

ON CERTAIN CLASSES OF p -VALENT FUNCTIONS

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ABSTRACT. Let $V_k^\lambda(\alpha, b, p)$ ($k \geq 2$, $b \neq 0$ is any complex number, $0 \leq \alpha < p$ and $|\lambda| < \pi/2$) denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ analytic in $U = \{z: |z| < 1\}$ having $(p-1)$ critical points in U and satisfying

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \left| \frac{\operatorname{Re}\{e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda\}}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda.$$

In this paper we generalize both those functions $f(z)$ which are p -valent convex of order α , $0 \leq \alpha < p$, with bounded boundary rotation and those p -valent functions $f(z)$ for which $zf'(z)/p$ is λ -spirallike of order α , $0 \leq \alpha < p$.

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1. INTRODUCTION.

Let $M_k(k \geq 2)$ denote the lower class of real-valued functions $m(t)$ of bounded variation on $[0, 2\pi]$ which satisfy the conditions $\int_0^{2\pi} dm(T) = 2$ and $\int_0^{2\pi} |dm(t)| \leq k$.

Let A_p , where p is a positive integer, denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$. For $f \in A_p$, we say that f belongs to the class $V_k^\lambda(\alpha, b, p)$ ($k \geq 2$, $b \neq 0$ is any complex number, $0 \leq \alpha < p$ and $|\lambda| < \pi/2$) if there exists $\delta > 0$ such that

$$\int_0^{2\pi} \operatorname{Re} \left\{ 1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta = 2p\pi \quad (1-\delta < r < 1) \quad (1.1)$$

and

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \left| \frac{\operatorname{Re}\{e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda\}}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda. \quad (1.2)$$

condition (1.1) implies that f has $(p-1)$ critical points in U .

It is noticed that, by giving specific values to k, α, b, p and λ in $V_k^\lambda(\alpha, b, p)$, we obtain the following important subclasses studied by various authors in earlier papers:

(1) $V_k^\lambda(o, 1, p) = V_k^\lambda(p)$, is the class of p -valent functions investigated by Silvia [1].

(2) $V_k^o(o, 1, 1) = V_k$, is the class of bounded boundary rotation introduced by Löwner [2] and Paatero [3, 4] $V_k^o(\alpha, 1, 1) = V_k(\alpha)$, $0 \leq \alpha < 1$, is the class of functions $f(z) \in A_1$ studied by Padmanabhan and Parvatham [5], $V_k^\lambda(o, 1, 1) = V_k^\lambda$, is the class of functions $f(z) \in A_1$ investigated by Moulis [6] and Silvia [7], $V_k^\lambda(\alpha, 1, 1) = V_k^\lambda(\alpha)$, $0 \leq \alpha < 1$, is the class of functions $f(z) \in A_1$ investigated by Moulis [8], $V_k^o(o, b, 1) = V_k(b)$, is the class of functions $f(z) \in A_1$ introduced by Nasr [9] and $V_k^\lambda(\alpha, b, 1) = V_k^\lambda(\alpha, b)$, is the class of functions $f(z) \in A_1$ investigated by Lakshma [10, 11].

(3) $V_2^o(o, 1, p) = C(p)$, is the class of p -valent convex functions considered by Goodman [12], $V_2^o(\alpha, 1, p) = C(\alpha, p)$, is the class of p -valent convex functions of order α , $0 \leq \alpha < p$, $V_2^\lambda(o, 1, p) = C^\lambda(p)$, is the class of functions $f(z) \in A_p$ for which $zf'(z)/p$ is λ -spirallike in U , $V_2^\lambda(\alpha, 1, p) = C^\lambda(\alpha, p)$, is the class of functions $f(z) \in A_p$ for which $zf'(z)/p$ is λ -spirallike of order α , $0 \leq \alpha < p$, $V_2^o(o, b, p) = C(b, p)$, is the class of p -valent functions satisfying

$$\operatorname{Re} \left[p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right] > 0, z \in U.$$

The class $C(b, p)$ introduced by Aouf [13] and finally for $k=2$ any function $f(z) \in V_2^\lambda(\alpha, b, p) = C^\lambda(\alpha, b, p)$ if and only if

$$\operatorname{Re} e^{i\lambda} \left[p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right] > \alpha \cos \lambda, z \in U.$$

Also from $V_k^\lambda(\alpha, b, p)$, we can obtain the following important subclasses:

(4) $V_k^o(\alpha, 1, p) = V_k(\alpha, p)$, is the class of p -valent functions $f(z) \in A_p$ satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \alpha}{p - \alpha} \right| d\theta \leq k\pi,$$

i.e., $f(z) \in V_k(\alpha, p)$ if and only if

$$f'(z) = pz^{p-1} \left\{ \exp \left[-(p-\alpha) \int_0^{2\pi} \log(1 - ze^{-it}) dm(t) \right] \right\}, m(t) \in M_k.$$

(5) $V_k^\lambda(\alpha, 1, p) = V_k^0(\alpha, \cos e^{-i\lambda}, p) = V_k^\lambda(\alpha, p)$, $|\lambda| < \frac{\pi}{2}$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} - \alpha \cos \lambda}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda$$

i.e., $f(z) \in V_k^\lambda(\alpha, p)$ if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -(p-\alpha) \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

(6) $V_k^0(o, b, p) = V_k(b, p)$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} \right| d\theta \leq k\pi p,$$

i.e., $f(z) \in V_k(b, p)$ if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -pb \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

(7) $V_k^0(\alpha, b, p) = V_k(\alpha, b, p)$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} - \alpha}{p - \alpha} \right| d\theta \leq k\pi,$$

i.e., $f(z) \in V_k(\alpha, b, p)$ if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -(p-\alpha)b \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

We state below some lemmas that are needed in our investigation.

LEMMA 1 [10]. $h(z) \in V_k^\lambda(\alpha, b)$ if and only if

$$(1) h'(z) = \exp \left\{ -(1-\alpha)b \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

$$(2) h'(z) = [f_1'(z)]^{(1-\alpha)b \cos \lambda e^{-i\lambda}}, f_1 \in V_k.$$

$$(3) h'(z) = [f_2'(z)]^b \cos \lambda e^{-i\lambda}, f_2 \in V_k(\alpha).$$

$$(4) h'(z) = [f_3'(z)]^{(1-\alpha)b}, f_3 \in V_k^\lambda.$$

$$(5) h'(z) = [f_4'(z)]^b, f_4 \in V_k^\lambda(\alpha).$$

$$(6) h'(z) = [f_5'(z)]^{(1-\alpha) \cos \lambda e^{-i\lambda}}, f_5 \in V_k(b).$$

(7) there exists two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = \left\{ \frac{s_1(z)}{z} \right\}^{\frac{k+2}{4}} (1-\alpha)b \cos \lambda e^{-i\lambda} \left\{ \frac{s_2(z)}{z} \right\}^{\frac{k-2}{4}}$$

LEMMA 2[10]. If $h(z)$ belongs to $V_k^\lambda(\alpha, b)$, then $H(z)$ is defined by

$$H'(z) = \frac{h'(\frac{z+a}{1+\bar{a}z})}{h'(a)(1+\bar{a}z)^2(1-\alpha)b \cos \lambda e^{-i\lambda}}$$

$H(0) = 0, |z| < 1, |a| < 1$ is also in the class $V_k^\lambda(\alpha, b)$.

LEMMA 3[10]. Suppose $h(z) = z + b_2z^2 + b_3z^3 + \dots$ belongs to $V_k^\lambda(\alpha, b)$ then

$$|b_2| \leq \frac{k}{2}(1-\alpha) |b| \cos \lambda,$$

and

$$|b_3| \leq \frac{1}{3}(1-\alpha) |b| \cos \lambda [1-(1-\alpha) |b| \cos \lambda \frac{k^2}{2}].$$

These bounds are sharp, with equality for the function $h(z) \in V_k^\lambda(\alpha, b)$ defined by

$$h'(z) = \left[\frac{(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}} \right]^{(1-\alpha)b \cos \lambda e^{-i\lambda}}.$$

LEMMA 4[1]. $g(z) \in V_k^\lambda(p), p \geq 1$, if and only if $g'(z) = pz^{p-1}[f_3'(z)]^p$ for some $f_3 \in V_k^\lambda$.

2. REPRESENTATION FORMULAS FOR THE CLASS $V_k^\lambda(\alpha, b, p)$.

LEMMA 5. $f(z) \in V_k^\lambda(\alpha, b, p)$ if and only if

$$f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}} = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1} \tag{2.1}$$

for some $h \in V_k^\lambda(\alpha, b)$.

PROOF. Let $f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}} = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}$ for

$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha, b), z \in U$.

By direct computation, we obtain

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} \{ e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda \}}{p-\alpha} \right| d\theta =$$

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} \{ e^{i\lambda} [1 + \frac{1}{b} \frac{zh''(z)}{h'(z)}] - \alpha \cos \lambda \}}{1-\alpha} \right| d\theta$$

and the result follows from (1.2).

An immediate consequence of Lemmas 5 and 1 is

THEOREM 1. $f(z) \in V_k^\lambda(\alpha, b, p)$ if and only if

$$(1) f'(z) = pz^{p-1} \exp \{ -(p-\alpha)b \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \}, m(t) \in M_k.$$

$$(2) \quad f'(z) = pz^{p-1} [f'_1(z)]^{(p-\alpha)b \cos \lambda} e^{-i\lambda}, \quad f_1 \in V_k.$$

$$(3) \quad f'(z) = pz^{p-1} [f'_2(z)]^{\left(\frac{p-\alpha}{1-\alpha}\right)b \cos \lambda} e^{-i\lambda}, \quad f_2 \in V_k(\alpha).$$

$$(4) \quad f'(z) = pz^{p-1} [f'_3(z)]^{(p-\alpha)b}, \quad f_3 \in V_k^\lambda.$$

$$(5) \quad f'(z) = pz^{p-1} [f'_4(z)]^{\left(\frac{p-\alpha}{1-\alpha}\right)b}, \quad f_4 \in V_k(\alpha)$$

$$(6) \quad f'(z) = pz^{p-1} [f'_5(z)]^{(p-\alpha)\cos \lambda} e^{-i\lambda}, \quad f_5 \in V_k(b).$$

(7) there exists two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = pz^{p-1} \frac{\left\{ \frac{s_1(z)}{z} \right\}^{\frac{k+2}{4}} (p-\alpha)b \cos \lambda e^{-i\lambda}}{\left\{ \frac{s_2(z)}{z} \right\}^{\frac{k-2}{4}}}$$

Also an immediate consequence of Theorem 1 (4) and Lemma 4 is

THEOREM 2. $f(z) \in V_k^\lambda(\alpha, b, p)$ if and only if

$$f'(z) = pz^{p-1} \left[\frac{g'(z)}{pz^{p-1}} \right]^{\left(\frac{p-\alpha}{p}\right)b}, \quad g \in V_k^\lambda(p)$$

3. COEFFICIENT ESTIMATES FOR THE CLASS $V_k^\lambda(\alpha, b, p)$.

THEOREM 3. If $f(z) = z^p + \sum_{n=p+1}^\infty a_n z^n \in V_k^\lambda(\alpha, b, p)$, then

$$|a_{p+1}| \leq \left[\frac{p(p-\alpha)}{p+1} \right] k |b| \cos \lambda, \tag{3.1}$$

$$|a_{p+2}| \leq p \left(\frac{p-\alpha}{p+2} \right) |b| \cos \lambda \left\{ [1 - (1-\alpha)|b| \cos \lambda \frac{k^2}{2}] + (p-1) |b| \cos \lambda \frac{k^2}{2} \right\} \tag{3.2}$$

These bounds are sharp with equality for

$$f'_*(z) = pz^{p-1} \left[\frac{\left(\frac{1-z}{k}\right)^{\frac{k}{2}-1} (p-\alpha)b \cos \lambda e^{-i\lambda}}{(1+z)^{\frac{k}{2}+1}} \right] \tag{3.3}$$

PROOF. By Lemma 5, there exists an $h(z) = z + \sum_{n=2}^\infty b_n z^n \in V_k^\lambda(\alpha, b)$ such that

$$f'(z) = pz^{p-1} + \sum_{n=p+1}^\infty n a_n z^{n-1} = pz^{p-1} \left[1 + \sum_{n=2}^\infty n b_n z^{n-1} \right]^{\left(\frac{p-\alpha}{1-\alpha}\right)} \tag{3.4}$$

Expanding the right side of (3.4), we obtain

$$f'(z) = pz^{p-1} + 2p \left(\frac{p-\alpha}{1-\alpha} \right) b_2 z^p + p \left\{ 3 \left(\frac{p-\alpha}{1-\alpha} \right) b_3 + 2 \left(\frac{p-\alpha}{1-\alpha} \right) \left(\frac{p-1}{1-\alpha} \right) b_2^2 \right\} z^{p+1} + \dots \tag{3.5}$$

Equating coefficients from (3.5) and (3.4), we have

$$(p+1)a_{p+1} = 2p\left(\frac{p-\alpha}{1-\alpha}\right)b_2,$$

$$(p+2)a_{p+2} = p\left\{3\left(\frac{p-\alpha}{1-\alpha}\right)b_3 + 2\left(\frac{p-\alpha}{1-\alpha}\right)\left(\frac{p-1}{1-\alpha}\right)b_2^2\right\}.$$

Thus the result follows from lemma 3.

4. SHARP RADIUS OF CONVEXITY OF THE CLASS $V_k^\lambda(\alpha, b, p)$.

LEMMA 6. If $f \in V_k^\lambda(\alpha, b, p)$, then the transformation F_a satisfying

$$F'_a(z) = \frac{p a^{p-1} z^{p-1} f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f'(a)(z+a)^{p-1}(1+\bar{a}z)^{2(p-\alpha)}b \cos \lambda e^{-i\lambda} - p+1}, \quad z \in U \quad (4.1)$$

and $F_a(0) = 0$, is in $V_k^\lambda(\alpha, b, p)$ for all $|a| < 1$.

The proof of this lemma follows by using lemmas 5 and 2.

LEMMA 7. For $|z| \leq r$ and f ranging over $V_k^\lambda(\alpha, b, p)$, the domain of values of $\frac{zf''(z)}{f'(z)}$ is the disc with center

$$\left(\frac{[2(p-\alpha)\operatorname{Re}\{b \cos \lambda e^{-i\lambda}\} - p+1] r^2 + (p-1)}{1-r^2}, \frac{2(p-\alpha)\operatorname{Im}\{b \cos \lambda e^{-i\lambda}\} r^2}{1-r^2} \right)$$

and radius $\frac{p(p-\alpha)k|b| \cos \lambda r}{1-r^2}$.

PROOF. Whenever

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^\lambda(\alpha, b, p), \quad \lim_{z \rightarrow 0} \left\{ \frac{f''(z) - p(p-1)z^{p-2}}{z^{p-1}} \right\} = p(p-1)a_{p+1}.$$

For $f \in V_k^\lambda(\alpha, b, p)$, let $F_a(z) = z^p + \sum_{n=p+1}^{\infty} A_n z^n \in V_k^\lambda(\alpha, b, p)$ be given by Lemma 6 for $|a| < 1$. By direct calculation we have

$$p(p+1)A_{p+1} = p(1-|a|^2) \frac{f''(a)}{f'(a)} - \frac{p[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]|a|^2 + p(p-1)}{a}. \quad (4.2)$$

Combining (3.1) and (4.2), we obtain

$$\left| \frac{f''(a)}{f'(a)} - \frac{[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]|a|^2 + p(p-1)}{a(1-|a|^2)} \right| \leq \frac{p(p-\alpha)k|b| \cos \lambda}{(1-|a|^2)} \quad (4.3)$$

From (4.3) it follows that for $|z| = r < 1$,

$$\left| \frac{zf''(z)}{f'(z)} - \frac{[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]r^2 + p(p-1)}{(1-r^2)} \right| \leq \frac{p(p-\alpha)k|b| \cos \lambda r}{(1-r^2)} \quad (4.4)$$

and the proof is completed.

THEOREM 4. The sharp radius of convexity of the class $V_k^\lambda(\alpha, b, p)$ is

$$r_0 = 2\{(p-\alpha)k|b| \cos \lambda + [(p-\alpha)^2 k^2 |b|^2 \cos^2 \lambda - 4(2(1-\frac{\alpha}{p})\operatorname{Re}\{b \cos \lambda e^{-i\lambda}\} - 1)]^{\frac{1}{2}}\}^{-1} \quad (4.5)$$

PROOF. From (4.4), we have

$$\left| \left(1 + \frac{zf''(z)}{f'(z)} \right)^{-p} \left(\frac{1 + (2(1 - \frac{\alpha}{p})b \cos \lambda e^{-i\lambda} - 1)r^2}{1-r^2} \right) \right| \leq \frac{p(p-\alpha)k|b|\cos \lambda r}{1-r^2}$$

which implies that

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq p \left[\frac{1 - (p-\alpha)k|b|\cos \lambda r + (2(1 - \frac{\alpha}{p}) \operatorname{Re}\{b \cos \lambda e^{-i\lambda} - 1\})r^2}{1-r^2} \right].$$

Therefore $\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$ for $|z| < r_0$ is given by (4.5). The function $f_x(z)$ defined by

$$f'_x(z) = pz^{p-1} \left[\frac{\frac{k}{2} - 1 - (p-\alpha)b \cos \lambda e^{-i\lambda}}{(1-\varepsilon z)^{\frac{k}{2} + 1}} \right] \tag{4.6}$$

when $z = r$, $e = \frac{r+p}{1+rpe} \frac{e^{i\lambda} \sqrt{\frac{b}{1-b}}}{\sqrt{\frac{b}{1-b}}}$ and $\varepsilon = \frac{r-pe}{1-rpe} \frac{e^{i\lambda} \sqrt{\frac{b}{1-b}}}{\sqrt{\frac{b}{1-b}}}$

shows that the bound in (4.5) is sharp.

5. DISTORTION AND ROTATION THEOREMS FOR THE CLASS $V_k^\lambda(\alpha, b, p)$.

In [11] Lakshma used variational method to solve the extremal problems for the class $V_k^\lambda(\alpha, b)$. He proved the following:

LEMMA 8[11]. Let $\xi \neq 0$ be a given point in U and let $H(x_1, x_2, \dots, x_{n+1})$ be analytic in a neighbourhood of each point $H(h'(\xi), h''(\xi), \dots, h^{(n)}(\xi), \xi)$, $h \in V_k^\lambda(\alpha, b)$.

Then the functional

$$J(h') = \operatorname{Re} H(h'(\xi), h''(\xi), \dots, h^{(n)}(\xi), \xi)$$

attains its maximum (or minimum) in $V_k^\lambda(\alpha, b)$ only for a function of the form

$$h'(z) = \prod_{j=1}^M (1-\varepsilon_j z)^{\beta_j} (1-\alpha)b \cos \lambda e^{-i\lambda} \prod_{j=1}^N (1-e_j z)^{-\alpha_j} (1-\alpha)b \cos \lambda e^{-i\lambda} \tag{5.1}$$

where $M \leq n$, $N \leq n$, $|\varepsilon_j| = |e_j| = 1$, $\sum_{j=1}^M \beta_j \leq \frac{k}{2} - 1$, $\sum_{j=1}^N \alpha_j \leq \frac{k}{2} + 1$.

LEMMA 9[11]. If $h(z) \in V_k^\lambda(\alpha, b)$, then we have

$$|e|^{\min} |\varepsilon| = 1 \left| \frac{(1-\varepsilon z)^{\frac{k}{2} - 1} A}{(1-\varepsilon z)^{\frac{k}{2} + 1} A} \right| \leq |h'(z)| \leq |e|^{\max} |\varepsilon| = 1 \left| \frac{(1-\varepsilon z)^{\frac{k}{2} - 1} A}{(1-\varepsilon z)^{\frac{k}{2} + 1} A} \right|,$$

where $A = (1-\alpha)b \cos \lambda e^{-i\lambda}$.

Bounds are sharp with properly selected e and ε with $M = N = 1$ in (5.1).

LEMMA 10[11]. If $h(z) \in V_k^\lambda(\alpha, b)$, then for $|z| = r < 1$, we obtain

$$|\arg h'(z)| \leq (1-\alpha)k|b|\cos \lambda \arcsin |z|.$$

The proof of the following two theorems follows from Lemma 5 and the above bounds.

THEOREM 5. If $f \in V_k^\lambda(\alpha, b, p)$, then for $|z| = r < 1$, we obtain

$$|e|^{\min} |\varepsilon| = 1 \left| \frac{\left(\frac{k}{2} - 1\right)B}{(1-\varepsilon z)} \frac{(1-\varepsilon z)}{\left(\frac{k}{2} + 1\right)B} \right| \leq \left| \frac{f'(z)}{pz^{p-1}} \right| \leq |e|^{\max} |\varepsilon| = 1 \left| \frac{\left(\frac{k}{2} - 1\right)B}{(1-\varepsilon z)} \frac{(1-\varepsilon z)}{\left(\frac{k}{2} + 1\right)B} \right| ,$$

where $B = (p-\alpha)b \cos \lambda e^{-i\lambda}$.

Bounds are sharp with equality for $f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}}$, where $h(z)$ is defined by (5.1) with properly selected e and ε with $M = N = 1$.

THEOREM 6. If $f \in V_k^\lambda(\alpha, b, p)$, then for $|z| = r < 1$, we obtain

$$|\arg f'(z)| \leq (p-1)^\theta + (p-\alpha)k |b| \cos \lambda \arcsin |z|.$$

6. HARDY CLASSES FOR THE CLASS $V_k^\lambda(\alpha, b)$.

In order to obtain the H^μ classes for the class $V_k^\lambda(\alpha, b)$, we will use the following lemmas.

LEMMA 11[7]. IF $f_3 \in V_k^\lambda$. Then for $|z|=r$, $|\arg f_3'(re^{i\theta})| \leq k \cos \lambda \arcsin r$.

LEMMA 12[14]. If $f' \in H^\mu (0 < \mu < 1)$ then $f \in H^{1-\mu}$ where, for $\mu=1$, H^∞ is the class of bounded functions.

LEMMA 13[15]. If $f(z) \in H^\mu (0 < \mu < 1)$ and $f(z) = \sum_{n=0}^\infty a_n z^n$, then $a_n = o(n^{\frac{1}{\mu} - 1})$.

LEMMA 14[1]. If $f_3 \in V_k^\lambda$, then $f_3' \in H^\mu$ for all $\mu < \frac{2}{\cos^2 \lambda (k+2)}$ and $f_3 \in H^\eta$ for $\eta < \frac{2}{[\cos^2 \lambda (k+2) - 2]}$, $\cos^2 \lambda > \frac{2}{(k+2)}$. Furthermore, if f_3' is not of the form

$$f_3'(z) = [f_0'(z)]^{\cos \lambda} e^{-i\lambda} \text{ where } f_0 \text{ is given by}$$

$$f_0'(z) = (1-ze^{-it})^{-\left(\frac{k}{2}+1\right)} \exp \left\{ \int_0^{2\pi} -\log(1-ze^{-it}) dm(t) \right\} \tag{6.1}$$

$(m(t))$ a probability measure on $[0, 2\pi]$, then there exists $\delta = \delta(f_3) > 0$ and $\varepsilon = \varepsilon(f_3) > 0$ such that

$$f_3'(z) \in H^{\cos^2 \lambda (k+2)} \text{ and } f_3 \in H^{\frac{(2+\varepsilon)}{[\cos^2 \lambda (k+2) - 2]}} \text{ for } \cos^2 \lambda > \frac{2}{(k+2)} .$$

THEOREM 7. If $h \in V_k^\lambda(\alpha, b)$, $\text{Re}\{b\} > 0$, then $h' \in H^\mu$ for all

$$\mu < \frac{2}{(1-\alpha)\text{Re}\{b\} \cos^2 \lambda (k+2)} \text{ and } h \in H^\eta \text{ for } \eta < \frac{2}{[(1-\alpha)\text{Re}\{b\} \cos^2 \lambda (k+2) - 2]} ,$$

where $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$. Furthermore, if h' is not of the form $h'(z) =$

$[f_3'(z)]^{(1-\alpha)b} = [f_0'(z)]^{(1-\alpha)b} \cos\lambda e^{-i\lambda}$ where $f_0(z)$ is given by (6.1), then there exists $\varepsilon = \varepsilon(h) > 0$ and $\delta = \delta(h) > 0$ such that

$$h' \in H^{\varepsilon+} \frac{2}{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)]} \quad \text{and} \quad h \in H^{\delta+} \frac{2}{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)-2]}$$

for $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$.

PROOF. If $h \in V_k^\lambda(\alpha, b)$, then it follows from Lemma 1(4) that

$$h'(z) = [f_3'(z)]^{(1-\alpha)b}, \quad f_3 \in V_k^\lambda.$$

Then

$$|h'(z)|^\mu = |f_3'(z)|^\mu (1-\alpha)\operatorname{Re}\{b\} \cdot \exp\{-\mu(1-\alpha)\operatorname{Im}\{b\}\arg f_3'(z)\}.$$

By Lemma 11, the exponential factor is bounded. Thus the result follows from Lemmas 12 and 14. From Theorem 7 and Lemma 13, we obtain a growth estimate for the Taylor's coefficients of $h \in V_k^\lambda(\alpha, b)$.

COROLLARY 1. If $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha, b)$ and $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$,

$$b_n = o\left(n^{\frac{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)-4]}{2}}\right).$$

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