Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2016, Article ID 4682168, 6 pages
http://dx.doi.org/10.1155/2016/4682168

Research Article

Number of Forts in Iterated Logistic Mapping

Kaixuan Yu[1] and Zhiheng Yu[2]

[1] *Chengdu Radio and TV University, Chengdu, Sichuan 610051, China*
[2] *Chengdu Technological University, Chengdu, Sichuan 611730, China*

Correspondence should be addressed to Zhiheng Yu; yuzhiheng9@163.com

Received 15 May 2016; Accepted 17 August 2016

Academic Editor: Daniele Fournier-Prunaret

Using the theory of complete discrimination system and the computer algebra system *MAPLE V.17*, we compute the number of forts for the logistic mapping $f_\lambda(x) = \lambda x(1-x)$ on $[0,1]$ parameterized by $\lambda \in (0,4]$. We prove that if $0 < \lambda \le 2$ then the number of forts does not increase under iteration and that if $\lambda > 2$ then the number of forts is not bounded under iteration. Furthermore, we focus on the case of $\lambda > 2$ and give for each $k = 1, \ldots, 7$ some critical values of λ for the change of numbers of forts.

1. Introduction

Iteration is the act of repeating a process with the aim of approaching a desired goal, target, or result. In mathematical sense, for a fixed integer $n \ge 1$, the nth iterate f^n of a mapping $f : E \to E$, where E is a nonempty set, is defined recursively by

$$f^k = f \circ f^{k-1}, \quad f^0 = \mathrm{id}, \tag{1}$$

where $\circ$ presents the composition of functions and id denote the identity mapping; that is, $\mathrm{id}(x) \equiv x$ for all $x \in E$. Being indispensable in the computer era, iteration brings many interesting but difficult problems to mathematics. Only from one-dimensional case, one can simply notice that an iterate of a linear function of any order remains linear but the degree of a polynomial may increase drastically, which shows that the nonlinear complexity is amplified by iteration.

Actually, in the one-dimensional case, the complexity of nonlinear functions is related to nonmonotonicity. For a continuous nonmonotonic self-mapping $f : I \to I$, where I is an interval, a point $x_0 \in I$ is called a *monotone point* of f if f is strictly monotone in a neighborhood of x_0; otherwise, x_0 is called a *nonmonotone point* or simply a *fort* of f. Obviously, a linear function does not have a fort generically. In 1980s, Zhang and Yang (see [1]) investigated the number of forts for a class of nonmonotonic functions called *strictly piecewise monotone functions* and simply *PM functions*, which are self-mapping on a compact interval and have at most finitely many forts each. Let $S(f)$ denote the set of all forts of f and let $N(f)$ denote the cardinality of $S(f)$. It is shown in [1, 2] that

$$\begin{aligned} 0 = N(f^0) \le N(f) \le N(f^2) \le \cdots \le N(f^k) \\ \le N(f^{k+1}) \le \cdots ; \end{aligned} \tag{2}$$

that is, the number $N(f^k)$ of forts is nondecreasing as k is increasing. One can similarly prove that (2) also holds for functions defined on the whole $\mathbb{R}$. It is easy to find nonlinear functions whose number of forts, regarded as the damagers of monotonicity, increases rapidly under iteration. Consider the quadratic function

$$f(x) = 4x(1-x), \quad x \in [0,1], \tag{3}$$

for example. Computing derivatives of f^i, $i = 1, \ldots, 5$, and counting the number of real zeros with odd multiplicity for the derivatives $(f^i)'$ (as done in [3]), we get $N(f) = 1$, $N(f^2) = 3$, $N(f^3) = 7$, $N(f^4) = 15$, and $N(f^5) = 31$. From the increasing tendency, without continuing the tedious computation, we have the following question: *Does $N(f^k)$ have a bound or approach infinity as k tends to ∞? How can we compute the number of forts for nonmonotonic functions?*

Polynomials, a special class of nonmonotonic functions, possess the advantage that each fort of a polynomial of degree

≥ 1 is either a peak or a valley although the notion is not true in general. In this paper, we focus on the family of logistic mappings:

$$f_\lambda(x) = \lambda x(1-x), \quad x \in [0,1], \tag{4}$$

where $\lambda \in (0,4]$ is a parameter, which is one of the simplest polynomial mappings, and a typical example used to show chaos and some complicated dynamics, for those problems. First of all, we introduce the theory of *complete discrimination system* (see [4]) and then use it to give a method for the computation of $N(f)$ with f polynomial in Section 2. In Section 3, we employ the method in the computer algebra system *MAPLE V.17* for the family of logistic mappings. We prove in Theorem 4 that $N(f_\lambda^k) = N(f_\lambda) = 1$ for all integer $k \geq 2$ if $0 < \lambda \leq 2$ and that $N(f_\lambda^k)$ approaches ∞ as $k \to \infty$ if $2 < \lambda \leq 4$. Furthermore, for various choices of $\lambda \in (2,4]$, we compute the number $N(f_\lambda^k)$ for each fixed $k = 2, 3, \ldots, 7$ in Theorem 5.

2. Preliminaries

In general, for polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i, \tag{5}$$

where $n \geq 2$ and $a_n \neq 0$. $N(f^k)$, $k = 1, 2, \ldots$, is decided by real zeros of the derivatives $(f^k)'$.

Lemma 1 (see [3, Lemma 2.1]). *x_0 is a fort of a real polynomial f if and only if x_0 is a real zero of the derivative f' of odd multiplicity. Moreover, $N(f)$ is odd (resp., even) if the degree n of f is even (resp., odd).*

Actually, the above lemma shows how a real zero of the derivative f' can be a fort of f. Note that $(f^k)' = (f' \circ f^{k-1})(f' \circ f^{k-2}) \cdots (f' \circ f)f'$. Then the set of zeros of $(f^k)'$ is a union of the set of zeros of $f' \circ f^{k-1}$ and the set of zeros of $(f' \circ f^{k-2}) \cdots (f' \circ f)f'$. Therefore, in order to know if f^k have more forts than f^{k-1}, we need to judge if $f' \circ f^{k-1}$ have real zeros different from $(f' \circ f^{k-2}) \cdots (f' \circ f)f'$'s with odd multiplicities; the following lemma gives the answer.

Lemma 2 (see [3, Lemma 2.5]). *Let G and P be real polynomials and $G(x) \not\equiv 0$. Then the composition $G \circ P$ and the derivative P' do not have a common real zero with odd multiplicity.*

Taking $G = f'$ and $P = f^{k-1}$, by Lemmas 1 and 2, we see that $(f^k)'$ has more real zeros of odd multiplicities than $(f^{k-1})'$ if and only if $f' \circ f^{k-1}$ has real zeros with odd multiplicities. Hence, in the process of computing $N(f^k)$, we only need to find out the number of real zeros for $f' \circ f^{k-1}$ with odd multiplicities. For this reason, we first introduce some notations of the theory of complete discrimination system (see [4, 5]) which will lead us to solve this problem.

Discriminants of polynomials are useful in determining the number of zeros for polynomials. Let $\mathrm{Discr}(f)$ denote the *discriminant matrix* of the polynomial f, which is constructed by the Sylvester matrix of f and f' as seen in [4, Definition 1]. For each $\tau = 1, \ldots, n$, let $D_\tau(f)$ denote the determinant of its submatrix formed by the first $2i$ rows and the first $2i$ columns. The n-tuple $(D_1, D_2, \ldots, D_n)$ is called the *discriminant sequence* of f and the list

$$(\mathrm{sign}(D_1), \mathrm{sign}(D_2), \ldots, \mathrm{sign}(D_n)) \tag{6}$$

is called the *sign list* of f, where $\mathrm{sign}(x)$ is defined to be equal to either 1 if $x > 0$, 0 if $x = 0$, or -1 if $x < 0$. Given a sign list $(s_1, s_2, \ldots, s_n)$ of f, we make a new list $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, called the *revised sign list* of f, in the following regulation:

> If $(s_i, s_{i+1}, \ldots, s_{i+j})$ is a section of the given list such that $s_i \neq 0$, $s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0$, and $s_{i+j} \neq 0$, then replace the subsection $(s_{i+1}, s_{i+2}, \ldots, s_{i+j-1})$ with $(\varepsilon_{i+1}, \varepsilon_{i+2}, \ldots, \varepsilon_{i+j-1})$, where $\varepsilon_{i+r} = (-1)^{[(r+1)/2]} s_i$ for $r = 1, 2, \ldots, j-1$; otherwise, let $\varepsilon_\tau = s_\tau$.

The following lemma tells us how to find the number of distinct zeros by means of the revised sign list.

Lemma 3 (see [4, Theorem 1]). *Let f be a real polynomial and suppose that the number of the sign changes in the revised sign list of f is ν. Then the number of pairs of distinct conjugate complex zeros of f equals ν. Furthermore, if the number of nonvanishing members in the revised sign list is l, then F has $l - 2\nu$ distinct real zeros.*

Now, we are ready to apply the above lemmas to detail the process in computing $N(f^k)$, $k = 1, 2, \ldots$. First of all, we compute f^k and $f' \circ f^{k-1}$ and the discriminant sequence for $f' \circ f^{k-1}$. Secondly, under algebraic relations among coefficients a_i's in the discriminant sequence, discuss the sign of each component of the discriminant sequence and list the sign lists. Then, compute the revised sign lists for $f' \circ f^{k-1}$ through the sign lists. According to the revised sign lists, we find out the number of real zeros of $f' \circ f^{k-1}$ with odd multiplicities and finally obtain $N(f^k)$.

The above idea can be implemented in the computer algebra system *MAPLE V.17*, and we will use this method for the logistic mappings up to iteration index $k = 7$ in next section.

3. Number of Forts

In this section, we first draw a conclusion for the logistic mappings which describe that the numbers of forts can be preserved or approach ∞ as λ varies under iteration and then compute $N(f_\lambda^k)$ for $f_\lambda(x)$ up to iteration index $k = 7$ with different choice of λ.

Theorem 4. *The logistic mappings f_λ defined in (4) have $N(f_\lambda^k) = N(f_\lambda) = 1$ for all integer $k \geq 2$ if and only if $0 < \lambda \leq 2$. Otherwise, $N(f_\lambda^k)$ approaches ∞ as $k \to \infty$.*

Table 1: The number of forts of f^i, $i = 1, 2, \ldots, 7$.

Ranges of λ	$N(f_\lambda)$	$N(f_\lambda^2)$	$N(f_\lambda^3)$	$N(f_\lambda^4)$	$N(f_\lambda^5)$	$N(f_\lambda^6)$	$N(f_\lambda^7)$
$\lambda_{1,0} < \lambda \le \lambda_{2,1}$	1	1	1	1	1	1	1
$\lambda_{2,1} < \lambda \le \lambda_{3,2}$	1	3	5	7	9	11	13
$\lambda_{3,2} < \lambda \le \lambda_{3,4}$	1	3	7	13	21	31	43
$\lambda_{3,4} < \lambda \le \lambda_{3,6}$	1	3	7	13	23	37	57
$\lambda_{3,6} < \lambda \le \lambda_{3,7}$	1	3	7	13	23	37	59
$\lambda_{3,7} < \lambda \le \lambda_{4,3}$	1	3	7	13	23	39	65
$\lambda_{4,3} < \lambda \le \lambda_{4,9}$	1	3	7	15	29	53	93
$\lambda_{4,9} < \lambda \le \lambda_{4,10}$	1	3	7	15	29	53	95
$\lambda_{4,10} < \lambda \le \lambda_{4,11}$	1	3	7	15	29	55	101
$\lambda_{4,11} < \lambda \le \lambda_{5,5}$	1	3	7	15	29	55	103
$\lambda_{5,5} < \lambda \le \lambda_{5,12}$	1	3	7	15	31	61	117
$\lambda_{5,12} < \lambda \le \lambda_{6,8}$	1	3	7	15	31	61	119
$\lambda_{6,8} < \lambda \le \lambda_{7,13}$	1	3	7	15	31	63	125
$\lambda_{7,13} < \lambda \le 4$	1	3	7	15	31	63	127

Proof. In order to obtain the condition for $N(f_\lambda^k) = N(f_\lambda) = 1$, from the method mentioned in the end of Section 2, we need to compute $N(f_\lambda^2)$. Simple computation shows that

$$f_\lambda' \circ f_\lambda = 2\lambda x^2 - 2\lambda x + 1. \tag{7}$$

Then computing the discriminant sequence of (7), we have

$$\left(\lambda^2, \lambda^3 (\lambda - 2)\right). \tag{8}$$

From (8), if $0 < \lambda < 2$, the revised sign list is $(1, -1)$, implying that $f_\lambda' \circ f_\lambda$ has one pair of complex zeros, which leads to $N(f_\lambda^2) = 1$; if $\lambda = 2$, the revised sign list is $(1, 0)$, implying that $f_\lambda' \circ f_\lambda$ has a double real zero, which leads to $N(f_\lambda^2) = 1$; therefore, $N(f_\lambda) = N(f_\lambda^k) = 1$ if and only if $0 < \lambda \le 2$.

Without loss of generality, we can turn the general form $f_\lambda(x) = -\lambda x^2 + \lambda x$ into

$$f_\lambda(x) = -\lambda\left(x - \frac{1}{2}\right)^2 + \frac{\lambda}{4}. \tag{9}$$

So the vertex of the parabola is

$$(x_0, y_0) := \left(\frac{1}{2}, \frac{\lambda}{4}\right). \tag{10}$$

If $2 < \lambda \le 4$, then f_λ and a diagonal line intersect at two points ξ_1 and ξ_2, where $\xi_1 < x_0 < \xi_2$. Obviously, ξ_1 and ξ_2 are fixed points of f_λ; $\xi_1 = 0$ and $\xi_2 = (\lambda - 1)/\lambda$ and f_λ is strictly increasing on the subinterval $[\xi_1, x_0]$. Thus,

$$\begin{aligned} &\xi_1 < f_\lambda^{-k-1}(x_0) < f_\lambda^{-k}(x_0) < x_0, \\ &\qquad \forall k = 1, 2, \ldots, \quad \lim_{k\to+\infty} f_\lambda^{-k}(x_0) = \xi_1. \end{aligned} \tag{11}$$

Since

$$S\left(f_\lambda^k\right) = S\left(f_\lambda^{k-1}\right) \cup \left\{x \in [0,1] : f_\lambda^{k-1} \in S(f_\lambda)\right\}, \tag{12}$$

we get

$$\begin{aligned} \mathcal{S}\left(f_\lambda^k\right) \setminus \mathcal{S}\left(f_\lambda^{k-1}\right) &= \mathcal{S}\left(f_\lambda \circ f_\lambda^{k-1}\right) \setminus \mathcal{S}\left(f_\lambda^{k-1}\right) \\ &= \left(\mathcal{S}\left(f_\lambda^{k-1}\right)\right. \\ &\quad \left.\cup \left\{x \in [0,1] : f_\lambda^{k-1}(x) \in \mathcal{S}(f_\lambda)\right\}\right) \setminus \mathcal{S}\left(f_\lambda^{k-1}\right) \\ &= \left\{x \in [0,1] : f_\lambda^{k-1}(x) \in \mathcal{S}(f_\lambda)\right\} \setminus \mathcal{S}\left(f_\lambda^{k-1}\right) \\ &\ne \emptyset \quad \forall k \in \mathbb{N}^+, \end{aligned} \tag{13}$$

which implies that $N(f_\lambda^k)$ approaches ∞ as $k \to \infty$. This completes the proof. □

Theorem 4 shows that the number $N(f_\lambda^k)$ approaches ∞ as $k \to \infty$ for each fixed $\lambda \in (2, 4]$. It is also interesting to see for each fixed k how the number $N(f_\lambda^k)$ varies as the parameter λ changes in $(2, 4]$. The following theorem shows the change of numbers $N(f_\lambda^k)$ as λ varies for each $k = 2, \ldots, 7$ (but larger k can be considered if the computational capacity of our computer is better). It gives a sequence of parameter values at which new forts arise.

Theorem 5. *$N(f_\lambda^2) = 3$ for all $\lambda \in (2, 4]$ and $N(f_\lambda^3) = 5$ and 7 for $\lambda \in (2, \lambda_{3,2}]$ and $\lambda \in (\lambda_{3,2}, 4]$, respectively, where $\lambda_{3,2} := \sqrt{5} + 1 \approx 3.236067977$. For more details, with the convenient notations $\lambda_{1,0} := 0$ and $\lambda_{2,1} := 2$, numbers $N(f_\lambda^k)$, $k = 1, 2, \ldots, 7$, are given in Table 1, where*

$$\lambda_{4,3} \approx 3.831874056,$$

$$\lambda_{3,4} \approx 3.498561699,$$

$$\lambda_{5,5} \approx 3.960270127,$$

$$\lambda_{3,6} \approx 3.627557530,$$

$$\lambda_{3,7} \approx 3.738914913,$$

$$\lambda_{6,8} \approx 3.990267047,$$

$$\begin{aligned}\lambda_{4,9} &\approx 3.844568792,\\ \lambda_{4,10} &\approx 3.905706470,\\ \lambda_{4,11} &\approx 3.937536445,\\ \lambda_{5,12} &\approx 3.977766422,\\ \lambda_{7,13} &\approx 3.997583118.\end{aligned} \tag{14}$$

Proof. By (8), if $2 < \lambda \leq 4$, the revised sign list of $f'_\lambda \circ f_\lambda$ is $(1, 1)$, implying that it has two distinct real zeros, which shows that $N(f^2) = 3$.

Furthermore, in order to obtain $N(f^3_\lambda)$, we compute

$$f'_\lambda \circ f^2_\lambda = 2\lambda^3 x^4 - 4\lambda^3 x^3 + 2\lambda^3 x^2 + 2\lambda^2 x^2 - 2\lambda^2 x + 1. \tag{15}$$

As shown in Section 2, we give the discriminant sequence for (15):

$$\left(\lambda^6, \lambda^{11}(\lambda-2), \lambda^{14}(\lambda-2)^2, \lambda^{15}\left(\lambda^2-2\lambda-4\right) \cdot (\lambda-2)^3\right). \tag{16}$$

Then, the revised sign list for (16) is

(i) $(1, 1, 1, 1)$, if $\lambda_{3,2} < \lambda \leq 4$, which implies that $f'_\lambda \circ f^2_\lambda$ has 4 distinct simple real zeros;

(ii) $(1, 1, 1, 0)$, if $\lambda = \lambda_{3,2}$, which implies that $f'_\lambda \circ f^2_\lambda$ has 3 distinct real zeros, 2 of which are simple zeros and the remaining one is a double zero;

(iii) $(1, 1, 1, -1)$, if $\lambda_{2,1} < \lambda < \lambda_{3,2}$, which implies that $f'_\lambda \circ f^2_\lambda$ has one pair of complex zeros and 2 distinct simple real zeros.

Here $\lambda_{2,1} = 2$ and $\lambda_{3,2} = \sqrt{5} + 1$, as defined in the theorem. $\lambda_{3,2}$ is the real zero of $g(\lambda) := \lambda^2 - 2\lambda - 4$ in $(2, 4]$. By Lemmas 1 and 2, $N(f^3_\lambda) = 5$ if $\lambda_{2,1} < \lambda \leq \lambda_{3,2}$ and $N(f^3_\lambda) = 7$ if $\lambda_{3,2} < \lambda \leq 4$.

Similarly, compute

$$\begin{aligned}f'_\lambda \circ f^3_\lambda = {} & 2\lambda^7 x^8 - 8\lambda^7 x^7 + \left(12\lambda^7 + 4\lambda^6\right) x^6 \\ & + \left(-8\lambda^7 - 12\lambda^6\right) x^5 \\ & + \left(2\lambda^7 + 12\lambda^6 + 2\lambda^5 + 2\lambda^4\right) x^4 \\ & + \left(-4\lambda^6 - 4\lambda^5 - 4\lambda^4\right) x^3 \\ & + \left(2\lambda^5 + 2\lambda^4 + 2\lambda^3\right) x^2 - 2\lambda^3 x + 1.\end{aligned} \tag{17}$$

Then we obtain the discriminant sequence for $f'_\lambda \circ f^3_\lambda$:

$$\begin{aligned}&\left(\lambda^{14}, \lambda^{27}(\lambda-2), \lambda^{38}(\lambda-2)^2, \lambda^{47}\left(\lambda^2-2\lambda-4\right)\right. \\ &\quad \cdot (\lambda-2)^3, \lambda^{54}\left(\lambda^2-2\lambda-4\right) \\ &\quad \cdot (\lambda-2)^4, \lambda^{59}\left(\lambda(\lambda-2)\left(\lambda^2-2\lambda-4\right)-16\right) \\ &\quad \cdot (\lambda-2)^5, \lambda^{62}\left(\lambda^2-2\lambda-4\right) \\ &\quad \cdot \left(\lambda(\lambda-2)\left(\lambda^2-2\lambda-4\right)-16\right) \\ &\quad \cdot (\lambda-2)^6, \lambda^{63}\left(\lambda(\lambda-2)\left(\lambda^2-2\lambda-4\right)^2-64\right) \\ &\quad \left.\cdot \left(\lambda^2-2\lambda-4\right)^2(\lambda-2)^7\right).\end{aligned} \tag{18}$$

Hence,

(i) for $\lambda_{2,1} < \lambda < \lambda_{3,2}$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, -1, -1, -1, 1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has 3 pairs of complex zeros and 2 distinct simple real zeros;

(ii) for $\lambda = \lambda_{3,2}$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, -1, -1, -1, 0, 0)$ and $f'_\lambda \circ f^3_\lambda$ can be factorized as

$$\begin{aligned}f'_\lambda \circ f^3_\lambda = {} & \frac{1}{64}\left(13\sqrt{5}+29\right) \\ & \cdot \left(32x^2 - 32x + 4\sqrt{5} + 12\sqrt{2} - 4\sqrt{10} - 4\right) \\ & \cdot \left(x - \frac{1}{2} - \frac{1}{4}\sqrt{6 - 2\sqrt{5} + 6\sqrt{2} - 2\sqrt{10}}\right) \\ & \cdot \left(x - \frac{1}{2} + \frac{1}{4}\sqrt{6 - 2\sqrt{5} + 6\sqrt{2} - 2\sqrt{10}}\right) \\ & \cdot \left(4x - 3 + \sqrt{5}\right)^2 \left(-4x + 1 + \sqrt{5}\right)^2,\end{aligned} \tag{19}$$

implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 4 distinct real zeros and two of the 4 distinct real zeros are simple and the rest are both double zeros;

(iii) for $\lambda_{3,2} < \lambda < 1 + (3 + 2\sqrt{5})^{1/2}$, where $1 + (3 + 2\sqrt{5})^{1/2}$ is the real zero of $h(\lambda) := \lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4) - 16$ in $(\lambda_{3,2}, 4]$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, 1, 1, -1, -1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 6 distinct simple real zeros;

(iv) for $\lambda = 1 + (3 + 2\sqrt{5})^{1/2}$, the revised sign list for $f' \circ f^3$ is $(1, 1, 1, 1, 1, -1, -1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 6 distinct simple real zeros;

(v) for $1 + (3 + 2\sqrt{5})^{1/2} < \lambda < \lambda_{4,3}$, where

$$\lambda_{4,3} := \frac{\sqrt{3}}{3}\left(\frac{2\left(100 + 12\sqrt{69}\right)^{2/3} + 11\left(100 + 12\sqrt{69}\right)^{1/3} + 8}{\left(100 + 12\sqrt{69}\right)^{1/3}}\right)^{1/2} + 1 \approx 3.831874056 \tag{20}$$

is the real zero of $p(\lambda) := \lambda(\lambda-2)(\lambda^2-2\lambda-4)^2-64$ in $(1+(3+2\sqrt{5})^{1/2}, 4]$, the revised sign list for $f'\circ f^3$ is $(1,1,1,1,1,1,1,-1)$, implying that $f_\lambda'\circ f_\lambda^3$ has a pair of complex zeros and 6 distinct simple real zeros;

(vi) for $\lambda = \lambda_{4,3}$, the revised sign list for $f_\lambda'\circ f_\lambda^3$ is $(1,1,1,1,1,1,1,0)$, implying that $f_\lambda'\circ f_\lambda^3$ has 7 distinct real zeros, one of which is a double zero but the rest are all simple;

(vii) for $\lambda_{4,3} < \lambda \le 4$, the revised sign list for $f_\lambda'\circ f_\lambda^3$ is $(1,1,1,1,1,1,1,1)$, implying that $f_\lambda'\circ f_\lambda^3$ has 8 distinct simple real zeros.

It follows that $N(f_\lambda^4) = 7$ if $\lambda_{2,1} < \lambda \le \lambda_{3,2}$, $N(f_\lambda^4) = 13$ if $\lambda_{3,2} < \lambda \le \lambda_{4,3}$, and $N(f_\lambda^4) = 15$ if $\lambda_{4,3} < \lambda \le 4$.

We similarly compute $f_\lambda'\circ f_\lambda^4$ and obtain the discriminant sequence

$$\begin{aligned}
&\Big(\lambda^{30}, \lambda^{59}(\lambda-2), \lambda^{86}(\lambda-2)^2, \lambda^{111}\left(\lambda^2-2\lambda-4\right)(\lambda \\
&\quad -2)^3, \lambda^{134}\left(\lambda^2-2\lambda-4\right)(\lambda-2)^4, \lambda^{155}\left(\lambda^4-4\lambda^3\right. \\
&\quad \left.+8\lambda-16\right)(\lambda-2)^5, \lambda^{174}\left(\lambda^2-2\lambda-4\right)\left(\lambda^4-4\lambda^3\right. \\
&\quad \left.+8\lambda-16\right)(\lambda-2)^6, \lambda^{191}\left(\lambda^6-6\lambda^5+4\lambda^4+24\lambda^3\right. \\
&\quad \left.-16\lambda^2-32\lambda-64\right)\left(\lambda^2-2\lambda-4\right)^2(\lambda-2)^7, \\
&\quad \lambda^{206}\left(\lambda^6-6\lambda^5+4\lambda^4+24\lambda^3-16\lambda^2-32\lambda-64\right) \\
&\quad \cdot\left(\lambda^2-2\lambda-4\right)^2(\lambda-2)^8, \lambda^{219}\left(\lambda^2-2\lambda-4\right)\left(\lambda^{10}\right. \\
&\quad -10\lambda^9+28\lambda^8+16\lambda^7-160\lambda^6+64\lambda^5+192\lambda^4 \\
&\quad \left.+384\lambda^3-256\lambda^2-1024\lambda+1024\right)(\lambda-2)^9, \\
&\quad \lambda^{230}\left(\lambda^4-4\lambda^3+8\lambda-16\right)\left(\lambda^{10}-10\lambda^9+28\lambda^8\right. \\
&\quad +16\lambda^7-160\lambda^6+64\lambda^5+192\lambda^4+384\lambda^3-256\lambda^2 \\
&\quad \left.-1024\lambda+1024\right)(\lambda-2)^{10}, \lambda^{239}\left(\lambda^2-2\lambda-4\right)\left(\lambda^8\right. \\
&\quad \left.-8\lambda^7+16\lambda^6+16\lambda^5-64\lambda^4+128\lambda-256\right)\left(\lambda^4\right. \\
&\quad \left.-4\lambda^3+8\lambda-16\right)^2(\lambda-2)^{11}, \lambda^{246}\left(\lambda^8-8\lambda^7+16\lambda^6\right. \\
&\quad \left.+16\lambda^5-64\lambda^4+128\lambda-256\right)\left(\lambda^2-2\lambda-4\right)^2\left(\lambda^4\right. \\
&\quad \left.-4\lambda^3+8\lambda-16\right)^2(\lambda-2)^{12}, \lambda^{251}\left(\lambda^4-4\lambda^3+8\lambda\right. \\
&\quad \left.-16\right)\left(\lambda^{14}-14\lambda^{13}+72\lambda^{12}-136\lambda^{11}-144\lambda^{10}\right. \\
&\quad +1088\lambda^9-1600\lambda^8-640\lambda^7+4096\lambda^6-512\lambda^5 \\
&\quad \left.-4096\lambda^4-12288\lambda^3+16384\lambda^2+8192\lambda-16384\right) \\
&\quad \cdot\left(\lambda^2-2\lambda-4\right)^3(\lambda-2)^{13}, \lambda^{254}\left(\lambda^6-6\lambda^5+4\lambda^4\right. \\
&\quad \left.+24\lambda^3-16\lambda^2-32\lambda-64\right)\left(\lambda^{14}-14\lambda^{13}+72\lambda^{12}\right. \\
&\quad -136\lambda^{11}-144\lambda^{10}+1088\lambda^9-1600\lambda^8-640\lambda^7 \\
&\quad +4096\lambda^6-512\lambda^5-4096\lambda^4-12288\lambda^3+16384\lambda^2 \\
&\quad \left.+8192\lambda-16384\right)\left(\lambda^2-2\lambda-4\right)^4(\lambda-2)^{14}, \\
&\quad \lambda^{255}\left(\lambda^{12}-12\lambda^{11}+48\lambda^{10}-40\lambda^9-192\lambda^8+384\lambda^7\right. \\
&\quad \left.+64\lambda^6-1024\lambda^4-512\lambda^3+2048\lambda^2+4096\right)\left(\lambda^6\right. \\
&\quad \left.-6\lambda^5+4\lambda^4+24\lambda^3-16\lambda^2-32\lambda-64\right)^2\left(\lambda^2-2\lambda\right. \\
&\quad \left.-4\right)^5(\lambda-2)^{15}\Big).
\end{aligned} \tag{21}$$

Similar discussion gives $N(f_\lambda^5)$ for various λ as shown in Table 1.

It is more complicated to compute discriminant sequences of $f_\lambda'\circ f_\lambda^5$ and $f_\lambda'\circ f_\lambda^6$ because the two discriminant sequences contain 32 and 64 components, respectively, and the biggest component in the discriminant sequence of $f_\lambda'\circ f_\lambda^5$ is a polynomial in the single variable λ of degree 1152 with 130 terms. Using a similar discussion as for $f_\lambda'\circ f_\lambda^k$, $k=1,2,3,4$, we obtain parameter values $\lambda_{3,6}$, $\lambda_{3,7}$, $\lambda_{6,8}$, $\lambda_{4,9}$, $\lambda_{4,10}$, $\lambda_{4,11}$, $\lambda_{5,12}$, and $\lambda_{7,13}$ as well as the numbers $N(f_\lambda^6)$ and $N(f_\lambda^7)$ on intervals between them as shown in Table 1. This completes the proof. □

Although we are not able to compute for all k those parameter values for changes of $N(f_\lambda^k)$ in Theorem 5, those data of Table 1 for $k=1,2,\ldots,7$ show that the number $N(f_\lambda^k)$ can reach its maximum 2^k-1 if $\lambda > \lambda_{k,l}$, $l = 0,1,2,3,5,8,13$. We naturally have the following.

Question 1. Does $N(f_\lambda^k)$ reach the maximum 2^k-1 for any integer $k \ge 1$?

Additionally, the well-known Feigenbaum sequence (see [6, 7]) is $\lambda_1 = 3$, $\lambda_2 \approx 3.449490$, $\lambda_3 \approx 3.544090$, $\lambda_4 \approx 3.564407$, $\lambda_5 \approx 3.568750$, $\lambda_6 = 3.569690,\ldots$, at each of which a period-doubling bifurcation happens in the logistic mapping. This suggests the following.

Question 2. Is there any relation between our sequence $\{\lambda_{k,l}\}$ and the Feigenbaum sequence $\{\lambda_k\}$?

A related work can be found from [8], but the question is not answered yet.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

This work is supported by FP7-PEOPLE-2012-IRSES-316338.

References

[1] J. Zhang and L. Yang, "Discussion on iterative roots of piecewise monotone functions, in Chinese," *Acta Mathematica Sinica*, vol. 26, pp. 398–412, 1983.

[2] W. Zhang, "PM functions, their characteristic intervals and iterative roots," *Annales Polonici Mathematici*, vol. 65, no. 2, pp. 119–128, 1997.

[3] L. L. Yang, L. Yang, Z. Yu, and W. Zhang, "Real polynomial iterative roots in the case of nonmonotonicity height ≥2," *Science China Mathematics*, vol. 55, no. 12, pp. 2433–2446, 2012.

[4] L. Yang, X. Hou, and Z. Zeng, "A complete discrimination system for polynomials," *Science China E*, vol. 39, no. 6, pp. 628–646, 1996.

[5] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser, Boston, Mass, USA, 1994.

[6] K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, New York, NY, USA, 1996.

[7] R. D. Smith, "Period doubling, information entropy, and estimates for Feigenbaum's constants," *International Journal of Bifurcation and Chaos*, vol. 23, no. 11, Article ID 1350190, 2013.

[8] G. Linage, F. Montoya, A. Sarmiento, K. Showalter, and P. Parmananda, "Fibonacci order in the period-doubling cascade to chaos," *Physics Letters A*, vol. 359, no. 6, pp. 638–639, 2006.