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Research Article

Approximate Analytical Solutions of the Fractional-Order Brusselator System Using the Polynomial Least Squares Method

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The paper presents a new method, called the Polynomial Least Squares Method (PLSM). PLSM allows us to compute approximate analytical solutions for the Brusselator system, which is a fractional-order system of nonlinear differential equations.

1. Introduction

In recent years, in many practical applications in various fields such as physics, mechanics, chemistry, and biology (see, e.g., [1–6]), the problems being studied are modeled using fractional nonlinear equations.

For most of such fractional nonlinear equations, the exact solutions cannot be found and, as a consequence, a numerical solution or, if possible, an analytical approximate solution of these equations is sought.

Due to the complexity of this type of problems, a general approximation algorithm does not exist and, thus, various approximation methods, each with its strong and weak points, were proposed, including, among others:

(i) the Adomian Decomposition Method [7–9],

(ii) the Homotopy Analysis Method [10–12],

(iii) the Homotopy Perturbation Method [13, 14],

(iv) the Laplace Transform Method [15, 16],

(v) the Fourier Transform Method [17],

(vi) the Variational Iteration Method [18–20].

The objective of our paper is to present the Polynomial Least Squares Method (PLSM), which allows us to compute approximate analytical solutions for the Brusselator system. The fractional order Brusselator system was recently studied by several authors [21–23] and can be expressed as follows.

We consider the following Brusselator system:

$$\begin{aligned} D_t^{\alpha_1} x(t) &= a - (\mu + 1) \cdot x(t) + x(t)^2 \cdot y(t), \\ D_t^{\alpha_2} y(t) &= \mu \cdot x(t) - x(t)^2 \cdot y(t), \end{aligned} \tag{1}$$

together with the initial conditions:

$$x(0) = c_1, \qquad y(0) = c_2, \tag{2}$$

where $a > 0$, $\mu > 0$, $0 < \alpha_1 \le 1$, $0 < \alpha_2 \le 1$, c_1, c_2 are real constants, and D_t^{α} denotes Caputo's fractional derivative [15]:

$$D_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^t (t-\zeta)^{-\alpha} \cdot x'(\zeta)\, d\zeta, \quad 0 < \alpha \le 1. \tag{3}$$

In the next section we will introduce PLSM for the Brusselator system and in the third section we will compare the approximate solutions obtained by using PLSM with the approximate solutions from [20]. The computations show that the approximations computed by using our method present an error smaller than the error of the corresponding solutions from [20].

2. The Polynomial Least Squares Method

For problem ((1), (2)) we consider the remainder operators:

$$\mathfrak{D}_1(x(t), y(t)) = D_t^{\alpha_1} x(t) - \left[a - (\mu+1)\cdot x(t) + x(t)^2 \cdot y(t)\right]$$
$$\mathfrak{D}_2(x(t), y(t)) = D_t^{\alpha_2} x(t) - \left[\mu \cdot x(t) - x(t)^2 \cdot y(t)\right]. \tag{4}$$

We will find approximate polynomial solutions $\tilde{x}(t), \tilde{y}(t)$ of problem ((1), (2)) on the $[0, b]$ interval, solutions which satisfy the following conditions:

$$\left|\mathfrak{D}_j(\tilde{x}(t), \tilde{y}(t))\right| < \epsilon, \quad j = 1, 2, \ \epsilon > 0 \tag{5}$$

$$\tilde{x}(0) = c_1, \qquad \tilde{y}(0) = c_2. \tag{6}$$

Definition 1. One calls an *ϵ-approximate polynomial solution* of problem ((1), (2)) an approximate polynomial solution $(\tilde{x}(t), \tilde{y}(t))$ satisfying relations ((5), (6)).

Definition 2. One calls a *weak δ-approximate polynomial solution* of problem ((1), (2)) an approximate polynomial solution $(\tilde{x}(t), \tilde{y}(t))$ satisfying the relations:

$$\int_0^b \mathfrak{D}_j^2(\tilde{x}(t), \tilde{y}(t))\, dt \le \delta, \quad j = 1, 2, \tag{7}$$

together with initial conditions (6).

Definition 3. One considers the sequence of polynomials: $P_m^j(t) = a_0^j + a_1^j t + \cdots + a_m^j t^m$, $a_i^j \in \mathbb{R}$, $i = 0, 1, \ldots, m$, $j = 1, 2$, satisfying the conditions:

$$P_m^1(0) = c_1, \quad P_m^2(0) = c_2, \quad m > 1, \ m \in \mathbb{N}. \tag{8}$$

One calls the sequence of polynomials $P_m^j(t)$ convergent to the solution of problem ((1), (2)) if $\lim_{m\to\infty}\mathfrak{D}_j(P_m^1(t), P_m^2(t)) = 0$.

We will find weak ϵ-polynomial solutions of the type:

$$\tilde{x}(t) = \sum_{k=0}^{m} d_k^1 \cdot t^k, \quad \tilde{y}(t) = \sum_{k=0}^{m} d_k^2 \cdot t^k, \quad m > 1, \tag{9}$$

where the constants $d_0^j, d_1^j, \ldots, d_m^j$, $j = 1, 2$ are calculated using the steps outlined as follows.

(i) We attach to problem ((1), (2)) the following real functional:

$$J\left(d_2^1, \ldots, d_m^1, d_2^2, \ldots, d_m^2\right) = \sum_{j=1}^{2} \int_0^b \mathfrak{D}_j^2(\tilde{x}(t), \tilde{y}(t))\, dt, \tag{10}$$

where d_0^1, d_0^2 are computed as functions of $d_1^1, d_2^1, \ldots, d_m^1, d_1^2, d_2^2, \ldots, d_m^2$ by using initial conditions (6).

(ii) We computes the values of $\overline{d}_1^1, \overline{d}_2^1, \ldots, \overline{d}_m^1, \overline{d}_1^2, \overline{d}_2^2, \ldots, \overline{d}_m^2$ as the values which give the minimum of functional (10) and the values of $\overline{d}_0^1, \overline{d}_0^2$ again as functions of $\overline{d}_1^1, \overline{d}_2^1, \ldots, \overline{d}_m^1, \overline{d}_1^2, \overline{d}_2^2, \ldots, \overline{d}_m^2$ by using the initial conditions.

(iii) Using the constants $\overline{d}_0^1, \overline{d}_1^1, \ldots, \overline{d}_m^1, \overline{d}_0^2, \overline{d}_1^2, \ldots, \overline{d}_m^2$ thus determined, we consider the polynomials:

$$T_m^1(t) = \sum_{k=0}^{m} \overline{d}_k^1 \cdot t^k, \quad T_m^2(t) = \sum_{k=0}^{m} \overline{d}_k^2 \cdot t^k, \quad m > 1. \tag{11}$$

The following convergence theorem holds.

Theorem 4. *The necessary condition for problem ((1), (2)) to admit sequences of polynomials $P_m^j(t)$ convergent to the solution of this problem is*

$$\lim_{m\to\infty} \int_0^b \mathfrak{D}_j^2\left(T_m^1(t), T_m^2(t)\right) dt = 0. \tag{12}$$

Moreover, $\forall\epsilon > 0$, $\exists m_0 \in \mathbb{N}$, such that, $\forall m \in \mathbb{N}$, $m > m_0$, it follows that $T_m^j(t)$, $j = 1, 2$, are weak ϵ-approximate polynomial solutions of problem ((1), (2)).

Proof. Based on the way the coefficients of the polynomials $T_m^j(t)$ are computed and taking into account relations ((9)–(11)), the following inequality holds:

$$0 \le \int_0^b \mathfrak{D}_j^2\left(T_m^1(t), T_m^2(t)\right) dt \le \int_0^b \sum_{j=1}^{n} \mathfrak{D}_j^2\left(P_m^1(t), P_m^2(t)\right) dt, \quad \forall m \in \mathbb{N}. \tag{13}$$

It follows that

$$0 \le \lim_{m\to\infty} \int_0^b \mathfrak{D}_j^2\left(T_m^1(t), T_m^2(t)\right) dt \le \lim_{m\to\infty} \int_0^b \sum_{j=1}^{n} \mathfrak{D}_j^2\left(P_m^1(t), P_m^2(t)\right) dt = 0, \quad \forall m \in \mathbb{N}. \tag{14}$$

We obtain

$$\lim_{m\to\infty} \int_0^b \mathfrak{D}_j^2\left(T_m^1(t), T_m^2(t)\right) dt = 0. \tag{15}$$

From this limit we obtain that, $\forall\epsilon > 0$, $\exists m_0 \in \mathbb{N}$ such that, $\forall m \in \mathbb{N}$, $m > m_0$, it follows that $T_m^j(t)$ are weak ϵ-approximate polynomial solutions of problem ((1), (2)), $j = 1, 2$. □

Remark 5. Any ϵ-approximate polynomial solutions of problem ((1), (2)) are also weak $\epsilon^2 \cdot b$-approximate polynomial solutions, but the opposite is not always true. It follows that the set of weak approximate solutions of problem ((1), (2)) also contains the approximate solutions of the system.

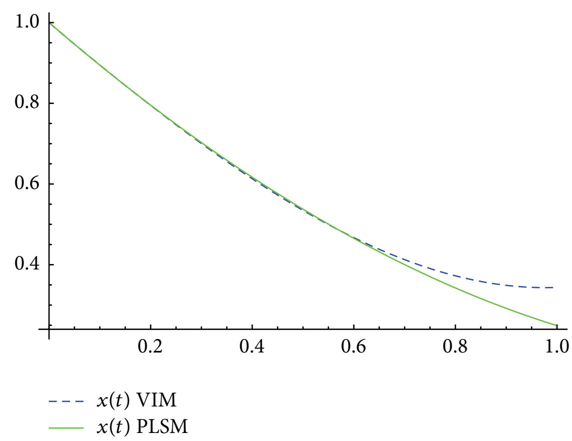

FIGURE 1: Approximations for $x(t)$ in the case $\alpha_1 = \alpha_2 = 0.98$.

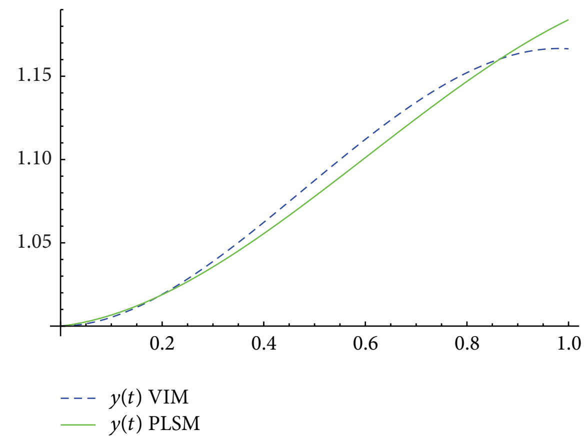

FIGURE 2: Approximations for $y(t)$ in the case $\alpha_1 = \alpha_2 = 0.98$.

Taking into account the above remark, in order to find ϵ-approximate polynomial solutions of problem ((1), (2)) by PLSM we will first determine weak approximate polynomial solutions, $\tilde{x}(t), \tilde{y}(t)$. If $|\mathfrak{D}_j(\tilde{x}(t), \tilde{y}(t))| < \epsilon$, $j = 1, 2$, then $\tilde{x}(t), \tilde{y}(t)$ are also ϵ-approximate polynomial solutions of the system.

3. Application: the Fractional-Order Brusselator System

We consider the following fractional-order Brusselator system [20]:

$$D_t^{\alpha_1} x(t) = -2 \cdot x(t) + x(t)^2 \cdot y(t),$$
$$D_t^{\alpha_2} y(t) = x(t) - x(t)^2 \cdot y(t), \tag{16}$$

together with the initial conditions:

$$x(0) = 1, \qquad y(0) = 1. \tag{17}$$

In [20] approximate solutions of (17) are computed using the Variational Iteration Method (VIM) for the case $\alpha_1 = \alpha_2 = 0.98$. Also, a comparison with numerical solutions is presented for the particular case $\alpha_1 = \alpha_2 = 1$, illustrating the applicability of the method.

3.1. The Case $\alpha_1 = \alpha_2 = 0.98$. For the case $\alpha_1 = \alpha_2 = 0.98$, using PLSM with $m = 3$, we obtain the following approximate polynomial solutions:

$$x_{\mathrm{PLSM}}(t) = 0.0243682 \cdot t^3 + 0.311138 \cdot t^2 - 1.08655 \cdot t + 1$$
$$y_{\mathrm{PLSM}}(t) = -0.184414 \cdot t^3 + 0.333424 \cdot t^2 + 0.0349127 \cdot t + 1. \tag{18}$$

In Figures 1 and 2 we compare these approximations with the corresponding approximations of the same order computed by VIM (relations (15) in [20]), obtaining a good agreement. In Figures 3 and 4 we compare the expressions of remainders (4) obtained by replacing the approximate solutions back in the equations. It is easy to observe that the errors obtained by using PLSM are smaller than the ones obtained by using VIM.

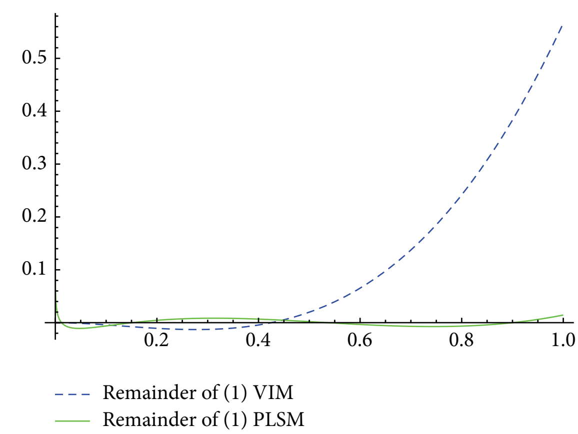

FIGURE 3: The remainders corresponding to the first equation for the case $\alpha_1 = \alpha_2 = 0.98$.

3.2. The Case $\alpha_1 = \alpha_2 = 1$. For the case $\alpha_1 = \alpha_2 = 1$, using PLSM with $m = 3$ we obtain the following approximate polynomial solutions:

$$x_{\mathrm{PLSM}}(t) = 0.0750974 \cdot t^3 + 0.201028 \cdot t^2 - 1.02827 \cdot t + 1$$
$$y_{\mathrm{PLSM}}(t) = -0.180088 \cdot t^3 + 0.334087 \cdot t^2 + 0.0271107 \cdot t + 1. \tag{19}$$

In this case both approximations (VIM and PLSM) consist of third-order polynomials.

We omitted the figures which compare our approximations with the ones given by VIM since they look almost the same as the corresponding ones from the case $\alpha_1 = \alpha_2 = 0.98$.

However, in the case $\alpha_1 = \alpha_2 = 1$ it is possible to compute the absolute error corresponding to an approximate solution

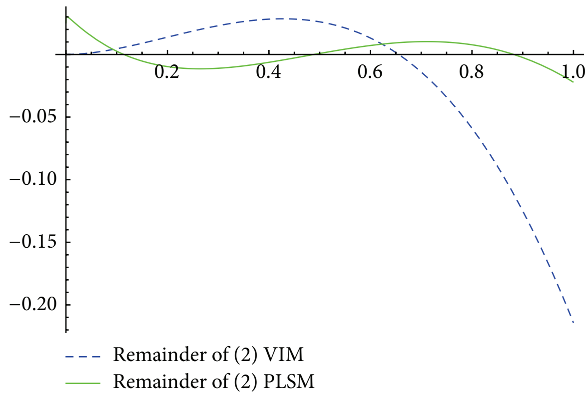

FIGURE 4: The remainders corresponding to the second equation for the case $\alpha_1 = \alpha_2 = 0.98$.

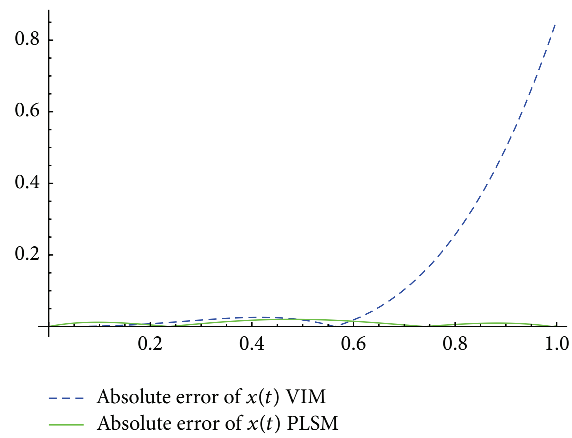

FIGURE 5: The absolute errors corresponding to $x(t)$ for the case $\alpha_1 = \alpha_2 = 1$.

as the difference in absolute value between the approximate solution and the numerical solution (in this case computed by using the Wolfram Mathematica software).

Figures 5 and 6 present the comparison between the absolute errors corresponding to the approximate solutions from [20] obtained by VIM and the absolute errors corresponding to our approximate solutions.

Again, it is easy to observe that the errors obtained by using PLSM are smaller than the ones obtained by using VIM.

4. Conclusions

In this paper we present the Polynomial Least Squares Method, which is a relatively straightforward and efficient method to compute approximate solutions for the fractional-order Brusselator system.

The comparison with previous results illustrates the accuracy of the method, since we were able to compute more precise approximations than the previously computed ones.

In closing we mention the fact that, due to the nature of the method, it is relatively easy to extend PLSM for the general

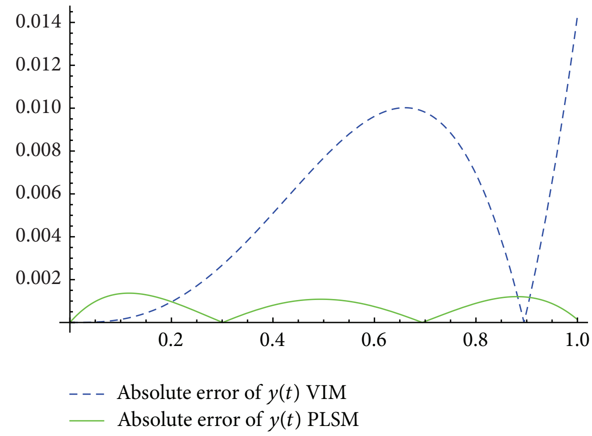

FIGURE 6: The absolute errors corresponding to $y(t)$ for the case $\alpha_1 = \alpha_2 = 1$.

case of fractional systems of $n \geq 3$ nonlinear differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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