Research Article

A 3D Fractional-Order Chaotic System with Only One Stable Equilibrium and Controlling Chaos

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One 3D fractional-order chaotic system with only one locally asymptotically stable equilibrium is reported. To verify the chaoticity, the maximum Lyapunov exponent (MAXLE) with respect to the fractional-order and chaotic attractors are obtained by numerical calculation for this system. Furthermore, by linear scalar controller consisting of a single state variable, one control scheme for stabilization of the 3D fractional-order chaotic system is suggested. The numerical simulations show the feasibility of the control scheme.

1. Introduction

Fractional-order calculus is an old branch of mathematics, which can be dated back to the 17th century [1, 2]. Now, it is well-known that many real-world physical systems [1–4] can be more accurately described by fractional-order differential equations, for example, dielectric polarization, viscoelasticity, electrode-electrolyte polarization, electromagnetic waves, diffusion-wave, superdiffusion, heat conduction. Meanwhile, chaotic behavior has been found in many fractional-order systems like the fractional-order brushless DC motor chaotic system [5, 6], the fractional-order gyroscopes chaotic system [7], the fractional-order microelectromechanical chaotic system [8], the fractional-order electronic circuits [9, 10], and so forth [11–16].

Recently, a simple three-dimensional autonomous chaotic system [17] with only one stable node-focus equilibrium has been reported by Wang and Chen. Due to the impossibility of existence of homoclinic orbit and the unique stable node-focus equilibrium in this striking chaotic system, the well-known Si'lnikov criterions are not applicable. To verify the chaoticity in this system, Wang and Chen [17] calculated the largest Lyapunov exponent, fractional dimension, and continuous broad frequency spectrum by

numerical calculation. Huan et al. presented a rigorous computer-assisted verification of horseshoe chaos by virtue of topological horseshoe theory [18]. Up to now, some integer order chaotic systems with stable node-focus equilibrium have been presented. To the best of our knowledge, many previous fractional-order chaotic systems like the fractional-order Lorenz chaotic system [19], the fractional-order Chen chaotic system [20], the fractional-order Lu chaotic system [21], the fractional-order brushless DC motor chaotic system [5, 6], the fractional-order gyroscopes chaotic system [7], the fractional-order microelectromechanical chaotic system [8], and so forth [9–14, 22, 23] have unstable equilibrium. There are seldom results on fractional-order chaotic systems with stable equilibrium. Hence, the finding of fractional-order chaotic systems with stable equilibrium is still an open problem.

Motivated by the above discussions, a three-dimensional autonomous fractional-order chaotic system with only one locally asymptotically stable equilibrium is proposed in this paper. The argument of all eigenvalues at equilibrium point satisfies $|\arg(\lambda_i)| \geq 0.5\pi$ ($i = 1, 2, 3$). Up to now, to the best of our knowledge, there are few results about the fractional-order chaotic systems with stable equilibrium. To verify the chaoticity in this fractional-order system, the maximum

Lyapunov exponent and chaotic attractors are yielded by numerical calculation. Furthermore, one control scheme for stabilization of this fractional-order chaotic system is suggested via linear scalar controller consisting of a single state variable. The numerical simulations show the feasibility of the control scheme. The organization of this paper is as follows: in Section 2, a new fractional-order chaotic system with only one stable equilibrium is presented, and the maximum Lyapunov exponent and chaotic attractors are obtained. In Section 3, in order to stabilize this fractional-order chaotic system, one control scheme is suggested via linear scalar controller consisting of a single state variable. The conclusion is finally drawn in Section 4.

2. A New Fractional-Order Chaotic Systems with Only One Stable Equilibrium

In this paper, the q-order Caputo derivative for function $y(t)$ is defined as follows:

$$
{}_{0}^{C}D_{t}^{q}y(t) = \frac{1}{\Gamma(n-q)} \int_{0}^{t} \left[\frac{d^{n}y(\tau)}{d\tau^{n}} \right] (t-\tau)^{n-q-1} d\tau, \quad (1)
$$

where $n - 1 \le q < n \in Z^{+}$ and $\Gamma(n-q) = \int_{0}^{+\infty} t^{n-q-1}e^{-t}dt$ is the Gamma function.

Now, we consider the following 3D nonlinear fractional-order system:

$$
\begin{aligned}
{}_{0}^{C}D_{t}^{q}x_{1}(t) &= \left(x_{2}(t) + \frac{1}{16} \right) x_{3}(t), \\
{}_{0}^{C}D_{t}^{q}x_{2}(t) &= x_{1}^{2}(t) + 0.5x_{1}(t) - x_{2}(t), \\
{}_{0}^{C}D_{t}^{q}x_{3}(t) &= -2x_{1}(t)
\end{aligned} \quad (2)
$$

and, here, the fractional-order is $0 < q < 1$.

First, let us recall the stability theorem for nonlinear commensurate fractional-order systems.

Lemma 1 (see [24, 25]). *Consider the following fractional-order nonlinear system:*

$$
{}_{0}^{C}D_{t}^{q}x(t) = g(x(t)), \quad (3)
$$

where $0 < q < 1$ is fractional-order, $x(t) \in R^{n}$ are state variables, and $g : R^{n} \rightarrow R^{n}$ is a nonlinear continuous vector function. The equilibrium point $\overline{x}$ (i.e., $g(\overline{x}) = 0$) in nonlinear fractional-order system (3) is locally asymptotically stable if $|\arg(\lambda_{i}(J))| > 0.5\pi q$ $(i = 1, 2, \ldots, n)$. Here, J is the Jacobian matrix of $g(x(t))$ at equilibrium point $\overline{x}$ and λ_{i} $(i = 1, 2, \ldots, n)$ are the eigenvalues of matrix J.

Now, we can obtain that system (2) has only one equilibrium point; that is, $(\overline{x}_{1}, \overline{x}_{2}, \overline{x}_{3}) = (0, 0, 0)$. Meanwhile, we can yield the eigenvalues of Jacobian matrix at equilibrium point as follows: $\lambda_{1} = -1$, and $\lambda_{\pm} = \pm 0.25\sqrt{2}i$. So, we can obtain $|\arg(\lambda_{i})| \ge 0.5\pi > 0.5\pi q$ $(i = 1, 2, 3)$. According to the lemma, the equilibrium point $(\overline{x}_{1}, \overline{x}_{2}, \overline{x}_{3}) = (0, 0, 0)$ is locally asymptotically stable. Therefore, the equilibrium in system (2) is asymptotically stable.

Next, we discuss the numerical solution for system (2). Based on [20], we set $h = T/N$ and $t_{n} = nh$ $(n = 0, 1, 2, \ldots, N)$ and let initial condition be $(x_{1}(0), x_{2}(0), x_{3}(0))$. So, the fractional-order system (2) can be discretized as follows:

$$
\begin{aligned}
x_{1}(n+1) &= x_{1}(0) + \frac{h^{q}}{\Gamma(q+2)} \left\{ \left[x_{2}^{p}(n+1) + \frac{1}{16} \right] \right. \\
&\quad \left. \cdot x_{3}^{p}(n+1) + \sum_{j=0}^{n} \alpha_{1,j,n+1} \left[x_{2}(j) + \frac{1}{16} \right] x_{3}(j) \right\}, \\
x_{2}(n+1) &= x_{2}(0) + \frac{h^{q}}{\Gamma(q+2)} \left\{ \left[\left(x_{1}^{p}(n+1) \right)^{2} \right. \right. \\
&\quad \left. + 0.5x_{1}^{p}(n+1) - x_{2}^{p}(n+1) \right] \\
&\quad \left. + \sum_{j=0}^{n} \alpha_{2,j,n+1} \left[(x_{1}(j))^{2} + 0.5x_{1}(j) - x_{2}(j) \right] \right\}, \\
x_{3}(n+1) &= x_{3}(0) + \frac{h^{q}}{\Gamma(q+2)} \left\{ \left[-2x_{1}^{p}(n+1) \right] \right. \\
&\quad \left. + \sum_{j=0}^{n} \alpha_{3,j,n+1} \left[-2x_{1}(j) \right] \right\},
\end{aligned} \quad (4)
$$

Where

$$
\begin{aligned}
x_{1}^{p}(n+1) &= x_{1}(0) + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{1,j,n+1} \left[x_{2}(j) + \frac{1}{16} \right] x_{3}(j), \\
x_{2}^{p}&(n+1) \\
&= x_{2}(0) + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{2,j,n+1} \left[(x_{1}(j))^{2} + 0.5x_{1}(j) - x_{2}(j) \right], \\
x_{3}^{p}(n+1) &= x_{3}(0) + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{3,j,n+1} \left[-2x_{1}(j) \right],
\end{aligned} \quad (5)
$$

$$
\begin{aligned}
&\alpha_{i,j,n+1} \\
&= \begin{cases} n^{q+1} - (n-q)(n+1)^{q}, & j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} \\
&\hspace{6cm} (i = 1, 2, 3)
\end{aligned}
$$

$$
b_{i,j,n+1} = \frac{h^{q}}{q} \left[(n-j+1)^{q} - (n-j)^{q} \right], \quad 0 \le j \le n.
$$

The error of this approximation is described as

$$
\begin{aligned}
\left| x_{i}(t_{n}) - x_{i}(n) \right| &= o(h^{p}) \\
(i = 1, 2, 3), \quad p &= \min(2, 1+q).
\end{aligned} \quad (6)
$$

Now, some results are obtained by numerical calculation. Let $q = 0.96$; Figures 1(a) and 1(b) show the results with initial condition $(x_{1}(0), x_{2}(0), x_{3}(0)) = (-1.2, 1, 1)$.

(a)

(b)

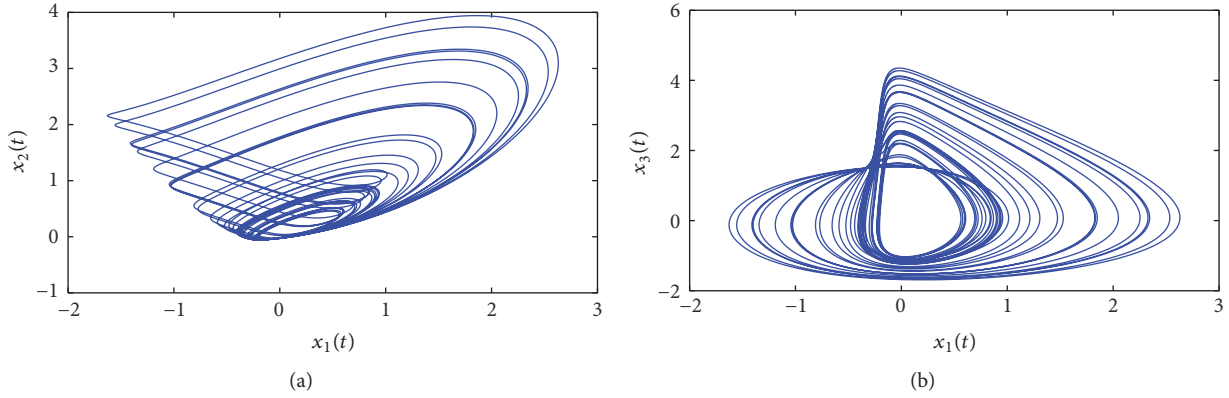FIGURE 1: A chaotic attractor in system (2) for $q = 0.96$.
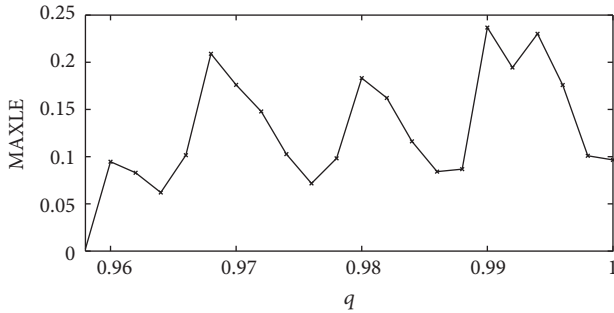

FIGURE 2: The maximum Lyapunov exponent (MAXLE) varies as fractional-order q.

The results in Figure 1 indicate that system (2) has a chaotic attractor if initial conditions are chosen as $(x_1(0), x_2(0), x_3(0)) = (-1.2, 1, 1)$.

To verify the chaoticity in system (2), we choose the initial conditions as $(x_1(0), x_2(0), x_3(0)) = (-1.2, 1, 1)$ and calculate the maximum Lyapunov exponent (MAXLE) of system (2) with respect to the fractional-order q by numerical calculation. We obtain that the maximum Lyapunov exponent (MAXLE) is larger than zero for $0.958 \leq q \leq 1$. Figure 2 shows the maximum Lyapunov exponent (MAXLE) varies as fractional-order q. So, the chaotic attractor is emerged in system (2) for $0.958 \leq q \leq 1$.

For example, the MAXLE is 0.0022 when $q = 0.958$, and its chaotic attractor is shown as Figures 3(a) and 3(b), while the MAXLE is 0.0946 when $q = 0.96$, and its chaotic attractor is shown as Figure 1.

3. Controlling Chaos via Linear Scalar Controller Consisting of a Single State Variable

Now, we discuss how to stabilize the fractional-order chaotic system (2) via linear scalar controller consisting of a single state variable. We suggest the following controlled fractional-order system:

$$
{}_0^C D_t^q x_1(t) = \left(x_2(t) + \frac{1}{16} \right) x_3(t) + K x_2(t),
$$

$$
{}_0^C D_t^q x_2(t) = x_1^2(t) + 0.5 x_1(t) - x_2(t),
$$

$$
{}_0^C D_t^q x_3(t) = -2 x_1(t).
$$

(7)

Here $K x_2(t)$ is the linear scalar controller consisting of a single state variable and $K < 0$ is control parameter.

The controlled fractional-order system (7) is stable if the maximum Lyapunov exponent (MAXLE) of system (7) is negative. So, we can choose suitable control parameter K such that the MAXLE of system (7) is negative. Now, we can calculate the MAXLE of system (7) by numerical calculation. The MAXLE of system (7) varying with respect to fractional-order q for control parameter K is shown in Figure 4, and here the initial condition is $(x_1(0), x_2(0), x_3(0)) = (-1.2, 1, 1)$.

The results in Figure 4 indicate that the controlled fractional-order system (7) is stable if $K = -1, -2, -3$, and -4. It means that the fractional-order chaotic system (2) can achieve stability via scalar controller consisting of a single state variable $x_2(t)$.

For example, let $q = 0.98$ and $K = -1$, and we can obtain MAXLE $= -0.2125$. So, the controlled fractional-order system (7) is stable. Figure 5 shows the simulative results. Let $q = 0.96$ and $K = -2$, and we have MAXLE $= -0.1236$. So, the controlled fractional-order system (7) is stable. Figure 6 shows the simulative results. Let $q = 0.97$ and $K = -3$, and we can obtain MAXLE $= -0.0774$. So, the controlled fractional-order system (7) is stable. Figure 7 shows the simulative results. Let $q = 0.99$ and $K = -4$, and we have obtained MAXLE $= -0.0805$. So, the controlled fractional-order system (7) is stable. Figure 8 shows the simulative results. Here, the initial condition is $(x_1(0), x_2(0), x_3(0)) = (-1.2, 1, 1)$.
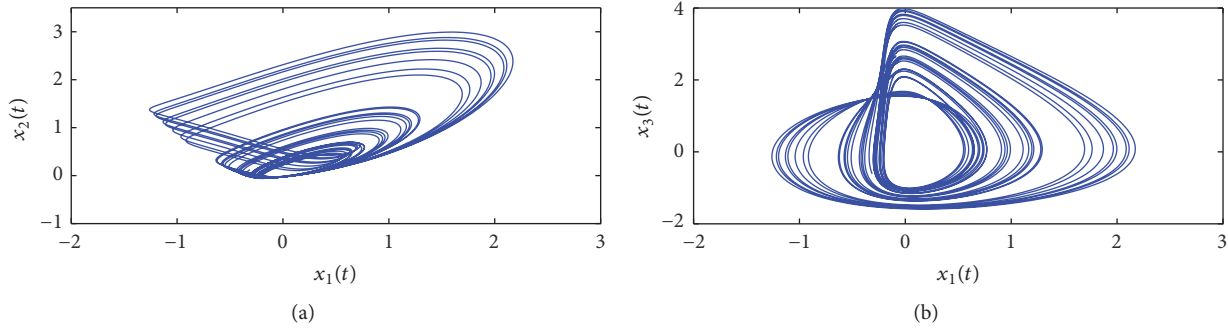
(a)

(b)

FIGURE 3: A chaotic attractor in system (2) for $q = 0.958$.

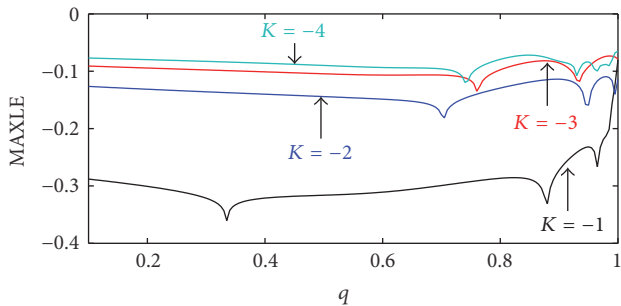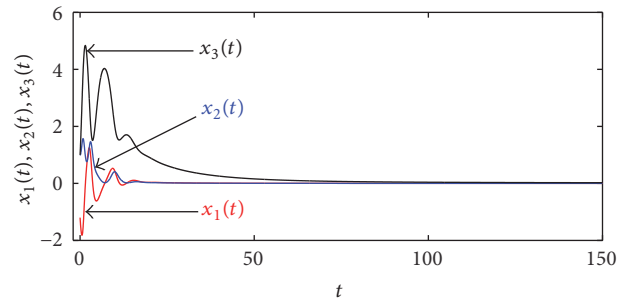FIGURE 4: The MAXLE of system (7) varies as fractional-order q for some control parameters K.

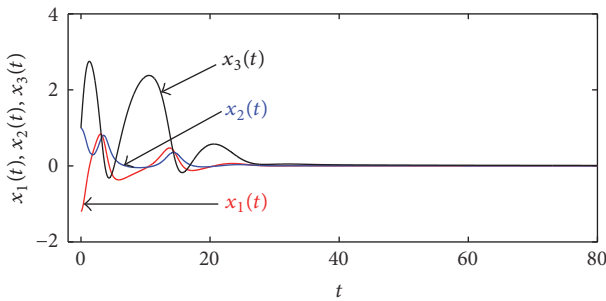FIGURE 7: Stabilization of system (7) for $q = 0.97$ and $K = -3$.

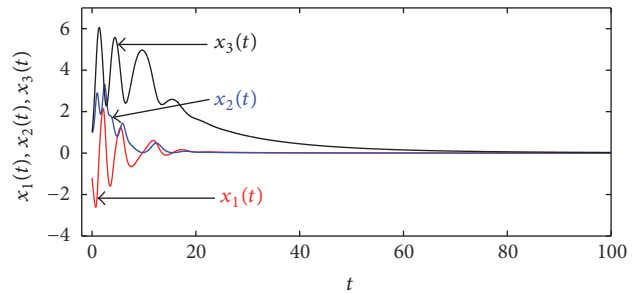FIGURE 5: Stabilization of system (7) for $q = 0.98$ and $K = -1$.

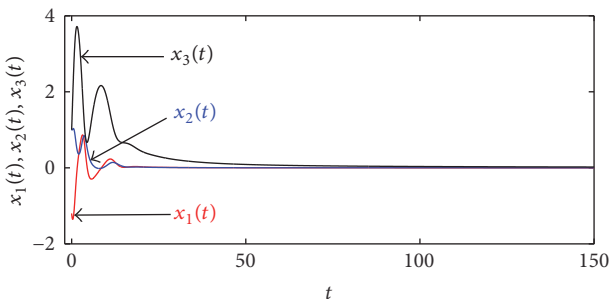FIGURE 8: Stabilization of system (7) for $q = 0.99$ and $K = -4$.

FIGURE 6: Stabilization of system (7) for $q = 0.96$ and $K = -2$.

4. Conclusions

One new fractional-order chaotic system with only one stable equilibria point is reported in this paper. By numerical calculation, we yield the maximum Lyapunov exponent spectrum for this new fractional-order chaotic system, and the chaotic attractor can been found when $0.958 \leq q \leq 1$. The chaotic attractors for $q = 0.958$ and $q = 0.96$ are given. Meanwhile, in order to stabilize this fractional-order chaotic system, one control scheme is proposed via linear scalar controller consisting of a single state variable. By calculating the maximum Lyapunov exponent spectrum of the controlled system, we find that the maximum Lyapunov exponent (MAXLE) of system (7) is negative for suitable control parameter K. This result means that the new fractional-order chaotic system can achieve stability by the proposed control scheme. Some simulation results are given to show the validity and feasibility of our control scheme.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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