Research Article

Quantum-Corrected Two-Dimensional Horava-Lifshitz Black Hole Entropy

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We focus on the Hamilton-Jacobi method to determine several thermodynamic quantities such as temperature, entropy, and specific heat of two-dimensional Horava-Lifshitz black holes by using the generalized uncertainty principles (GUP). We also address the product of horizons, mainly concerning the event, Cauchy, and cosmological and virtual horizons.

1. Introduction

Theories of gravity in two space-time dimensions have received much attention in the literature [1–7] and provide an excellent theoretical basis for understanding issues relevant to quantum gravity. Such theories in recent years have presented a very rich structure and an interesting relationship with conformal field theory [8], the Liouville model [9–11], random lattice models [12], and sigma models [13–16]. Formally, gravity has similarity to four-dimensional general relativity. In fact, the solution of this theory has a nontrivial event horizon structure that enables the existence of black holes in two space-time dimensions. Recently, a new theory of gravity has been presented by Horava in [17], and this is now the well-known Horava-Lifshitz (HL) gravity. Many aspects of the theory have been considered in the literature [18–28]. In [29], a new Horava-Lifshitz black hole solution was proposed in two dimensions in the slow varying dilatonic field regime.

The main objective of the present study is to address the issues of quantum-corrected entropy in the two-dimensional HL black hole. A semiclassical approach considering the Hawking radiation as a tunneling phenomenon across the horizon has been proposed in [30, 31], in addition to the Hamilton-Jacobi method [32–39] to determine the Hawking radiation and the entropy of black holes. The tunneling formalism has also been applied to HL gravity, for instance,

in [40–44]. In [30, 45, 46], the method of radial null geodesic was used by the authors for calculating the Hawking temperature. In [47], applying the tunneling formalism, Hawking radiation has been investigated considering self-gravitation and back reaction effects. In [48, 49], the information loss paradox in the WKB/tunneling picture of Hawking radiation considering the back reaction effects was presented. More recently, using this formalism, the back reaction effects for self-dual black hole have also been investigated [50]. In [51], the quantum-corrected Hawking temperature and entropy of a Schwarzschild black hole considering the effects of the generalized uncertainty principle (GUP) in the tunneling formalism were calculated. Moreover, using the Hamilton-Jacobi version of the tunneling formalism, the Hawking radiation for acoustic black hole was investigated [52–58] and in [59] the thermodynamic properties of self-dual black holes and noncommutative BTZ black hole have been discussed [60, 61]. In [62], the corrections for the thermodynamics of black holes were analyzed assuming that the GUP corrected entropy-area relation is universal for all black objects.

A lot of work has been proposed in the literature in order to understand the quantum aspects of the black hole entropy—see, for instance, [63–72]. In [73, 74], the authors have shown that the quantum corrections to the Bekenstein-Hawking entropy are logarithmic and dependent on the area. Furthermore, in [75], an additional correction term was

obtained to the entropy that depends on conserved charges. In addition, in [76–78], using the brick-wall method, entropy of acoustic black hole in two dimensions was investigated. However, when determining the entropy by this method, an ultraviolet cut-off must be inserted in the calculations to eliminate the divergence of states density near black hole horizon. On the other hand, considering models in which the Heisenberg uncertainty relation is modified, the divergence that arises in the brick-wall model is eliminated [79–95]. For example, in one-dimensional space, we have the following modified Heisenberg uncertainty relation:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \alpha^2 \left(\Delta p \right)^2 \right], \tag{1}$$

where Δx and Δp are uncertainties for position and momentum, respectively, and α is a positive constant which is independent of Δx and Δp. The thermodynamic properties of black holes are modified due to the GUP [96, 97]. See, for instance, [98, 99], which review how to approach space-time noncommutativity that leads to an effective GUP to quantum gravity.

In order to obtain a change in the temperature of the black hole, first the bound on the maximum momentum in GUP is identified with a bound on maximum energy of the system and the uncertainty in the position can be taken to be proportional to the radius of the event horizon of the black hole [100].

In this paper, inspired by all of these previous works, we will mainly focus on new Horava-Lifshitz black hole solutions in two dimensions and the Hamilton-Jacobi method in order to determine the temperature and the entropy of a black hole using the GUP.

2. Tunneling Formalism for 2D HL Black Hole

In this section, we will use the tunneling formalism to derive the Hawking temperature for a two-dimensional HL black hole. In our calculations, we assume that the classical action satisfies the relativistic Hamilton-Jacobi equation to leading order in the energy. The metric in (Arnowitt-Deser-Misner) ADM decomposition is

$$ds^2 = -N^2 dt^2 + g_{ij} \left(dx^i + N^i dt \right) \left(dx^j + N^j dt \right) \tag{2}$$

with anisotropic scaling between space and time, $t \to b^{-z} t$, $x^i \to b^{-1} x^i$, $i = 1, 2, \ldots, D$. The power-counting renormalizability requires $z \geq D$. In $1 + 1$ dimensions, this means $z \geq 1$ such that we will assume $z = 1$, that is, infrared regime. Despite this choice, the HL gravity does not coincide with Einstein gravity in general—see [17, 43, 44] for further discussions on IR issues in HL gravity. Now, by using the gauge $N_1 = 0$ and considering $g_{ij} \equiv g_{11} = N^{-2}$, we have

$$ds^2 = -f(x) dt^2 + f(x)^{-1} dx^2, \tag{3}$$

where we have redefined $N^2 \equiv f(x)$. The two-dimensional HL black hole solutions are explicitly given by [29]

$$f(x) = 2C_2 + \frac{A}{\eta} x^2 - 2C_1 x + \frac{B}{\eta x} + \frac{C}{3\eta x^2}, \tag{4}$$

where η is related to the nonprojectable version of HL gravity. In the sequel, we consider the Klein-Gordon (KG) equation

$$\hbar^2 g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - m^2 \phi = 0. \tag{5}$$

Notice that the KG equation preserves its usual form because we have fixed $z = 1$ and the gauge $N_1 = 0$ as discussed above. Similar considerations in higher dimensions have been considered both in arbitrary [40–42] and in fixed $z = 1$ (IR regime) [43, 44]. Now, considering metric (3), we have

$$-\partial_t^2 \phi + f(x)^2 \partial_x^2 \phi + \frac{1}{2} f(x)^{2\prime} \partial_x \phi - \frac{m^2}{\hbar} f(x) \phi = 0. \tag{6}$$

Next, we apply the WKB approximation to ϕ given by

$$\phi(x, t) = \exp\left[-\frac{i}{\hbar} \mathcal{I}(x, t) \right], \tag{7}$$

and, for the lowest order in $\hbar$, we have

$$\left(\partial_t \mathcal{I} \right)^2 - f(x)^2 \left(\partial_x \mathcal{I} \right)^2 - m^2 f(x) = 0. \tag{8}$$

Because of the symmetries of the metric, we can write a solution to $\mathcal{I}(x, t)$ in the form

$$\mathcal{I}(x, t) = -\omega t + W(r), \tag{9}$$

where for $W(r)$ we have

$$W = \int \frac{dx}{f(x)} \sqrt{\omega^2 - m^2 f(x)}. \tag{10}$$

At this point, we can apply near the horizon the following approximation:

$$f(x) = f\left(x_{\mathrm{h}}^+ \right) + f'\left(x_{\mathrm{h}}^+ \right) \left(x - x_{\mathrm{h}}^+ \right) + \cdots. \tag{11}$$

In this way, for the spatial part of the action function, we find

$$\begin{aligned} W &= \int \frac{dx}{f'\left(x_{\mathrm{h}}^+ \right)} \frac{\sqrt{\omega^2 - m^2 f'\left(x_{\mathrm{h}}^+ \right) \left(x - x_{\mathrm{h}}^+ \right)}}{\left(x - x_{\mathrm{h}}^+ \right)} \\ &= \frac{2\pi i \omega}{f'\left(x_{\mathrm{h}}^+ \right)}. \end{aligned} \tag{12}$$

Therefore, the tunneling probability for a particle with energy ω is given by

$$\Gamma \cong \exp\left[-2 \operatorname{Im} \mathcal{I} \right] = \exp\left[-\frac{4\pi\omega}{f'\left(x_{\mathrm{h}}^+ \right)} \right]. \tag{13}$$

Thus, comparing (13) with the Boltzmann factor ($e^{-\omega/T}$), we obtain the general Hawking temperature formula for the black hole solution (4) as

$$T_{\mathrm{HL}} = \frac{\omega}{2 \operatorname{Im} \mathcal{I}} = \frac{f'\left(x_{\mathrm{h}}^+ \right)}{4\pi}. \tag{14}$$

Below we will mainly consider three cases.

2.1. First Case: Schwarzschild-Like Black Hole. In this case, we consider $C_1 \neq 0$, $C_2 \neq 0$, $B \neq 0$, and $A = C = 0$ into (4) and the metric becomes

$$f(x) = 2C_2 - 2C_1 x + \frac{B}{\eta x}. \tag{15}$$

The event horizons can be obtained at $f(x) = 0$, such that we have

$$x_h^{\pm} = \frac{C_2}{2C_1} \pm \sqrt{\frac{C_2^2}{4C_1^2} + \frac{B}{2C_1 \eta}}. \tag{16}$$

For special case where $C_2 = 0$, $C_1 = -M$, and $B = -4M\Lambda^2$ (where Λ is a parameter with dimension of length), the horizons are

$$x_h^{\pm} = \pm \Lambda \sqrt{\frac{2}{\eta}}. \tag{17}$$

Thus, considering (15) and substituting (17) into (14), we obtain the temperature given by

$$T_{HL1} = \frac{M}{\pi}. \tag{18}$$

Since the radius of the horizon in (17) is independent of the mass M, the Hawking temperature is directly proportional to the mass parameter M, contrary to the case in four dimensions where Hawking temperature is inversely proportional to the mass parameter M. To compute the entropy, we use

$$S_{HL1} = \int \frac{dM}{T_{HL1}}, \tag{19}$$

where, substituting (18) into (19), we find

$$S_{HL1} = \pi \ln \left(\frac{M}{M_0} \right) = \pi \ln \left(\pi M^2 \right) - \pi \ln \left(\pi M M_0 \right). \tag{20}$$

This is the expected entropy in $(1+1)$-dimensional black holes [1]. Furthermore, the first term in (20) resembles a correction term which is of type $\ln(A/4) = \ln(4\pi M^2)$ for the entropy of black holes in four dimensions. Notice that the meaning of entropy for two-dimensional black holes is different from higher dimensional cases. This is because the event horizon is a point, that is, has no area. However, it still enjoys the thermodynamic relationship [101–103]

$$dM = TdS - \Phi dQ, \tag{21}$$

where Φ is the electric potential and there is no angular momentum term. Thus, the horizon has its own associated temperature and entropy and we can use (21) to define the entropy [1]. Since the constant M_0 plays the role of a fundamental length, the thermodynamic properties of a two-dimensional black hole require this length [1]. This seems to have the characteristic of theories that break down at the semiclassical regime. So, a minimum measurable length implies a major revision of quantum physics [95]. These approaches are precisely the GUP, as we see in Section 3.

Several other interesting thermodynamic quantities can also be found as follows. The specific heat is given by

$$C = T \frac{\partial S}{\partial T}, \tag{22}$$

which from (18) and (20) reads

$$C_{HL1} = \pi. \tag{23}$$

Furthermore, the Hawking temperature can be used to compute the emission rate. Let us assume that in the black hole the energy loss is dominated by photons [104, 105]. Then, using the Stefan-Boltzmann law in two-dimensional space-time, we have

$$\frac{dM}{dt} \propto T^2. \tag{24}$$

Thus, the emission rate in this case is

$$\frac{dM_{HL1}}{dt} \propto \frac{M^2}{\pi^2}. \tag{25}$$

2.2. Second Case: Reissner-Nordström-Like Black Hole. In this second case, we will make $B = C_1 = C_2 = 0$ and $C = -3Q^2\Lambda^2$ in (4), so for the function $f(x)$ we have

$$f(x) = \frac{A}{\eta} x^2 - \frac{Q^2 \Lambda^2}{\eta x^2}. \tag{26}$$

Now, choosing $A = \Lambda^{-2}$ and $Q^2 = M^2 \Lambda^2$ (for an extreme-like case), we have that the event horizons are

$$x_h^{\pm} = \pm \left(\frac{Q^2 \Lambda^2}{A} \right)^{1/4} = \pm \Lambda \sqrt{M\Lambda}. \tag{27}$$

Using (14), the temperature is given by

$$T_{HL2} = \frac{1}{\pi \eta} \left(Q^2 \Lambda^2 A^3 \right)^{1/4} = \frac{1}{\pi \eta} \sqrt{\frac{M}{\Lambda}}. \tag{28}$$

Here, unlike the first case, the Hawking temperature is proportional to $\sqrt{M}$. For the entropy, we find the following result:

$$S_{HL2} = \int \frac{dM}{T_{HL2}} = 2\pi \eta \sqrt{M\Lambda}. \tag{29}$$

Now, following the same steps of the previous case, we can also compute identical thermodynamic quantities. From (28) and (29), the specific heat now reads

$$C_{HL2} = 2\pi \eta \sqrt{M\Lambda}, \tag{30}$$

and the emission rate is

$$\frac{dM_{HL2}}{dt} \propto \frac{1}{(\pi \eta)^2} \frac{M}{\Lambda}. \tag{31}$$

2.3. The AdS-Schwarzschild-Like Case. In this case, we consider $A \neq 0$, $B \neq 0$, and $C = C_1 = C_2 = 0$ into (4) and the metric becomes

$$f(x) = \frac{A}{\eta} x^2 + \frac{B}{\eta x}. \tag{32}$$

Now, choosing $A = \Lambda^{-2}$ and $B = -4M\Lambda^2$, we get the real event horizon which is

$$x_{\mathrm{h}} = \left(-\frac{B}{A}\right)^{1/3} = \left(4M\Lambda^4\right)^{1/3}. \tag{33}$$

Using (14), the temperature is given by

$$T_{\mathrm{AdS\text{-}S}} = \frac{3}{4\pi\eta} \left(-A^2 B\right)^{1/3} = \frac{3}{4^{2/3}\pi\eta} \left(\frac{M}{\Lambda^2}\right)^{1/3}. \tag{34}$$

For the entropy, we find the following result:

$$S_{\mathrm{AdS\text{-}S}} = 2^{1/3} \pi\eta \left(M\Lambda\right)^{2/3}. \tag{35}$$

Finally, in the present case, from (34) and (35), we find the specific heat

$$C_{\mathrm{AdS\text{-}S}} = 2^{4/3} \pi\eta\Lambda \frac{M}{(M\Lambda)^{1/3}}. \tag{36}$$

The emission rate now reads

$$\frac{dM_{\mathrm{AdS\text{-}S}}}{dt} \propto \frac{9}{4^{4/3}\pi^2\eta^2} \left(\frac{M}{\Lambda^2}\right)^{2/3}. \tag{37}$$

Notice that, except in the first example, all the thermodynamic quantities go to zero as $M \to 0$. This phenomenon prevents the existence of black hole remnants [96, 97, 104, 105]. In the next section, we will find black holes with richer thermodynamic scenarios due to the GUP.

3. Quantum Corrections to the Entropy

In this section, we will consider the GUP and we will apply the Hamilton-Jacobi method in tunneling formalism to calculate the quantum-corrected Hawking temperature and entropy for a two-dimensional Horava-Lifshitz black hole. Hence, for the GUP, we have [79–95, 100]

$$\Delta x \Delta p \geq \hbar \left(1 - \alpha \frac{l_{\mathrm{p}}}{\hbar} \Delta p + \frac{\alpha^2 l_{\mathrm{p}}^2}{\hbar^2} \left(\Delta p\right)^2\right), \tag{38}$$

where α is a dimensionless positive parameter and l_{p} is the Planck length. We can still write (38) as follows:

$$\Delta p \geq \frac{\left(\Delta x + \alpha l_{\mathrm{p}}\right)}{2\alpha^2 l_{\mathrm{p}}^2} \left(1 - \sqrt{1 - \frac{4\alpha^2 l_{\mathrm{p}}^2}{\left(\Delta x + \alpha l_{\mathrm{p}}\right)^2}}\right), \tag{39}$$

where we have chosen the negative sign and $\hbar = 1$. Here, since $l_{\mathrm{p}}/\Delta x \ll 1$, the above equation can be expanded into Taylor series as

$$\Delta p \geq \frac{1}{\Delta x} \left[1 - \frac{\alpha l_{\mathrm{p}}}{\Delta x} + \frac{2\alpha^2 l_{\mathrm{p}}^2}{(\Delta x)^2} + \cdots\right]. \tag{40}$$

Now, with $\hbar = 1$, the uncertainty principle becomes $\Delta x \Delta p \geq 1$ and applying the saturated form of the uncertainty principle we have $\omega \Delta x \geq 1$. Thus, (40) can be written as

$$\omega_{\mathrm{G}} \geq \omega \left[1 - \frac{\alpha l_{\mathrm{p}}}{2(\Delta x)} + \frac{\alpha^2 l_{\mathrm{p}}^2}{2(\Delta x)^2} + \cdots\right], \tag{41}$$

where ω is the energy of a quantum particle.

Therefore, for a particle with energy corrected ω_{G}, the tunneling probability reads

$$\Gamma_{\mathrm{G}} \cong \exp\left[-2 \operatorname{Im} I_{\mathrm{G}}\right]. \tag{42}$$

Consequently, the corrected temperature becomes

$$T_{\mathrm{HLG}} = \frac{\omega}{2 \operatorname{Im} I_{\mathrm{G}}}$$

$$= T_{\mathrm{HL}} \left[1 - \frac{\alpha l_{\mathrm{p}}}{2(\Delta x)} + \frac{\alpha^2 l_{\mathrm{p}}^2}{2(\Delta x)^2} + \cdots\right]^{-1}. \tag{43}$$

In the following, we will consider three cases.

3.1. The First Case with GUP. Here, we choose $\Delta x = 2x_{\mathrm{h}}^+ = 2\Lambda\sqrt{2/\eta}$. Thus, for the first case, the corrected temperature due to the GUP is

$$T_{\mathrm{HLG1}} = T_{\mathrm{HL1}} \left[1 - \frac{\alpha l_{\mathrm{p}}}{4\Lambda} \sqrt{\frac{\eta}{2}} + \frac{\eta\alpha^2 l_{\mathrm{p}}^2}{16\Lambda^2} + \cdots\right]^{-1}. \tag{44}$$

Hence, the corrected entropy becomes

$$S_{\mathrm{HLG1}} = \int \frac{dM}{T_{\mathrm{HLG1}}}$$

$$= \left[\pi - \frac{\pi\alpha l_{\mathrm{p}}}{4\Lambda} \sqrt{\frac{\eta}{2}} + \frac{\pi\eta\alpha^2 l_{\mathrm{p}}^2}{16\Lambda^2} + \cdots\right] \ln\left(\frac{M}{M_0}\right). \tag{45}$$

Corrections due to GUP for entropy do not change the dependence of the mass parameter that is always of the type $\ln(M)$.

As in the previous section, other thermodynamic quantities can also be found. Here, they appear to be corrected by the GUP. Namely, the corrected specific heat is

$$C_{\mathrm{HLG1}} = C_{\mathrm{HL1}} \left[1 - \frac{\alpha l_{\mathrm{p}}}{4\Lambda} \sqrt{\frac{\eta}{2}} + \frac{\eta\alpha^2 l_{\mathrm{p}}^2}{16\Lambda^2} + \cdots\right] \tag{46}$$

and the corrected emission rate reads

$$\frac{dM_{\mathrm{HLG1}}}{dt} \propto \frac{M^2}{\pi^2} \left[1 - \frac{\alpha l_{\mathrm{p}}}{4\Lambda} \sqrt{\frac{\eta}{2}} + \frac{\eta\alpha^2 l_{\mathrm{p}}^2}{16\Lambda^2} + \cdots\right]^{-2}. \tag{47}$$

Due to the GUP we can also address the issue of minimum mass of black holes. Thus, from (39), we can ensure the following inequality (notice that we have reinserted a factor such as $\Delta x \to 2\Delta x$ in the GUP):

$$4\alpha^2 l_{\mathrm{p}}^2 \leq \left(2\Delta x + \alpha l_{\mathrm{p}}\right)^2. \tag{48}$$

However, in the present case, since the horizon is mass independent, that is, $\Delta x = 2\Lambda\sqrt{2/\eta}$, we simply find a minimum length scale given by

$$\Lambda_{\min} = \frac{\alpha l_p}{4\sqrt{2/\eta}}. \tag{49}$$

3.2. The Second Case with GUP. Now, for the second case, we have $\Delta x = 2\Lambda\sqrt{M\Lambda}$ and the corrected temperature due to the GUP reads

$$T_{\text{HLG2}} = T_{\text{HL2}}\left[1 - \frac{\alpha l_p}{4\Lambda\sqrt{M\Lambda}} + \frac{\alpha^2 l_p^2}{8M\Lambda^3} + \cdots\right]^{-1}. \tag{50}$$

Consequently, for the corrected entropy, we obtain

$$S_{\text{HLG2}} = \int \frac{dM}{T_{\text{HLG2}}}$$

$$= 2\pi\eta\sqrt{M\Lambda} - \frac{\pi\eta\alpha l_p}{4\Lambda}\ln\left(\frac{M}{M_0}\right) - \frac{\pi\eta\alpha^2 l_p^2}{4\Lambda^3}\sqrt{\frac{\Lambda}{M}} \tag{51}$$

$$+ \cdots.$$

In this example, besides other types of corrections, a logarithmic correction to the entropy of the black hole has been obtained. This logarithmic correction arises from the contribution $\alpha l_p(\Delta p)$ in the GUP.

Again, computing other thermodynamic quantities, we have now the corrected specific heat given by

$$C_{\text{HLG2}} = 2\pi\eta\sqrt{M\Lambda} - \frac{\pi\eta\alpha l_p}{2\Lambda} + \frac{\pi\eta\alpha^2 l_p^2}{4\sqrt{\Lambda/M}M\Lambda^2} + \cdots \tag{52}$$

and the corrected emission rate now reads

$$\frac{dM_{\text{HLG2}}}{dt}$$

$$\propto \frac{1}{(\pi\eta)^2}\frac{M}{\Lambda}\left[1 - \frac{\alpha l_p}{4\Lambda\sqrt{M\Lambda}} + \frac{\alpha^2 l_p^2}{8M\Lambda^3} + \cdots\right]^{-2}. \tag{53}$$

In the present case, the horizon is mass dependent. Thus, substituting $\Delta x = 2\Lambda\sqrt{M\Lambda}$ into (48), the minimum mass is given by

$$M_{\min} = \frac{\alpha^2 l_p^2}{16\Lambda^3}. \tag{54}$$

3.3. The AdS-Schwarzschild-Like Case with GUP. Now, in this case, we have $\Delta x = 2(4M\Lambda^4)^{1/3}$ and the corrected temperature due to the GUP reads

$$T_{\text{AdS-S-G}} = T_{\text{AdS-S}}\left[1 - \frac{\alpha l_p 4^{2/3}}{16(M\Lambda^4)^{1/3}} + \frac{\alpha^2 l_p^2 4^{1/3}}{32(M\Lambda^4)^{2/3}}\right.$$

$$\left.+ \cdots\right]^{-1}. \tag{55}$$

Consequently, for the corrected entropy, we obtain

$$S_{\text{AdS-S-G}} = \int \frac{dM}{T_{\text{AdS-S-G}}}$$

$$= 2^{1/3}\pi\eta(M\Lambda)^{2/3} - \frac{1}{4}\frac{\pi\eta 2^{2/3}\alpha l_p(M\Lambda)^{1/3}}{\Lambda} \tag{56}$$

$$+ \frac{1}{24}\frac{\pi\eta\alpha^2 l_p^2}{\Lambda^2}\ln\left(\frac{M}{M_0}\right) + \cdots.$$

Note that the result has other terms besides the logarithmic contribution. $M^{1/3}$ and logarithmic terms come from $\alpha l_p(\Delta p)$ and $\alpha^2 l_p^2(\Delta p)^2$ corrections into the GUP, respectively.

Finally, in the third case, the corrected specific heat is

$$C_{\text{AdS-S-G}} = 2^{4/3}\pi\eta(\Lambda M)^{1/2} - \frac{2^{2/3}}{4}\frac{\pi\eta\alpha l_p M}{(M\Lambda)^{2/3}}$$

$$+ \frac{1}{8}\frac{\pi\eta\alpha^2 l_p^2}{\Lambda^2} + \cdots \tag{57}$$

and the corrected emission rate is given by

$$\frac{dM_{\text{AdS-S-G}}}{dt} \propto \frac{9}{4^{4/3}\pi^2\eta^2}\left(\frac{M}{\Lambda^2}\right)^{2/3}$$

$$\cdot\left[1 - \frac{\alpha l_p 4^{2/3}}{16(M\Lambda^4)^{1/3}} + \frac{\alpha^2 l_p^2 4^{1/3}}{32(M\Lambda^4)^{2/3}} + \cdots\right]^{-2}. \tag{58}$$

In this case, replacing $\Delta x = 2(4M\Lambda^4)^{1/3}$ into (48), we get the minimum mass

$$M_{\min} = \frac{\alpha^3 l_p^3}{256\Lambda^4}. \tag{59}$$

As we can easily check, in the three examples studied above, the temperature and emission rate go to zero as $M \to 0$, though neither the entropy (unless by considering back reaction effects [48, 49] in order to address the issue of information loss) nor specific heat vanishes at this limit. However, it has already been shown in the literature that at the minimal mass the specific heat indeed goes to zero. This is particularly clear as one considers the exact formula of the specific heat [96, 97, 104, 105] rather than the approximated formulas above. By properly working with the GUP, we can find an exact expression for the temperature as

$$T_{\text{HLG}} = 2T_H\left(1 + \frac{\alpha l_p}{2\Delta x}\right)^{-1}$$

$$\cdot\left[1 + \sqrt{1 - \frac{4}{(1 + 2\Delta x/\alpha l_p)^2}}\right]^{-1}, \tag{60}$$

which approaches the maximum $T_{\max} = T_H$ when bound (48) is saturated, that is, at $\Delta x = \alpha l_p/2$, where the black hole achieves the minimum mass, as we have discussed above.

As a consequence, we can also find an exact expression for the specific heat as follows. Let us first consider the last two cases above, where the horizons depend on black hole mass M. Since $T_H = 1/4\pi\Delta x$, then, making the scaling $\Delta x = \alpha l_p f(M)/2$ into (60), where $f(M)$ is a function of the mass whose first derivative $f'(M) \neq 0$, we find the specific heat

$$C_{HLG} = -\left(1 + f(M)\right)^2$$

$$\cdot \sqrt{\frac{(f(M) + 3)(f(M) - 1)}{(1 + f(M))^2}} \left(1 \right.$$

$$\left. + \sqrt{\frac{(f(M) + 3)(f(M) - 1)}{(1 + f(M))^2}}\right) \frac{\alpha l_p \pi}{f'(M)}. \tag{61}$$

Notice that, for minimum mass, that is, as $\Delta x \rightarrow \alpha l_p/2$, we have $f(M) \rightarrow 1$ and then the specific heat $C_{HLG} \rightarrow 0$. On the other hand, in the first case above, entropy (45) vanishes as $M \rightarrow M_0$, where M_0 (a minimum mass) is normally associated with the Planck scale. Interestingly, the minimum length scale found in (49) appears to be directly related to the Planck length l_p, which suggests that $M_0 \sim 1/\Lambda_{min}$ is a natural choice.

Thus, the studies of the three cases above show that the minimum masses (or length scale, in the first case) imply the existence of black hole remnants at which the specific heat (or entropy) vanishes and ceases to radiate even if the effective temperature (T_{HLG}) reaches a maximum [96, 97]. In other words, in such a scenario, one prevents black holes from entire evaporation [104, 105].

4. Product of Event Horizons

In this section, we will consider the products of horizon. Such products are often formulated in terms of the areas of inner (Cauchy) horizons and outer (event) horizons and sometimes include the effects of unphysical virtual horizons. It is conjectured that the product of the areas for multihorizon stationary black holes is in some cases independent of the mass of the black hole [106–112]. However, there are studies in the literature where the areas product is dependent on the mass [113]. It was also shown in [114] for acoustic black hole that the universal aspects of the areas product depend only on quantized quantities such as conserved electric charge and angular momentum. Recently, in [60], it has been shown for noncommutative BTZ black holes that the product of entropy is dependent on the mass parameter M up to linear order in the noncommutative parameter θ and becomes independent of the mass when $\theta = 0$. The areas product with the intriguing property of depending only on conserved charges has attracted much interest in string theory [106–112] microscopic description of black hole entropy once the area products in terms of quantized charges and quantized angular momenta may provide the basis of microstates counting. In the following, we are going to investigate such universal aspects with the introduction of the GUP.

Let us first start with metric (15) assuming $C_2 = 1/2$, $C_1 = -M$, and $B = -4M\Lambda^2$, so

$$x_h^{\pm} = \frac{1}{4M} \pm \sqrt{\frac{1}{16M^2} + \frac{2\Lambda^2}{\eta}}. \tag{62}$$

Note that

$$x_h^+ x_h^- = -\frac{2\Lambda^2}{\eta}. \tag{63}$$

The product of the radii of horizons is independent of the mass parameter M.

On the other hand, considering the quantum corrections due to the GUP, the horizon radius is changed. From (38) (with $\hbar = 1$) and solving for Δx, we have

$$\Delta x \geqslant \frac{1}{\Delta p}\left(1 - \alpha l_p \Delta p + \alpha^2 l_p^2 (\Delta p)^2\right), \tag{64}$$

which can be written as

$$r_G \geqslant r_h\left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2\right), \tag{65}$$

where we have identified $\Delta x = 2r_G$, $1/\Delta p = 2r_h$, and $\Delta p = M$. Thus,

$$x_G^{\pm} \geqslant x_h^{\pm}\left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2\right), \tag{66}$$

and the product $x_G^+ x_G^-$ becomes

$$x_G^+ x_G^- = -\frac{2\Lambda^2}{\eta}\left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2\right)^2. \tag{67}$$

This product is now dependent on the mass parameter M.

Now, we consider the case where $C_1 = 0$, $C_2 = 1/2$, $C = 3Q^2\Lambda^2$, and $A = -1/\Lambda^2$ in (4):

$$f(x) = 1 - \frac{1}{\Lambda^2\eta}x^2 + \frac{B}{\eta x} + \frac{Q^2\Lambda^2}{\eta x^2}. \tag{68}$$

At $f(x) = 0$, we find the quartic written as

$$x^4 - \Lambda^2\left(\eta x^2 + Bx + Q^2\Lambda^2\right) = 0, \tag{69}$$

or better

$$x^4 - \Lambda^2\left(x - x_+\right)\left(x - x_-\right) = 0, \tag{70}$$

where

$$x_{\pm} = \frac{-B \pm \sqrt{B^2 - 4\eta Q^2\Lambda^2}}{2\eta}. \tag{71}$$

4.1. Approximate Results. First, we rearrange the quartic to yield the exact equation

$$x = x_{\pm} + \frac{x^4}{\Lambda^2\left(x - x_{\mp}\right)} \tag{72}$$

and then try to solve it perturbatively, so for the event horizon we can write the following approximation:

$$x_E \approx x_+ + \frac{x_+^4}{\Lambda^2 (x_+ - x_-)} = x_+ \left(1 + \frac{x_+^3}{\Lambda^2 (x_+ - x_-)} \right). \quad (73)$$

On the other hand, for the inner (Cauchy) horizon, we find

$$x_C \approx x_- - \frac{x_-^4}{\Lambda^2 (x_+ - x_-)} = x_- \left(1 - \frac{x_-^3}{\Lambda^2 (x_+ - x_-)} \right). \quad (74)$$

Consequently, the product of horizons is

$$x_E x_C \approx x_+ x_- \left(1 + \frac{x_+^3 - x_-^3}{\Lambda^2 (x_+ - x_-)} \right), \quad (75)$$

or simply

$$x_E x_C \approx x_+ x_- \left(1 + \frac{x_+^2 + x_+ x_- + x_-^2}{\Lambda^2} \right). \quad (76)$$

In terms of B and η, we know that

$$x_+ x_- = \frac{Q^2 \Lambda^2}{\eta},$$

$$x_\pm^2 = \frac{2B^2 \mp 2B \sqrt{(B^2 - 4\eta Q^2 \Lambda^2)} + 4\eta Q^2 \Lambda^2}{4\eta^2}, \quad (77)$$

so that

$$x_+^2 + x_+ x_- + x_-^2 = \frac{B^2 + \eta Q^2 \Lambda^2}{\eta^2}. \quad (78)$$

This implies a product of horizons as follows:

$$x_E x_C \approx \frac{Q^2 \Lambda^2}{\eta} \left(1 + \frac{B^2 - 3\eta Q^2 \Lambda^2}{\Lambda^2 \eta^2} \right). \quad (79)$$

In addition, we also have the relationship

$$x_E + x_C \approx -\frac{B}{\eta} + \frac{x_+^4 - x_-^4}{\Lambda^2 (x_+ - x_-)}, \quad (80)$$

which again is explicitly dependent on parameters Q and B. For $\Lambda^2 \to \infty$, identifying Q as the charge and $B = -4M\Lambda^2$, the product of the radii of horizons is independent of the mass parameter M, that is, $x_E x_C \approx Q^2 \Lambda^2/\eta$, whereas the sum of the radii is dependent on the mass M, $x_E + x_C \approx 4M\Lambda^2/\eta$. In [113], it has been argued that the nondependence of the mass often fails when the cosmological constant is added to calculate the product of the radii of the horizons.

Now, we consider the quantum corrections due to the GUP for this product. The corrections to (73) and (74) are

$$x_{EG} = x_E \left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right),$$

$$x_{CG} = x_C \left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right), \quad (81)$$

so for the product $x_{EG} x_{CG}$ we have obtained

$$x_{EG} x_{CG} = x_E x_C \left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right)^2$$

$$\approx \frac{Q^2 \Lambda^2}{\eta} \left(1 + \frac{B^2 - 3\eta Q^2 \Lambda^2}{\Lambda^2 \eta^2} \right) \quad (82)$$

$$\cdot \left(1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right)^2.$$

For the sum of the radii, we have

$$x_{EG} + x_{CG} = (x_E + x_C) \left[1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right]$$

$$\approx \left(-\frac{B}{\eta} + \frac{x_+^4 - x_-^4}{\Lambda^2 (x_+ - x_-)} \right) \left[1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right]. \quad (83)$$

Before finishing this section, let us address the issues concerning cosmological and virtual horizons as in the following.

4.1.1. Cosmological Horizon. Let us go back to the quartic to write it as in the following:

$$x^2 = \frac{\Lambda^2 (x - x_+)(x - x_-)}{x^2}. \quad (84)$$

For a zero-order approximation, we have

$$x_{CH} = \Lambda, \quad (85)$$

and that for a first order of approximation gives

$$x_{CH} = \Lambda \sqrt{\frac{(\Lambda - x_+)(\Lambda - x_-)}{\Lambda^2}}$$

$$= \Lambda \left(1 - \frac{x_+ + x_-}{2\Lambda} + \cdots \right), \quad (86)$$

$$= \Lambda \left(1 + \frac{B}{2\eta\Lambda} + \cdots \right).$$

Hence, the cosmological horizon becomes

$$x_{CH} = \Lambda + \frac{B}{2\eta}. \quad (87)$$

Thus, in this order of approximation, the result for the cosmological horizon does not depend on Q but depends on $B = -4M\Lambda^2$ and consequently depends on the mass parameter M.

The corrected cosmological horizon is

$$x_{CHG} = x_{CH} \left[1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right]$$

$$= \left(\Lambda + \frac{B}{2\eta} \right) \left[1 - \alpha l_p M + \alpha^2 l_p^2 M^2 \right]. \quad (88)$$

4.1.2. Virtual Horizon. Therefore, considering the exact result, we have the following virtual horizon:

$$x_{\mathrm{V}} = -\left(x_{\mathrm{E}} + x_{\mathrm{C}} + x_{\mathrm{CH}}\right) = -\Lambda - \frac{B}{\eta}, \tag{89}$$

which is also dependent on the mass parameter M.

The corrected virtual horizon is

$$\begin{aligned} x_{\mathrm{VG}} &= x_{\mathrm{V}} \left[1 - \alpha l_{\mathrm{p}} M + \alpha^2 l_{\mathrm{p}}^2 M^2\right] \\ &= \left(-\Lambda - \frac{B}{\eta}\right) \left[1 - \alpha l_{\mathrm{p}} M + \alpha^2 l_{\mathrm{p}}^2 M^2\right]. \end{aligned} \tag{90}$$

Notice that the effect of the GUP is essential to reveal mass dependence of the event horizon products. As we previously mentioned, a similar role is played by noncommutative black holes. This effect at least shows more examples where the conjectured mass independence of the event horizon products often fails once either a cosmological constant is added or quantum corrections via GUP are present.

5. Conclusions

In summary, by considering the GUP, we derive the two-dimensional Horava-Lifshitz black hole temperature and entropy using the Hamilton-Jacobi version of the tunneling formalism. In our calculations, the Hamilton-Jacobi method was applied to calculate the imaginary part of the action and the GUP was introduced by the correction to the energy of a particle due to gravity near horizon. We apply this to essentially three types of black holes: Schwarzschild, Reissner-Nordström, and AdS-Schwarzschild-like black holes. Furthermore, we also addressed the issues of event horizon products that can find relevance in computation of the microscopic entropy in $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. Several other interesting issues, such as applying holography techniques in two-dimensional models of QCD by using 2D AdS-Schwarzschild-like black holes, should be addressed elsewhere.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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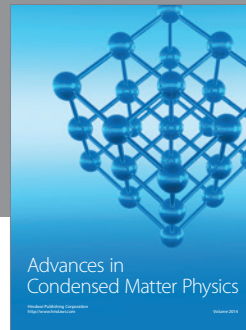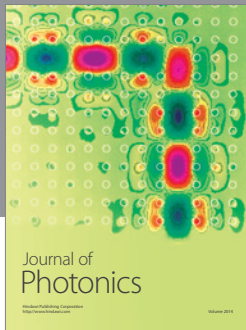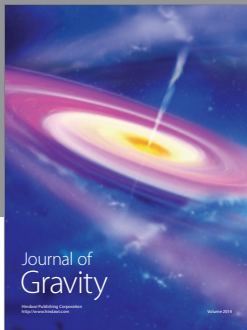
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