INVERTIBILITY-PRESERVING MAPS OF C^*-ALGEBRAS WITH REAL RANK ZERO

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In 1996, Harris and Kadison posed the following problem: show that a linear bijection between C^*-algebras that preserves the identity and the set of invertible elements is a Jordan isomorphism. In this paper, we show that if A and B are semisimple Banach algebras and $\Phi : A \to B$ is a linear map onto B that preserves the spectrum of elements, then Φ is a Jordan isomorphism if either A or B is a C^*-algebra of real rank zero. We also generalize a theorem of Russo.

1. Notation

In what follows, the term Banach algebra will mean a unital complex Banach algebra and a C^*-algebra will mean a unital complex C^*-algebra. The unit is denoted by 1 and the spectrum of an element x by $\sigma(x)$. The set of invertible elements of a Banach algebra A is denoted by A_{inv} and the closed unit ball of A by A_1. The density of a subset of a Banach algebra in another subset is meant to be in the norm topology. A linear map Φ from a Banach algebra A to a normed algebra B is a Jordan homomorphism if $\Phi(a^2) = \Phi(a)^2$ for every $a \in A$. Properties of Jordan homomorphisms are given in [7] or [9]. For C^*-algebras A and B, a C^*-homomorphism in the sense of Kadison is a selfadjoint linear mapping of A into B which is a Jordan homomorphism, that is, $\Phi(a^*) = \Phi(a)^*$ and $\Phi(a^2) = \Phi(a)^2$ for all $a \in A$ [13].

2. Introduction

There are many results on the conjecture of Harris and Kadison. A summary of these results can be found in [7]. One of the most important results is [2, Theorem 1.3] of Aupetit.

Theorem 2.1. *Let A and B be two von Neumann algebras and let Φ be a spectrum-preserving linear mapping from A onto B. Then Φ is a Jordan isomorphism.*

Among other theorems, Russo proved the following [12, Theorem 2] in 1996.

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THEOREM 2.2. *Let Φ be a linear mapping from a von Neumann algebra M into a C^*-algebra B such that $\Phi(M_{\text{inv}} \cap M_1) \subset B_{\text{inv}} \cap B_1$ and $\Phi(1) = 1$. Then Φ is a C^*-homomorphism.*

The definition of a C^*-algebra with real rank zero was given by Brown and Pedersen [3].

Definition 2.3. A C^*-algebra A has real rank zero if the set of invertible selfadjoint elements of A is dense in the set of selfadjoint elements of A.

Also in [3, Theorem 2.6] Brown and Pedersen prove the following.

THEOREM 2.4. *A C^*-algebra A has real rank zero exactly when the set of selfadjoint elements of A with finite spectra is dense in the set of selfadjoint elements of A.*

Theorem 2.4 enables us to generalize Theorems 2.1 and 2.2 and thus obtain our main results.

THEOREM 2.5. *Suppose A is a C^*-algebra with real rank zero and B is a semisimple Banach algebra. If Φ is a spectrum-preserving linear map from A onto B, then Φ is a Jordan isomorphism.*

THEOREM 2.6. *Let Φ be a linear mapping from a C^*-algebra A with real rank zero into a C^*-algebra B such that $\Phi(A_{\text{inv}} \cap A_1) \subset B_{\text{inv}} \cap B_1$ and $\Phi(1) = 1$. Then Φ is a C^*-homomorphism.*

3. Proofs

We use the following lemma to complete the proofs of both Theorems 2.5 and 2.6.

LEMMA 3.1. *Let Φ be a continuous linear mapping from a C^*-algebra A with real rank zero into a normed algebra B such that if p and q are mutually orthogonal projections in A, then $\Phi(p)$ and $\Phi(q)$ are mutually orthogonal idempotents in B. Then Φ is a Jordan homomorphism.*

Proof of Lemma 3.1. Let a be a selfadjoint element of A with finite spectrum and write $\sigma(a) = \{\lambda_1, \dots, \lambda_n\}$ where $\lambda_i \in \mathbb{R}$. Let further

$$p_j(\lambda) = \prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k}, \qquad p(\lambda) = \sum_{j=1}^{n} \lambda_j p_j(\lambda). \tag{3.1}$$

Let $e_j = p_j(a)$ for all j. We show that $\{e_1, \dots, e_n\}$ is a set of mutually orthogonal idempotents in A and $a = \sum_{j=1}^{n} \lambda_j e_j$. Each e_j is selfadjoint and

$$e_j^2 - e_j = (p_j^2 - p_j)(a). \tag{3.2}$$

By the spectral mapping theorem, if $i \neq j$,

$$\begin{aligned} \sigma(e_j^2 - e_j) &= (p_j^2 - p_j)(\sigma(a)) = \{0\}, \\ \sigma(e_i e_j) &= p_i p_j(\sigma(a)) = \{0\}, \\ \sigma(a - p(a)) &= (id - p)(\sigma(a)) = \{0\}. \end{aligned} \tag{3.3}$$

Hence, $e_j^2 - e_j = 0$, $e_i e_j = 0$ for $i \neq j$ and $a - p(a) = 0$.

Now put $f_j = \Phi(e_j)$ for all j. By assumption $\{f_1,\dots,f_n\}$ is a set of mutually orthogonal idempotents in B (containing possibly the zero idempotent). Then

$$\begin{aligned} a &= \sum_{j=1}^{n} \lambda_j e_j, & \Phi(a) &= \sum_{j=1}^{n} \lambda_j f_j, \\ a^2 &= \sum_{j=1}^{n} \lambda_j^2 e_j, & \Phi(a)^2 &= \sum_{j=1}^{n} \lambda_j^2 f_j. \end{aligned} \tag{3.4}$$

Hence, $\Phi(a^2) = \Phi(a)^2$.

Theorem 2.4 ensures that for any selfadjoint $a \in A$, there is a sequence a_n of selfadjoint elements of A with finite spectra such that $a_n \to a$ in norm. Then $a_n^2 \to a^2$. Hence, $\Phi(a_n) \to \Phi(a)$ and $\Phi(a_n^2) \to \Phi(a^2)$ by the continuity of Φ. Also

$$\Phi(a_n)^2 \longrightarrow \Phi(a)^2, \qquad \Phi(a_n^2) = \Phi(a_n)^2, \tag{3.5}$$

so $\Phi(a^2) = \Phi(a)^2$. It follows that $\Phi(x^2) = \Phi(x)^2$ for all $x \in A$ since $x = a + ib$ for some selfadjoint elements $a, b \in A$ and

$$(a+ib)^2 = a^2 - b^2 + i[(a+b)^2 - a^2 - b^2]. \tag{3.6}$$

This proves Lemma 3.1. □

The mapping Φ of Theorem 2.5 has the following properties given by Aupetit in [2].

Proposition 3.2. *Suppose A and B are semisimple Banach algebras and Φ is a spectrum-preserving linear map from A into B. Then Φ is injective, and if in addition Φ is onto, then $\Phi(1) = 1$ and Φ is continuous.*

Proof. To prove that Φ is injective, suppose $a \in A$ and $\Phi(a) = 0$. Then

$$\sigma(a+x) = \sigma(\Phi(a+x)) = \sigma(\Phi(x)) = \sigma(x) \tag{3.7}$$

for every $x \in A$. Hence, $a = 0$ by [8, Corollary 2.4].

To show that Φ preserves the identity write $\Phi(1) = 1 + q$ where $q \in B$. As Φ is spectrum-preserving, if $x \in A$, then

$$\begin{gathered} 1 + \sigma(\Phi(x)) = 1 + \sigma(x) = \sigma(1+x), \\ \sigma(\Phi(1+x)) = \sigma(1 + q + \Phi(x)) = 1 + \sigma(q + \Phi(x)), \end{gathered} \tag{3.8}$$

so $\sigma(\Phi(x)) = \sigma(q + \Phi(x))$. Then $q = 0$ again by [8, Corollary 2.4].

The continuity of Φ is proven in [1, Theorem 1].

The mappings of Theorems 2.5 and 2.6 both satisfy the assumptions of Lemma 3.1.

To prove Theorem 2.5, we need the next theorem of Aupetit [2, Theorem 1.2]. □

THEOREM 3.3. *If A and B are semisimple Banach algebras and if* Φ *is a spectrum-preserving operator from A onto B, then* Φ *transforms a set of mutually orthogonal idempotents of A to a set of mutually orthogonal idempotents of B.*

Lemma 3.1 completes the proof of Theorem 2.5.

Remarks 3.4. (a) Note that Φ is onto, so Proposition 3.2 implies that Φ is a homeomorphism and Φ^{-1} is spectrum-preserving. Hence, A and B are interchangeable in Theorem 2.5.

(b) The spectral resolution theorem [10, Theorem 5.5.2] ensures that in a von Neumann algebra a selfadjoint element is the norm limit of real linear combinations of orthogonal projections. Hence, von Neumann algebras have real rank zero.

Proof of Theorem 2.6. Let U denote the set of unitaries of A. In [6, Corollary 1], Harris gives an elegant proof of the fact that the open unit ball of A is the convex hull of U. A more elementary proof of Gardner can be found in [11, Proposition 3.2.23]. It follows easily that $\|a\|_u = \|a\|$ for $a \in A$ where

$$\|a\|_u := \inf\left\{\sum_{i=1}^{n} |\lambda_i| : a = \sum_{i=1}^{n} \lambda_i u_i,\ \lambda_i \in \mathbb{C},\ u \in U,\ n \in \mathbb{N}\right\}. \tag{3.9}$$

(See [13, Lemma 2].) For Φ satisfying the conditions of Theorem 2.6, we have that if $a \in A$ and

$$a = \sum_{j=1}^{n} \lambda_j u_j \tag{3.10}$$

then

$$\|\Phi(a)\| \leq \sum_{j=1}^{n} |\lambda_j|. \tag{3.11}$$

Hence, $\|\Phi(a)\| \leq \|a\|_u = \|a\|$ for every $a \in A$ and $\|\Phi\| = 1$.

As B is a C^*-algebra, this is enough to ensure $\Phi \geq 0$ by [13, Corollary 1], that is, $\Phi(a) \geq 0$ whenever $a \in A$ and $a \geq 0$.

Since Φ is an invertibility-preserving selfadjoint map from A into B, by [12, Lemma 3] Φ maps mutually orthogonal projections of A into mutually orthogonal idempotents of B. Hence, we can apply Lemma 3.1 and $\Phi(a^2) = \Phi(a)^2$ follows for $a \in A$. This proves Theorem 2.6. □

Remarks 3.5. (a) It follows from [4, Theorem 2] that the assumption that A has real rank zero can not be omitted in Theorem 2.6 even when A is commutative.

(b) It is known that if Φ is a linear bijection between C^*-algebras with $\Phi(A_{\text{inv}}) \subset B_{\text{inv}}$ and $\|\Phi\| \leq 1$, then Φ is a Jordan isomorphism (see [4, Theorem 6] and [7, Corollary 8]). Theorem 2.6 does not require bijectivity of the mapping.

(c) If in Theorem 2.6 we require only that $\Phi(1)$ is unitary, then Φ becomes a Jordan homomorphism followed by multiplication by $\Phi(1)$.

(d) The C^*-algebra generated by the compact operators $\mathcal{K}$ and the identity on an infinite-dimensional Hilbert space $\mathcal{H}$ has real rank zero, though it is not a von Neumann algebra. The Calkin algebra, which is the factor C^*-algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$, has real rank zero, though it is not a von Neumann algebra. All the Bunce-Deddens algebras, the Cuntz algebras, AF-algebras, and irrational rotation algebras have real rank zero (see [5]). The class of C^*-algebras with real rank zero is considerably wider than the class of von Neumann algebras. Thus Theorems 2.5 and 2.6 are nontrivial extensions of Theorems 2.1 and 2.2.

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