, it can be seen that the *detuning/resonant* approaches achieves a more accurate prediction at order ε^1 than the *tangent* approach for this example (compare (42) with the ε^2 accurate (44).

3.2 Small assumptions

At the start of this section it was shown graphically that the magnitude of the nonlinear contribution is reduced when γ is used to linearize the system about the resonant frequency ω_0 (the *resonant* approach) rather than ω_n (the *tangent* approach). Due to the small nonlinearity assumption this would suggest that the ε^1 solution for M using the *resonant* (or the equivalent *detuning*) approach is more accurate than that using the *tangent* approach. However, in addition to assuming that the nonlinear term in the dynamic equation for $\mathbf{x}$ is small there are other small assumptions. Considering an unforced one degree-of-freedom system, the small assumptions and the corresponding ratios of their size compared to accompanying non-small terms, r (see [23]), are

- x dynamics: $\mathbf{N}_x$ is small in (2) giving ratio $r_x = N_x/(Kx)$.
- v dynamics: $\mathbf{N}_v$ is small in (13) giving $r_v = \varepsilon n_{v1}/(\Lambda v)$. Note $r_v = r_x$ as the system is unforced.
- near-identity transform: $\mathbf{h}$ is small in (14) giving $r_{uv} = \varepsilon \mathbf{h}^*_1 \mathbf{u}^*/u$.
- u dynamics: $\mathbf{N}_u$ is small in (15) giving $r_u = \varepsilon \mathbf{n}^*_{u1}\mathbf{u}^*/(\Lambda u)$.

Table 1: Ratios of small:non-small terms for the ε^1 order normal form, expressed to order R^2, where $R = \alpha U^2/\omega_r^2$, (42) has been used to eliminate M and Taylor series expansions have been used assuming R is small, and noting that $\omega_0 = \omega_r$ as the system is unforced.

ratio	*resonant*/*detuning* approach $\omega_a = \omega_r = \omega_0$	*tangent* approach $\omega_a = \omega_n$
$r_x = r_v = \dfrac{\alpha U^2(1+M)^2 - \gamma}{\omega_\gamma^2}$	$r_x = \dfrac{R}{4} + \dfrac{R^2}{16}$	$r_x = R + \dfrac{13R^2}{16}$
$r_{uv} = \quad M$	$r_{uv} = \dfrac{R}{32}$	$r_{uv} = \dfrac{R}{32} - \dfrac{3R^2}{1024}$
$r_u = \dfrac{3\alpha U^2 - 4\gamma}{4\omega_\gamma^2}$	$r_u = 0$	$r_u = \dfrac{3R}{4} + \dfrac{9R^2}{16}$

Note that the maximum value of these ratios over a cycle of oscillation is of interest.

For the Duffing oscillator example, (32) with $V = v^3$, taking the oscillation amplitude at the fundamental frequency to be U, corresponding to an oscillation in x of amplitude $U(1+M)$ (to order ε^1), these ratios are summerised in table 3.2 for both the *tangent* and the *resonant* approaches. It can be seen that adopting the *resonant* approach (equivalent to the *detuning* approach) results in a marked reduction in r_x, as expected from Figure 1, and results in $r_u = 0$, with a cost that there is a slight increase in r_{uv} compared to the *tangent* version, however r_{uv} remains small compared with r_x so is not the limiting ratio for accuracy.

In summary, the detuning step of the second-order normal form calculation was presented as an approximation in [18]. In this example, a physical interpretation of this detuning step has been provided for an unforced system, namely that it is equivalent to taking the linearised natural frequency to be ω_0, the resonant frequency, rather than ω_n. The selection of the approach does not affect the prediction of the fundamental frequency response (to order ε^1), however the *resonant* approach does improve the ε^1 order prediction of the harmonic response (at an ε^2 order level).

4 A two-degree-of-freedom oscillator

We will now consider a two-mode system in which the second mode is auto-parametrically excited by the first mode. The equations of motion are

$$\ddot{x}_1 + 2\zeta\omega_{n1}\dot{x}_1 + \omega_{n1}^2 x_1 + \alpha x_1^3 = \frac{1}{m} f(t), \tag{45}$$

$$\ddot{x}_2 + 2\zeta\omega_{n2}\dot{x}_2 + \omega_{n2}^2 x_2 + \alpha(x_1 + x_2)^3 = 0, \tag{46}$$

where $\omega_{n2} = 3\omega_{n1}$, ζ is the modal damping parameter (for both modes) and α is the coefficient of the small nonlinear terms. The forcing function is given by $f(t) = F\cos(\Omega t)$, where F is the forcing amplitude and Ω is the forcing frequency. We assume that the forcing frequency is close to the resonance of the first mode such that $\Omega = \omega_{r1} \approx \omega_{n1}$. For forced systems the response frequency for mode n, ω_{rn}, is selected to be the closest harmonic of the forcing frequency Ω to the natural frequency, hence here $\omega_{r2} = 3\Omega$.

Since the linear terms in the equations are already in a modal form, we can write $\mathbf{x} = \mathbf{q}$. In addition the forcing is purely resonant therefore $\mathbf{q} = \mathbf{v}$. Here we are assuming that $\omega_{\gamma 1}^2 \approx \omega_{r1}^2$, so via (10), option F1 must be selected. We therefore define a further parameter that is assumed to be small in the derivation

$$r_q = \omega_{\gamma 1}^2 - {\omega_{r1}}^2. \tag{47}$$

Considering the near-identity transform, we write the nonlinear terms as functions of u_{1p}, u_{1m}, u_{2p} and u_{2m} (after replacing x_i with u_i) and these terms are used to define the vector $\mathbf{u}^*$ and the corresponding matrix of coefficients $\mathbf{n}_{u1}^*$. From $\mathbf{u}^*$, the matrix β_1 can be calculated using (25). The resulting matrices are

$$\mathbf{u}^* = \begin{bmatrix} u_{1p}^3 \\ u_{1p}^2 u_{1m} \\ u_{1p}^2 u_{2p} \\ u_{1p}^2 u_{2m} \\ u_{1p} u_{1m}^2 \\ u_{1p} u_{1m} u_{2p} \\ u_{1p} u_{1m} u_{2m} \\ u_{1p} u_{2p}^2 \\ u_{1p} u_{2p} u_{2m} \\ u_{1p} u_{2m}^2 \\ u_{1m}^3 \\ u_{1m}^2 u_{2p} \\ u_{1m}^2 u_{2m} \\ u_{1m} u_{2p}^2 \\ u_{1m} u_{2p} u_{2m} \\ u_{1m} u_{2m}^2 \\ u_{2p}^3 \\ u_{2p}^2 u_{2m} \\ u_{2p} u_{2m}^2 \\ u_{2m}^3 \end{bmatrix}, \quad \mathbf{n}_{u1}^* = \alpha \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 0 & 3 \\ 0 & 3 \\ 3 & 3 \\ 0 & 6 \\ 0 & 6 \\ 0 & 3 \\ 0 & 6 \\ 0 & 3 \\ 1 & 1 \\ 0 & 3 \\ 0 & 3 \\ 0 & 3 \\ 0 & 6 \\ 0 & 3 \\ 0 & 1 \\ 0 & 3 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}^T, \tag{48}$$

$$\beta_1 = \begin{bmatrix} 9\omega_{r1}^2-\omega_{a1}^2 & 9\omega_{r1}^2-\omega_{a2}^2 \\ \omega_{r1}^2-\omega_{a1}^2 & \omega_{r1}^2-\omega_{a2}^2 \\ (2\omega_{r1}+\omega_{r2})^2-\omega_{a1}^2 & (2\omega_{r1}+\omega_{r2})^2-\omega_{a2}^2 \\ (2\omega_{r1}-\omega_{r2})^2-\omega_{a1}^2 & (2\omega_{r1}-\omega_{r2})^2-\omega_{a2}^2 \\ \omega_{r1}^2-\omega_{a1}^2 & \omega_{r1}^2-\omega_{a2}^2 \\ \omega_{r2}^2-\omega_{a1}^2 & \omega_{r2}^2-\omega_{a2}^2 \\ \omega_{r2}^2-\omega_{a1}^2 & \omega_{r2}^2-\omega_{a2}^2 \\ (\omega_{r1}+2\omega_{r2})^2-\omega_{a1}^2 & (\omega_{r1}+2\omega_{r2})^2-\omega_{a2}^2 \\ \omega_{r1}^2-\omega_{a1}^2 & \omega_{r1}^2-\omega_{a2}^2 \\ (\omega_{r1}-2\omega_{r2})^2-\omega_{a1}^2 & (\omega_{r1}-2\omega_{r2})^2-\omega_{a2}^2 \\ 9\omega_{r1}^2-\omega_{a1}^2 & 9\omega_{r1}^2-\omega_{a2}^2 \\ (2\omega_{r1}-\omega_{r2})^2-\omega_{a1}^2 & (2\omega_{r1}-\omega_{r2})^2-\omega_{a2}^2 \\ (2\omega_{r1}+\omega_{r2})^2-\omega_{a1}^2 & (2\omega_{r1}+\omega_{r2})^2-\omega_{a2}^2 \\ (\omega_{r1}-2\omega_{r2})^2-\omega_{a1}^2 & (\omega_{r1}-2\omega_{r2})^2-\omega_{a2}^2 \\ \omega_{r1}^2-\omega_{a1}^2 & \omega_{r1}^2-\omega_{a2}^2 \\ (\omega_{r1}+2\omega_{r2})^2-\omega_{a1}^2 & (\omega_{r1}+2\omega_{r2})^2-\omega_{a2}^2 \\ 9\omega_{r2}^2-\omega_{a1}^2 & 9\omega_{r2}^2-\omega_{a2}^2 \\ \omega_{r2}^2-\omega_{a1}^2 & \omega_{r2}^2-\omega_{a2}^2 \\ \omega_{r2}^2-\omega_{a1}^2 & \omega_{r2}^2-\omega_{a2}^2 \\ 9\omega_{r2}^2-\omega_{a1}^2 & 9\omega_{r2}^2-\omega_{a2}^2 \end{bmatrix}^T . \quad (49)$$

Note that the damping terms have been excluded from these matrices as, from inspection, they are resonant terms and so will remain in the equations of motion once the transform has been applied (i.e. option N2, (27), must be applied).

From β_1 it can be seen that the resonant terms are $[1,2]$, $[1,4]$, $[1,5]$, $[1,9]$, $[1,12]$ and $[1,15]$ for mode 1 and $[2,1], [2,6], [2,7], [2,11], [2,18]$ and $[2,19]$ for mode 2. Applying option N2 to these terms gives the transformed equations of motion

$$\ddot{u}_1 + 2\zeta\omega_{n1}\dot{u}_1 + \omega_{n1}^2 u_1 + \frac{3}{4}\alpha U_1^2 u_1 = \frac{1}{m}f(t), \quad (50)$$

$$\ddot{u}_2 + 2\zeta\omega_{n2}\dot{u}_2 + \omega_{n2}^2 u_2 + \frac{3}{4}\alpha(2U_1^2+U_2^2)u_2 = -\frac{\alpha}{4}U_1^3\cos(3\Omega t - 3\phi_1), \quad (51)$$

where the substitutions $u_{np} = (U_n e^{-i\phi_n}/2)e^{i\omega_{rn}t}$ and $u_{nm} = (U_n e^{i\phi_n}/2)e^{-i\omega_{rn}t}$ have been made with $\omega_{r1} = \Omega$ and $\omega_{r2} = 3\Omega$. Note the right hand side of the second equation arises from the [2,1] and [2,11] resonance terms.

The first of these equations, (50), can be solved to give a relationship between forcing frequency and response amplitude U_1

$$\left(\frac{F}{mU_1}\right)^2 = (2\zeta\omega_{n1}\Omega)^2 + \left(\omega_{n1}^2 - \Omega^2 + \frac{3\alpha U_1^2}{4}\right)^2 \quad (52)$$

which is a quadratic equation in Ω, hence allowing Ω to be solved for given U_1 and F values.

Considering the second equation, (51), it is reasonable to assume that $U_2^2 \ll U_1^2$ since the second mode is parametrically excited from the first via a weak

nonlinearity, hence the U_2^2 can be ignored. This give an expression for the amplitude U_2 for a given U_1 and forcing frequency

$$U_2 = \frac{\alpha U_1^3}{4\sqrt{[9(\omega_{n1}^2 - \Omega^2) + 3\alpha U_1^2/2]^2 + [18\zeta\omega_{n1}\Omega]^2}}. \tag{53}$$

where we have used the relationship $\omega_{n2} = 3\omega_{n1}$. Recall that, as discussed before, these equations for the resonant dynamics are independent of γ and as a result the choice between the *tangent*, *detuning* and *resonant* approaches is immaterial to the resonant response prediction. In addition, in deriving (53), we find the phase relationship

$$\cos(\phi_2 - 3\phi_1) = \frac{-4[9(\omega_{n1}^2 - \Omega^2) + 3\alpha U_1^2/2]U_2}{\alpha U_1^3}. \tag{54}$$

The non-resonant terms in β_1 are placed in the transform equation from u to $v = q = x$ using option N1, (26). Using the transform equation the response at non-resonant frequencies can be found, here we consider the response of mode 1 at 3Ω and the response of mode 2 at Ω, $x_{1,3\Omega}$ and $x_{2,\Omega}$ respectively, which may be written as

$$x_{1,3\Omega} = \frac{\alpha}{4(9\Omega^2 - \omega_{a1}^2)} U_1^3 \cos(3\Omega t'), \tag{55}$$

$$x_{2,\Omega} = \frac{3\alpha}{4(\Omega^2 - \omega_{a2}^2)} \left[(U_1^3 + 2U_1U_2^2)\cos(\Omega t') + U_1^2U_2\cos(\Omega t' + 3\phi_1 - \phi_2)\right], \tag{56}$$

using the time-shift $\Omega t' = \Omega t - \phi_1$. These expressions have come from terms $[1,1]$ and $[1,11]$ in β_1 for mode 1 (note that terms $[1,6], [1,7], [1,18]$ and $[1,19]$ also result in a response at 3Ω however the corresponding terms in $\mathbf{n}_{u1}^*$ are zero) and terms $[2,2], [2,4], [2,5], [2,9], [2,12]$ and $[2,15]$ for mode 2. The resulting amplitudes of these sinusoidal responses are $X_{1,3\Omega}$ and $X_{2,\Omega}$ respectively, where to calculate $X_{2,\Omega}$ (54) is used.

As with the unforced case, the choice of approach effects the non-resonant response as the amplitudes of the higher harmonics are dependent on ω_a via the calculation of β_1, see (25).

4.1 Accuracy of the response prediction

First we consider the response of the first mode using the *tangent* and *resonant* approaches, then later the *detuning* approach is discussed. Figure 2(a) shows the normal form solution for the resonant response of the first mode, using (52), when $\alpha = 1$, $\omega_{n1} = 0.5$, $\zeta = 0.05$ with forcing amplitudes $F/m = 0.0025$, $F/m = 0.005$ and $F/m = 0.0075$. Note that all three approaches result in the same normal form prediction for the resonant response. This is compared to a numerical integration solution (using Matlab solver ode45), where the circles

represent the results for stepping up and stepping down in frequency respectively. It can be seen that the normal form approximation agrees well with the simulation results.

From the transform expression, (55), it can be seen that the ratio of the 3rd harmonic response to the forcing frequency response is given by

$$M = \frac{\alpha U_1^2}{4(9\Omega^2 - \omega_{a1}^2)}. \tag{57}$$

This can be used in conjunction with the solution to (52) to give a prediction of the harmonic response for the three approaches.

Figure 2(b) shows the normal form predictions along with timestepping results (shown as dots and circles). The solid, dashed and dotted lines show the *resonant* ($\omega_{ai} = \omega_{0i}$), *tangent* ($\omega_{ai} = \omega_{ni}$) and *detuning* ($\omega_{ai} = \omega_{ri}$) approaches respectively. The resonant frequencies are derived from (50) and (51) giving

$$\omega_{01} = \sqrt{\omega_{n1}^2 + \frac{3\alpha U_1^2}{4}}, \quad \omega_{02} = \sqrt{\omega_{n2}^2 + \frac{6\alpha U_1^2}{4}}, \tag{58}$$

and are valid for all three approaches. It can be seen that for $X_{1,3\Omega}$ the *resonant* and *detuning* approaches work well (the dotted line is obscured by the solid line) with the *tangent* approach working less well. Note that the *resonant* and *detuning* approaches are not exactly the same because for forced systems, the response frequency matches the forcing frequency, $\omega_{r1} = \Omega$ and only approximates (albeit closely for lightly damped systems) the resonant frequency.

To better understand the accuracy of the approaches in predicting the response of the first mode the small assumptions can be checked. The equations for the parameter r_x, r_q, r_{uv} and r_u are given by

$$r_x = r_v = \frac{\alpha U_1^2(1+M)^2 + \omega_{n1}^2 - \omega_{a1}^2}{\omega_{a1}^2}, \tag{59}$$

$$r_q = \omega_{a1}^2 - \Omega^2, \tag{60}$$

$$r_{uv} = M, \tag{61}$$

$$r_u = \frac{3\alpha U_1^2 + 4(\omega_{n1}^2 - \omega_{a1}^2)}{4\omega_{a1}^2}. \tag{62}$$

Figure 3 (a-d) show r_x, r_q, r_{uv} and r_u respectively. Despite the system having sufficient nonlinearity to shift the resonant peak frequency by 10% at this forcing level, the values for the *resonant* approach remain reasonable low. By contrast r_x and r_u for the *tangent* approach are large, indicating that the approximation using this technique is based on questionable assumptions. It can be seen that the only case where the *resonant* approach is larger than the *tangent* approach is r_{uv}, however the difference is small and for both cases this parameter is much smaller than r_x.

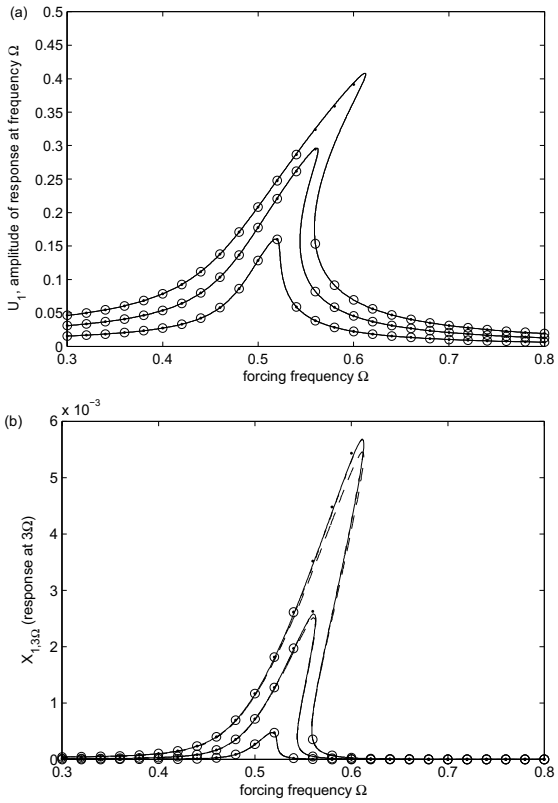

Figure 2: Normal form predictions compared with time-stepping results for three forcing levels $F/m = 0.0025$, $F/m = 0.005$ and $F/m = 0.0075$ showing (a) the resonant response of mode 1, i.e at frequency Ω, U_1 and (b) the non-resonant response of mode 1 at frequency 3Ω, $X_{1,3\Omega}$. The dashed, dotted and solid lines are the normal form prediction for the *tangent*, *detuning* and *resonant* approaches respectively and the dots and circles are numerical integration simulation results for stepping up and down in frequency respectively (note for the resonant response only the solid line is visible as the prediction is unaffected by the approach taken).

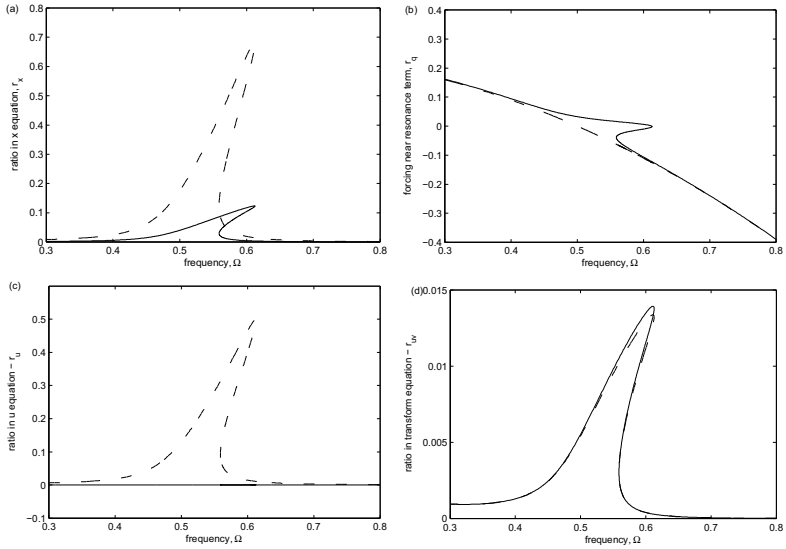

Figure 3: (a) r_x, (b) r_q, (c) r_{uv} and (d) r_u for the *tangent* (dashed line) and *resonant* (solid line) approaches for the forcing level $F/m = 0.0075$.

The resonant response prediction using the normal forms technique (regardless of approach adopted) for the second mode is shown in Figure 4(a) for three levels of forcing, $F/m = 0.0025$, $F/m = 0.005$ and $F/m = 0.0075$, with $\omega_{n1} = 0.5$, $\zeta = 0.03$ and $\alpha = 1$. In addition the dots and circles represent timestepping solutions with the frequency stepped up and down respectively. It can be seen that the agreement is good, with just a small deviation at the highest forcing for the second mode over the range $\Omega = 0.52$ to 0.68rad/s. Note the double resonance peak in the second mode due to its resonance near $\Omega = 0.52$ and the resonance of mode 1 near $\Omega = 0.58$ (which results in increased auto-parametric forcing).

Figure 4(b) shows the non-resonant response of the second mode at frequency Ω for the three forcing levels $F/m = 0.0025$, $F/m = 0.005$ and $F/m = 0.0075$. The predictions of $X_{2,\Omega}$ using the *resonant* approach is good, with poorer predictions using both the *tangent* and *detuning* approaches especially at the highest forcing level. Despite there being low levels of damping here the *detuning* approach are less good than for $X_{1,3\Omega}$ since the resonant peak is wide and so the response frequency does not approximate well to the resonant frequency.

5 Conclusions

In this paper we have considered the accuracy of the second-order normal form technique, a technique that allows the application of normal form transfor-

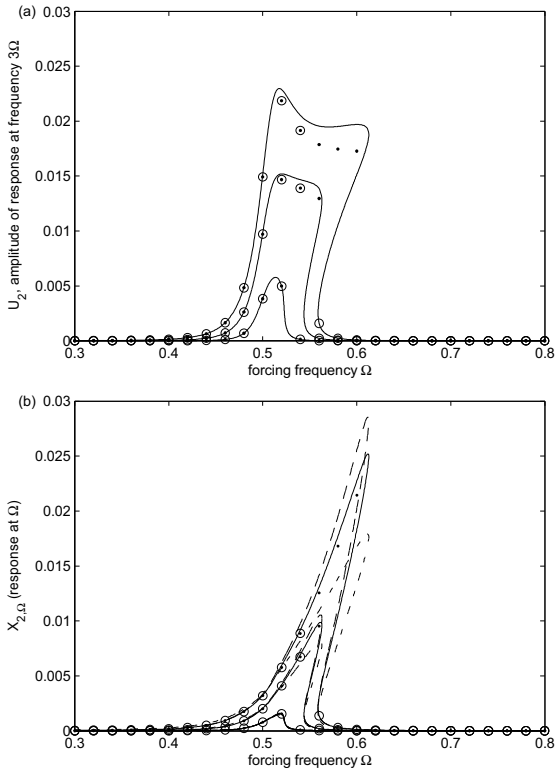

Figure 4: Normal form predictions compared with time-stepping results for three forcing levels $F/m = 0.0025$, $F/m = 0.005$ and $F/m = 0.0075$ showing (a) the resonant response of mode 2, i.e at 3Ω, U_2 and (b) the non-resonant response of mode 2 at frequency Ω, $X_{2,\Omega}$. The dashed, dotted and solid lines are the normal form prediction for the *tangent*, *detuning* and *resonant* approaches respectively and the dots and circles are numerical integration simulation results for stepping up and down in frequency respectively (note for the resonant response only the solid line is visible as the prediction is unaffected by the approach taken).

mations directly to second-order differential equations without first converting them into first-order differential equations. We have shown the selection of the linearised natural frequency affects the prediction of the nonlinear system dynamics, considering either using the natural frequency in the case where the nonlinearity is set to zero (the *tangent* approach) or using the nonlinear natural frequency (the *resonant* approach). In addition we relate this selection to the detuning approximation that is used in the existing second-order normal form technique (the *detuning* approach). We show that neither the choice of linearised natural frequencies about which we perform the normal form transformation nor the use of the detuning approximation affects the prediction of the resonant response of the system (provided the linearised natural frequencies selected are close to the response frequencies). However these choices do affect the non-resonant response (that is the sub and super-harmonics terms in the response). For a single degree-of-freedom unforced system the *detuning* and *resonant* approaches are shown to be identical. It is shown, via an example, that the small approximations used in the normal form technique are better satisfied using the *resonant* (or equivalent *detuning*) approach than using the *tangent* approach. This is why the *resonant* approach gives a superior prediction of the non-resonant response than the *tangent* approach.

The *detuning* approach is also approximately equivalent to the *resonant* approach for forced vibration provided the resonant peak is narrow as it is in this case where the response frequency is close to the nonlinear natural frequency. However as seen in the two degree-of-freedom example as the resonant peak becomes wider then linearising the system about the resonant frequency, using the *resonant* approach, is superior to both the *tangent* and *detuning* approaches in predicting the non-resonant response terms. It is therefore recommended that if the non-resonant response terms are of particular interest the *resonant* approach is adopted.

Appendix – ε^2-accuracy

Here the normal form technique is refined to have ε^2-accuracy. The technique is then applied to the unforced Duffing equation.

Revisiting (16), which has already be satisfied to ε^0 and ε^1, yields the ε^2 equation

$$\mathbf{n}_{u2}(\mathbf{u},\dot{\mathbf{u}},\mathbf{r}) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{h}_2(\mathbf{u},\dot{\mathbf{u}},\mathbf{r})\right) = \mathbf{\Lambda}\mathbf{h}_2(\mathbf{u},\dot{\mathbf{u}},\mathbf{r}) + D\{\mathbf{n}_{v1}(\mathbf{u},\dot{\mathbf{u}},\mathbf{r})\}\mathbf{h}_1(\mathbf{u},\dot{\mathbf{u}},\mathbf{r}), \tag{63}$$

where $D\{\mathbf{n}_{v1}\}$ is the Jacobian of $\mathbf{n}_{v1}$ resulting from the Taylor series expansion of $\mathbf{n}_{v1}$ and where $\mathbf{n}_{v2}$ has been taken to be zero. For the Duffing oscillator example the Jacobian may be written as

$$D\{\mathbf{n}_{v1}(u)\} = D\{-\hat{\gamma}u + \hat{\alpha}u^3\} = -\hat{\gamma} + 3\hat{\alpha}(u_p^2 + 2u_p u_m + u_m^2). \tag{64}$$

As a result the last term in (63) may be written in matrix form as

$$
\begin{aligned}
D\{\mathbf{n}_{v1}\}\mathbf{h}_1 &= \mathbf{D}\mathbf{h}\mathbf{u}^+; \\
\mathbf{D}\mathbf{h} &= \frac{\hat{\alpha}}{9\omega_r^2-\omega_a^2}\left[\begin{array}{cccccccc} 3\hat{\alpha} & 6\hat{\alpha} & 3\hat{\alpha} & 3\hat{\alpha} & 6\hat{\alpha} & 3\hat{\alpha} & -\hat{\gamma} & -\hat{\gamma} \end{array}\right], \\
\mathbf{u}^+ &= \left[\begin{array}{cccccccc} u_p^5 & u_p^4u_m & u_p^3u_m^2 & u_p^2u_m^3 & u_pu_m^4 & u_m^5 & u_p^3 & u_m^3 \end{array}\right]^T.
\end{aligned}
\tag{65}
$$

The, as yet unknown, order ε^2 terms in the transform $\mathbf{h}$ and the resulting nonlinear term $\mathbf{N}_u$ may also be expressed in matrix form; $\mathbf{n}_{u2} = \mathbf{n}_{u2}^+\mathbf{u}^+$ and $\mathbf{h}_2 = \mathbf{h}_2^+\mathbf{u}^+$ respectively. Using a similar approach to that for the ε^1 expression, (18), gives

$$
\begin{aligned}
\beta_2 = \Big[\ & 25\omega_r^2-\omega_a^2 \quad 9\omega_r^2-\omega_a^2 \quad \omega_r^2-\omega_a^2 \quad \omega_r^2-\omega_a^2 \quad 9\omega_r^2-\omega_a^2 \\
& \qquad\qquad\qquad\qquad 25\omega_r^2-\omega_a^2 \quad 9\omega_r^2-\omega_a^2 \quad 9\omega_r^2-\omega_a^2\ \Big].
\end{aligned}
\tag{66}
$$

Then by identifying the resonant terms expressions for the transform and the nonlinear terms can be found. The transform term is given by

$$
\begin{aligned}
\mathbf{h}_2^+ = \frac{\hat{\alpha}}{9\omega_r^2-\omega_a^2}\Bigg[\ & \frac{3\hat{\alpha}}{25\omega_r^2-\omega_a^2} \quad \frac{6\hat{\alpha}}{9\omega_r^2-\omega_a^2} \quad 0 \quad 0 \quad \frac{6\hat{\alpha}}{9\omega_r^2-\omega_a^2} \\
& \qquad\qquad \frac{3\hat{\alpha}}{25\omega_r^2-\omega_a^2} \quad \frac{-\hat{\gamma}}{9\omega_r^2-\omega_a^2} \quad \frac{-\hat{\gamma}}{9\omega_r^2-\omega_a^2}\ \Bigg].
\end{aligned}
\tag{67}
$$

Considering the transform equation to order ε^2, $\mathbf{x} = \mathbf{v} = \mathbf{u}+\varepsilon^1\mathbf{h}_1^*\mathbf{u}^*+\varepsilon^2\mathbf{h}_2^+\mathbf{u}^+$, the amplitude ratio between the fundamental and third harmonic response is

$$
M = \frac{\alpha U^2}{4(9\omega_r^2-\omega_a^2)} + \frac{3\alpha^2U^4-2\gamma\alpha U^2}{8(9\omega_r^2-\omega_a^2)^2},
\tag{68}
$$

To simplify this expression we again use $R = \alpha U^2/\omega_{r,\varepsilon^1}^2$, where $\omega_n^2/\omega_{r,\varepsilon^1}^2 = 1-3R/4$ is ω_r to order ε^1, such that this expression is directly comparable to the equivalent order ε^1 expressions given in (42). In addition the response frequency, now at order ε^2, is needed. By calculating $\mathbf{n}_{u2}$, this can be found to be

$$
\omega_r^2 = \omega_{r,\varepsilon^1}^2\left(1+\frac{3R^2}{128}\right)
\tag{69}
$$

for both the *tangent* and the *resonant* approaches to order ε^2. The expression for M can now be written as

$$
M = \frac{R}{4(9\omega_r^2/\omega_{r,\varepsilon^1}^2-\omega_a^2/\omega_{r,\varepsilon^1}^2)} + \frac{3R^2-2\gamma R/\omega_{r,\varepsilon^1}^2}{8(9\omega_r^2/\omega_{r,\varepsilon^1}^2-\omega_a^2/\omega_{r,\varepsilon^1}^2)^2},
\tag{70}
$$

For the *tangent* approach we set $\omega_a^2 = \omega_n^2$ and $\gamma = 0$ to give

$$
M = \frac{R}{32}\left(1+\frac{3R}{32}\right)^{-1} + \frac{3R^2}{512}\left(1+\frac{3R}{32}\right)^{-2},
\tag{71}
$$

where $\mathcal{O}\{R^3\}$ terms have been dropped. Whereas for the *resonant* approach we set $\omega_a^2 = \omega_r^2$ and $\gamma = 3R\omega_{r,\varepsilon^1}^2/4 + \mathcal{O}\{R^2\}$ such that

$$M = \frac{R}{32}\left(1 + \frac{3R^2}{128}\right)^{-1} + \frac{3R^2}{1024}\left(1 + \frac{3R^2}{128}\right)^{-2}. \tag{72}$$

For both the *tangent* and the *resonant* approaches, the expressions for M, (71) and (72) respectively, may be written as

$$M = \frac{R}{32} + \frac{3R^2}{1024}, \tag{73}$$

to order ε^2 after Taylor series expansions.

References

[1] McIntyre, M.E., Schumacher, R.T.,Woodhouse, J.: On the oscillations of musical instuments. Journal of the Acoustical Society of America **74**(5), 1325–1345 (1983)

[2] Thompson, J.M.T., Elvey, J.S.N.: Elimination of subharmonic resonances of compliant marine structures. International Journal of Mechanical Sciences, **26**, 419–425 (1984)

[3] Cartmell, M.: Introduction to linear, parametric and nonlinear vibrations. Chapman and Hall (1990)

[4] Schwartz, I.B., Erneux., T.: Subharmonic hysteresis and period-doubling bifurcations for a periodically driven laser. SIAM Journal on Applied Mathematics, **54**(4), 1083–1100 (1994)

[5] Bishop, S.R., Xu., D.: Use of control to eliminate subharmonic and chaotic impacting motions of a driven beam. Journal of Sound and Vibration, **205**(2), 223–234 (1997)

[6] Tondl, A., Ruijgrok, T., Verhulst, F., Nabergoj, R.: Autoparametric Resonance in Mechanical Systems. Cambridge (2000)

[7] Zhou, Q., Larsen, J.W., Nielsen, S.R.K, Qu, W.L.: Nonlinear stochastic analysis of subharmonic response of a shallow cable. Nonlinear Dynamics, **48**(1-2), 97–114 (2007)

[8] Nayfeh, A.H., Mook, D.T.: Nonlinear oscillations. John Wiley, New York (1995)

[9] Jordan, D.W., Smith, P.: Nonlinear ordinary differential equationsan introduction to dynamical systems. Oxford University Press, 3rd Edition, (1999)

[10] Strogatz, S.H.: Nonlinear Dynamics and Chaos. Perseus Books Group (2001)

[11] Szemplińska, W.: Behaviour on nonlinear vibrating systems Volume II: Advanced concepts and applications to multi-degree-of-freedom systems. Kluwer (1990)

[12] Kahn, P.B., Zarmi, Y.: Nonlinear dynamics: a tutorial on the method of normal forms. American Journal of Physics **68**(10), 907–919 (2000)

[13] Stolovitch, L.: Progress in normal form theory. Nonlinearity, **22** R77–R99 (2009)

[14] Murdock, J.: Normal forms and unfoldings for local dynamical systems. Springer (2002)

[15] Touzé, C., Thomas, O., Huberdeau, A.: Asymptotic non-linear normal modes for large-amplitude vibrations of continuous structures. Computers and Structures **82**, 2671–2682 (2004)

[16] Wang, F., Bajaj, A.K.: On the Formal Equivalence of Normal Form Theory and the Method of Multiple Time Scales. Journal of Computational and Nonlinear Dynamics **4**, article 021005 (2009)

[17] Lamarque, C.-H., Touzé, T., Thomas, O.: An upper bound for validity limits of asymptotic analytical approaches based on normal form theory. Nonlinear Dynamics **70**, 1931–1949 (2012)

[18] Neild, S.A., Wagg, D.J.,: Applying the method of normal forms to second-order nonlinear vibration problems. Proceedings of the Royal Society Part A **467**, 1141–1163 (2011)

[19] Neild, S.A.: Approximate Methods for Analysing Nonlinear Structures, in Exploiting Nonlinear Behavior in Structural Dynamics, Wagg, D.J., Virgin, L.N., Eds.. Springer (2012)

[20] Nayfeh, A.H.: Method of normal forms. Wiley (1993)

[21] Xin, Z., Zuo, Z., Feng, H., Wagg, D., Neild, S.: Higher order accuracy analysis of the second order normal form method. Nonlinear Dynamics, **70**, 2175–2185 (2012)

[22] Davis, H.T.: Introduction to Nonlinear Differential and Integral Equations. Dover (1962)

[23] Xin, Z.F., Neild, S.A., Wagg, D.J.: The selection of the linearized natural frequency for the second-order normal form method. Proceedings of IDETC/CIE ASME International Design Engineering Technical Conferences, Washington, DC. USA, Paper DETC2011-47654 (2011)