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## NONWANDERING SETS OF MAPS ON THE CIRCLE

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**ABSTRACT.** Let  $f$  be a continuous map of the circle  $S^1$  into itself. And let  $R(f), \Lambda(f), \Gamma(f)$ , and  $\Omega(f)$  denote the set of recurrent points,  $\omega$ -limit points,  $\gamma$ -limit points, and nonwandering points of  $f$ , respectively. In this paper, we show that each point of  $\Omega(f) \setminus \overline{R(f)}$  is one-side isolated, and prove that

- (1)  $\Omega(f) \setminus \Gamma(f)$  is countable and
- (2)  $\Lambda(f) \setminus \Gamma(f)$  and  $\overline{R(f)} \setminus \Gamma(f)$  are either empty or countably infinite.

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**1. Introduction.** Let  $I$  be the unit interval,  $S^1$  the circle, and  $X$  a topological space. And let  $C^0(X, X)$  denote the set of continuous maps from  $X$  into itself. For any  $f \in C^0(X, X)$ , let  $P(f), R(f), \Lambda(f), \Gamma(f)$ , and  $\Omega(f)$  denote the set of periodic points, recurrent points,  $\omega$ -limit points,  $\gamma$ -limit points and nonwandering points of  $f$ , respectively.

For any  $f \in C^0(I, I)$ , in 1980, Z. Nitecki [6] has proved that each point of  $\Omega(f) \setminus \overline{P(f)}$  is isolated in  $\Omega(f)$  if  $f$  is piecewise monotone and is not flat on any subinterval of  $I$ . In 1984, J. C. Xiong [7] has proved that each point of  $\Omega(f) \setminus \overline{P(f)}$  is one-side isolated in  $\Omega(f)$ , for a continuous self map of interval  $I$ . And, in 1988, J. C. Xiong [9] also showed that  $\Omega(f) \setminus \Gamma(f)$  is countable and that  $\Lambda(f) \setminus \Gamma(f)$  and  $\overline{P(f)} \setminus \Gamma(f)$  are either empty or countably infinite.

In this paper, we obtain the following similar results for maps of the circle:

**THEOREM 1.1.** *Let  $f \in C^0(S^1, S^1)$ . Then each point of  $\Omega(f) \setminus \overline{R(f)}$  is one-side isolated in  $\Omega(f)$ .*

**THEOREM 1.2.** *Let  $f \in C^0(S^1, S^1)$ . Then*

- (1)  $\Omega(f) \setminus \Gamma(f)$  is countable.
- (2)  $\Lambda(f) \setminus \Gamma(f)$  and  $\overline{R(f)} \setminus \Gamma(f)$  are either empty or countably infinite.

**2. Preliminaries and definitions.** Let  $X$  be a compact metric space and  $f \in C^0(X, X)$ . For any positive integer  $n$ , we define  $f^n$  inductively by  $f^1 = f$  and  $f^{n+1} = f \circ f^n$ . Let  $f^0$  denote the identity map of  $X$ . The *forward orbit*  $\text{Orb}(x)$  of  $x \in X$  is the set  $\{f^k(x) \mid k = 0, 1, 2, \dots\}$ . Usually, the forward orbit of  $x$  is simply called the *orbit* of  $x$ .

A point  $x \in X$  is called a *periodic point* of  $f$  if, for some positive integer  $n$ ,  $f^n(x) = x$ . The period of  $x$  is the least such integer  $n$ . We denote the set of periodic points of  $f$  by  $P(f)$ . A point  $x \in X$  is called a *recurrent point* of  $f$  if there exists a sequence  $\{n_i\}$  of positive integers with  $n_i \rightarrow \infty$  such that  $f^{n_i}(x) \rightarrow x$ . We denote the set of recurrent

points of  $f$  by  $R(f)$ . A point  $x \in X$  is called a *nonwandering point* of  $f$  if, for every neighborhood  $U$  of  $x$ , there exists a positive integer  $m$  such that  $f^m(U) \cap U \neq \emptyset$ . We denote the set of nonwandering points of  $f$  by  $\Omega(f)$ .

A point  $y \in X$  is called an  $\omega$ -*limit point* of  $x$  if there exists a sequence  $\{n_i\}$  of positive integers with  $n_i \rightarrow \infty$  such that  $f^{n_i}(x) \rightarrow y$ . We denote the set of  $\omega$ -limit points of  $x$  by  $\omega(x)$ . Define  $\Lambda(f) = \bigcup_{x \in X} \omega(x)$ . A point  $y \in X$  is called an  $\alpha$ -*limit point* of  $x$  if there exists a sequence  $\{n_i\}$  of positive integers with  $n_i \rightarrow \infty$  and a sequence  $\{y_i\}$  of points such that  $f^{n_i}(y_i) = x$  and  $y_i \rightarrow y$ . The symbol  $\alpha(x)$  denotes the set of  $\alpha$ -limit points of  $x$ . A point  $y \in X$  is called a  $\gamma$ -*limit point* of  $x$  if  $y \in \omega(x) \cap \alpha(x)$ . The symbol  $\gamma(x)$  denotes the set of  $\gamma$ -limit points of  $x$  and  $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$ .

Let  $R$  be the set of reals and  $Z$  be the set of integers. Formally, we think of the circle  $S^1$  as  $R/Z$  and use  $\pi: R \rightarrow R/Z$  to denote the canonical projection. In fact, the map  $\pi: R \rightarrow S^1$  is an example of a covering map since it wraps  $R$  around  $S^1$  without doubling back (i.e., without critical points). To study the dynamics of the circle map, it is helpful to use a *lifting*. Let  $f$  be a continuous map on the circle. We say that a continuous map  $F$  from  $R$  into itself is a lifting of  $f$  if  $f \circ \pi = \pi \circ F$ . We use the following notations throughout this paper.

Let  $a, b \in S^1$  with  $a \neq b$ , and let  $A \in \pi^{-1}(a), B \in \pi^{-1}(b)$  with  $|A - B| < 1$  and  $A < B$ . Then we write  $\pi((A, B)), \pi([A, B]), \pi([A, B]),$  and  $\pi((A, B])$  to denote the open, closed, and half-open arcs from  $a$  counterclockwise to  $b$ , respectively, and we denote it by  $(a, b), [a, b], [a, b),$  and  $(a, b]$ . For  $x, y \in [a, b]$  with  $a \neq b$ , let  $X \in \pi^{-1}(x), Y \in \pi^{-1}(y)$  with  $X, Y \in [A, B]$ , then we define for  $x, y \in [a, b], x > y$  if and only if  $X > Y$ . Let  $C$  be a subset of a closed arc  $[a, b]$ , then we define  $\sup C = \pi(\sup(\pi^{-1}(C) \cap [A, B]))$  and  $\inf C = \pi(\inf(\pi^{-1}(C) \cap [A, B]))$ .

In particular, for  $a, b, c \in S^1, a < b < c$  means that  $b$  lies in the open arc  $(a, c)$ , that is,  $b \in (a, c)$ .

Let  $X$  be  $I$  or  $S^1$  and  $Y \subset X$ . Let  $x \in Y$ . A point  $x \in X$  is said to be *left-sided isolated* (resp., *right-sided isolated*) in  $Y$  if, for some  $\epsilon > 0$ ,  $(x - \epsilon, x) \cap Y = \emptyset$  (resp.,  $(x, x + \epsilon) \cap Y = \emptyset$ ). A point  $x$  is said to be *one-side isolated* in  $Y$  if  $x$  is either left-side or right-side isolated in  $Y$ , and a point  $x$  which is both a right-sided and a left-sided isolated in  $Y$  is said to be *isolated* in  $Y$ .

Let  $x \in S^1$  and  $f \in C^0(S^1, S^1)$  be given. Then we use the symbols  $\omega_+(x)$  (resp.,  $\omega_-(x)$ ) to denote the set of all points  $y \in S^1$  such that there exists a sequence  $\{n_i\}$  of positive integers with  $n_i \rightarrow \infty$  such that  $f^{n_i}(x) \rightarrow y$  and  $y < \dots < f^{n_i}(x) < \dots < f^{n_2}(x) < f^{n_1}(x)$  (resp.  $f^{n_1}(x) < f^{n_2}(x) < \dots < f^{n_i}(x) < \dots < y$ ). It is clear that if  $x \notin P(f)$ , then  $\omega(x) = \omega_+(x) \cup \omega_-(x)$ . Define  $\Lambda_+(f) = \bigcup_{x \in S^1} \omega_+(x)$  and  $\Lambda_-(f) = \bigcup_{x \in S^1} \omega_-(x)$ .

Also, we use the symbols  $\alpha_+(x)$  (resp.  $\alpha_-(x)$ ) to denote the set of all points  $y \in S^1$  such that there exists a sequence  $\{n_i\}$  of positive integers with  $n_i \rightarrow \infty$  and a sequence  $\{x_i\}$  of points such that  $x_i \rightarrow y, f^{n_i}(x_i) = x$  for every  $i > 0$  and  $y < \dots < x_i < \dots < x_2 < x_1$  (resp.  $x_1 < x_2 < \dots < x_i < \dots < y$ ). It is clear that if  $x \notin P(f)$ , then  $\alpha(x) = \alpha_+(x) \cup \alpha_-(x)$ .

Define  $\gamma_+(x) = \omega_+(x) \cap \alpha_+(x)$  and  $\gamma_-(x) = \omega_-(x) \cap \alpha_-(x)$ . Also, we define  $\Gamma_+(f) = \bigcup_{x \in S^1} \gamma_+(x)$  and  $\Gamma_-(f) = \bigcup_{x \in S^1} \gamma_-(x)$ .

Let  $Y$  be an arc in  $S^1$  and let  $\bar{Y}$  denote the closure of  $Y$  as usual. A point  $y \in S^1$  is

called a *right-sided* (resp., *left-sided*) *accumulation point* of  $Y$  if, for any  $z \in S^1$ ,  $(y, z) \cap Y \neq \emptyset$  (resp.  $(z, y) \cap Y \neq \emptyset$ ).

The right-side closure  $\bar{Y}_+$  (resp. left-side closure  $\bar{Y}_-$ ) is the union of  $Y$  and the set of right-sided (resp. left-sided) accumulation points of  $Y$ . A point which is both a right-sided and a left-sided accumulation point of  $Y$  is called a *two-sided accumulation point* of  $Y$ .

**3. Main results.** The following lemmas are founded in [3].

**LEMMA 3.1.** *Let  $f \in C^0(S^1, S^1)$  and  $x \in \Omega(f)$ . Then we have  $x \in \alpha(x)$ .*

**LEMMA 3.2.** *Let  $f \in C^0(S^1, S^1)$  and  $I = [a, b]$  be an arc for some  $a, b \in S^1$  with  $a \neq b$ , and let  $I \cap P(f) = \emptyset$ .*

- (a) *Suppose that there exists  $x \in I$  such that  $f(x) \in I$  and  $x < f(x)$ . Then*
- (i) *if  $y \in I, x < y$ , and  $f(y) \notin [y, b]$ , then  $[x, y]$   $f$ -covers  $[f(x), b]$ , and*
  - (ii) *if  $y \in I, y < x$ , and  $f(y) \notin [y, b]$ , then  $[y, x]$   $f$ -covers  $[f(x), b]$ .*
- (b) *Suppose that there exists  $x \in I$  such that  $f(x) \in I$  and  $x > f(x)$ . Then*
- (i) *if  $y \in I, x < y$ , and  $f(y) \notin [a, y]$ , then  $[x, y]$   $f$ -covers  $[a, f(x)]$ , and*
  - (ii) *if  $y \in I, y < x$ , and  $f(y) \notin [a, y]$ , then  $[y, x]$   $f$ -covers  $[a, f(x)]$ .*

**LEMMA 3.3.** *Let  $f \in C^0(S^1, S^1)$ . Then we have*

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f) \subset CR(f). \quad (1)$$

The following lemma is due to [5]

**LEMMA 3.4.** *Let  $f \in C^0(S^1, S^1)$ , and let  $K \subset S^1$  with  $f(K) \subset K$ . If  $x \in \Omega(f) \setminus K$ , then  $f^n(x) \notin K^\circ$  for any  $n \geq 1$ .*

The idea of the proof of the following lemma is due to [7].

**LEMMA 3.5.** *Let  $f \in C^0(S^1, S^1)$ , and let  $K \subset S^1$  have only finitely many connected components and  $f(K) = K$ . Then we have  $\bar{K} \setminus K \subset P(f)$ .*

**PROOF.** By continuity of  $f$ , we have  $f(\bar{K}) \subset \overline{f(K)}$ . And by the compactness of  $\bar{K}$ ,  $f(\bar{K}) \subset S^1$  is closed. Thus,  $\overline{f(K)} \subset \overline{f(\bar{K})} = f(\bar{K})$ . Therefore,  $f(\bar{K}) = \overline{f(K)} = \bar{K}$ . Hence, for each  $x \in \bar{K} \setminus K$ , there exists  $x' \in \bar{K} \setminus K$  such that  $f(x') = x$ , i.e.,  $f(\bar{K} \setminus K) = \bar{K} \setminus K$ . By the finiteness of  $\bar{K} \setminus K$ ,  $\bar{K} \setminus K \subset P(f)$ .  $\square$

**PROPOSITION 3.6.** *Let  $f \in C^0(S^1, S^1)$ . Suppose that  $x \in \Omega(f) \setminus \overline{R(f)}$ .*

- (1) *If  $x \in \alpha_+(x)$ , then there exists  $z \in S^1$  such that  $f^i(z, x) \cap (z, x) = \emptyset$  for all  $i \geq 1$ .*
- (2) *If  $x \in \alpha_-(x)$ , then there exists  $u \in S^1$  such that  $f^i(x, u) \cap (x, u) = \emptyset$  for all  $i \geq 1$ .*

**PROOF.** We only need to prove part (1). There exists  $a, b \in S^1$  such that  $x \in (a, b)$  and  $(a, b) \cap \text{Orb}(x) = \emptyset$ . Let  $V = (a, x)$  and let  $W = \cup_{i=0}^{\infty} f^i(V)$ . Then  $x \in \bar{W}$ . Since  $x \in \alpha_+(x)$ , there exist a positive integer  $m$  and a point  $y \in (x, b)$  such that  $f^m(y) = x$ . By Lemma 3.2,

$$[x, y] f^m\text{-covers } [a, x]. \quad (2)$$

We claim that  $x \notin W$ . To show this, suppose that  $x \in W$ . Then there exist a positive integer  $j$  and a point  $x_0 \in (a, x)$  such that  $f^j(x_0) = x$ . By Lemma 3.2,

$$[x_0, x] f^j\text{-covers } [x, b]. \quad (3)$$

In particular,  $[x_0, x] f^j\text{-covers } [x, y]$ .

By (2),

$$[x_0, x] f^j\text{-covers } [x_0, y]. \quad (4)$$

Thus,

$$[x_0, x] f^{j+m}\text{-covers itself}, \quad (5)$$

and, hence,  $f^{j+m}$  has a periodic point in  $(a, b)$ , a contradiction. Hence, we have  $x \in \overline{W} \setminus W$ .

Assume that the proposition is false, i.e., for each  $z \in (a, x)$ , there is some  $i \geq 1$  such that  $(z, x) \cap f^i(z, x) \neq \emptyset$ . Note that  $V \subset f(W)$ . Because, for each  $y' \in V$ , there is some  $i \geq 1$  such that  $(y', x) \cap f^i(y', x) \neq \emptyset$ . There exists  $x_0 \in (y', x)$  such that  $f^i(x_0) \in (y', x)$ . By Lemma 3.2, either

$$[x_0, x] f^i\text{-covers } [f^i(x_0), b] \quad \text{or} \quad [x_0, x] f^i\text{-covers } [a, f^i(x_0)]. \quad (6)$$

Particularly, either

$$[x_0, x] f^i\text{-covers } [x, b] \quad \text{or} \quad [x_0, x] f^j\text{-covers } [a, f^i(x_0)]. \quad (7)$$

If

$$[x_0, x] f^i\text{-covers } [x, b], \quad (8)$$

then

$$[x_0, x] f^j\text{-covers } [x, y]. \quad (9)$$

By (2),

$$[x, y] f^m\text{-covers } [x_0, x]. \quad (10)$$

Hence,

$$[x_0, x] f^{j+m}\text{-covers itself}. \quad (11)$$

Thus,  $f^{j+m}$  has a periodic point in  $(a, b)$ . This is a contradiction. Therefore,

$$[x_0, x] f^j\text{-covers } [a, f^i(x_0)]. \quad (12)$$

Thus,  $y' \in f^i(x_0, x) \subset f^i(V) \subset f(W)$  since  $y' \in (a, f^i(x_0))$ . Thus, for each  $i = 1, 2, 3, \dots, l-1$ ,  $f^i(V) \cap f^{l+i}(V) \neq \emptyset$ , and  $f^{l+i}(V) \cap f^{2l+i}(V) \neq \emptyset, \dots$ . Therefore,  $U_i = \cup_{m=0}^{\infty} f^{ml+i}(V)$  is connected and  $W = \cup_{i=0}^{l-1} U_i$  has only finitely many connected components. Now, by Lemma 3.5,  $x \in \overline{W} \setminus W \subset P(f)$ . This is in contradiction with the assumption of this proposition.  $\square$

The following theorem follows immediately from the proposition.

**THEOREM 3.7.** *Let  $f \in C^0(S^1, S^1)$ . Then each point of  $\Omega(f) \setminus \overline{R(f)}$  is one-side isolated in  $\Omega(f)$ .*

**COROLLARY 3.8.** *Let  $f \in C^0(S^1, S^1)$ . Then  $\Omega(f) \setminus \overline{R(f)}$  is countable which is nowhere dense in  $S^1$ .*

The following proposition is found in [1].

**PROPOSITION 3.9.** *Let  $f \in C^0(S^1, S^1)$ . Then we have*

- (1)  $\overline{R(f)}_+ \setminus R(f) \subset \Lambda(f)_+$ .
- (2)  $\overline{R(f)}_- \setminus R(f) \subset \Lambda(f)_-$ .

**PROPOSITION 3.10.** *Let  $f \in C^0(S^1, S^1)$ . Then we have  $\overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f) \subset \Gamma(f)$ .*

**PROOF.** If  $P(f) = \phi$ , then we have the desired results since  $\overline{R(f)} = \Gamma(f)$  [2]. Suppose that  $P(f) \neq \phi$ . Let  $z \in \overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f)$ . Then there exist  $a, b \in S^1$  with  $a < b$  such that  $z \in (a, b)$  and  $(a, b) \cap \text{Orb}(z) = \phi$ . By Proposition 3.9,  $z \in \Lambda(f)_+ \cap \Lambda(f)_-$ . Then there exist  $\gamma_1, \gamma_2$  such that  $a < \gamma_1 < z < \gamma_2 < b$  with  $z \in \omega(\gamma_1) \cap \omega(\gamma_2)$ . Since  $\overline{P(f)} = \overline{R(f)}$  [4],  $z \in \overline{P(f)}_+ \cap \overline{P(f)}_- \setminus P(f)$ . Then there exists  $u_i$  of periodic point of  $f$  with  $a < \gamma_1 < u_1 < u_2 < \dots < z$  and  $u_i \rightarrow z$ . Let  $p_i$  be the period of  $u_i$  with respect to  $f$ . Then  $f^{p_i}(u_i) = u_i$  for all  $i \geq 1$ . Then either  $[u_i, z]$   $f^{p_i}$ -covers  $[a, u_i]$  or  $[u_i, z]$   $f^{p_i}$ -covers  $[u_i, b]$ .

We may assume that, for infinitely many  $i$ , either

$$[u_i, z] f^{p_i}\text{-covers } [a, u_i] \quad \text{or} \quad [u_i, z] f^{p_i}\text{-covers } [u_i, b]. \quad (13)$$

Then we consider two cases.

**CASE I.**  $[u_i, z]$   $f^{p_i}$ -covers  $[a, u_i]$  for infinitely many  $i$ . There exists  $z_i \in [u_i, z]$  such that  $f^{p_i}(z_i) = \gamma_1$ . Since  $u_i \rightarrow z$ ,  $z_i \rightarrow z$ . Thus,  $z \in \alpha(\gamma_1)$  and, hence,  $z \in \omega(\gamma_1) \cap \alpha(\gamma_1) \subset \Gamma(f)$ .

**CASE II.**  $[u_i, z]$   $f^{p_i}$ -covers  $[u_i, b]$  for infinitely many  $i$ . There exists  $z'_i \in [u_i, z]$  such that  $f^{p_i}(z'_i) = \gamma_1$ . Since  $u_i \rightarrow z$ ,  $z'_i \rightarrow z$ . Thus,  $z \in \alpha(\gamma_2)$  and, hence,  $z \in \omega(\gamma_2) \cap \alpha(\gamma_2) \subset \Gamma(f)$ .  $\square$

The idea of the proof of the following lemma is due to [8].

**LEMMA 3.11.** *Let  $f \in C^0(S^1, S^1)$  and  $Y \subset S^1$ . Then  $\overline{Y} \setminus (\overline{Y}_+ \cap \overline{Y}_-)$  is countable.*

**PROOF.** For each  $\gamma \in \overline{Y}_+ \setminus \overline{Y}_-$ , there is some  $u_\gamma \in S^1$  such that  $(u_\gamma, \gamma) \cap Y = \phi$ . The family of  $\{(u_\gamma, \gamma) \mid \gamma \in \overline{Y}_+ \setminus \overline{Y}_-\}$  is countable because it is disjoint. Hence,  $\overline{Y}_+ \setminus \overline{Y}_-$  is countable. Similarly,  $\overline{Y}_- \setminus \overline{Y}_+$  is also countable. Therefore,

$$\overline{Y} \setminus (\overline{Y}_+ \cap \overline{Y}_-) = (\overline{Y}_+ \setminus \overline{Y}_-) \cup (\overline{Y}_- \setminus \overline{Y}_+) \quad (14)$$

is countable.  $\square$

**THEOREM 3.12.** *Let  $f \in C^0(S^1, S^1)$ . Then*

- (1)  $\Omega(f) \setminus \Gamma(f)$  is countable.
- (2)  $\Lambda(f) \setminus \Gamma(f)$  and  $\overline{R(f)} \setminus \Gamma(f)$  are either empty or countably infinite.

**PROOF.** (1) We know that  $\overline{R(f)} \setminus (\overline{R(f)}_+ \cap \overline{R(f)}_-)$  is countable by Lemma 3.11. By Proposition 3.10,  $\overline{R(f)} \setminus \Gamma(f)$  is also countable. By Corollary 3.8,  $\Omega(f) \setminus \overline{R(f)}$  is countable. Hence,  $\Omega(f) \setminus \Gamma(f)$  is countable.

(2) It is easy to prove that  $f(\omega(x)) = \omega(x)$  and  $f(\overline{R(f)}) = \overline{R(f)}$  for  $x \in S^1$ . Hence,  $f(\Lambda(f)) = \Lambda(f)$ . Suppose that  $\Lambda(f) \setminus \Gamma(f) \neq \emptyset$  (resp.,  $\overline{R(f)} \setminus \Gamma(f) \neq \emptyset$ ). Then we take  $z_1 \in \Lambda(f) \setminus \Gamma(f)$  (resp.,  $z_1 \in \overline{R(f)} \setminus \Gamma(f)$ ). We can take  $z_2 \in \Lambda(f) \setminus \Gamma(f)$  (resp.,  $z_2 \in \overline{R(f)} \setminus \Gamma(f)$ ) such that  $z_1 = f(z_2)$ . Continuing this process, we can take  $z_i \in \Lambda(f) \setminus \Gamma(f)$  (resp.,  $z_i \in \overline{R(f)} \setminus \Gamma(f)$ ) such that  $z_i = f(z_{i+1})$  for all  $i = 1, 2, \dots$ . Since  $z_i \notin \Gamma(f)$  for all  $i \geq 1$ , the points  $z_1, z_2, \dots$  are pairwise disjoint. Hence,  $\Lambda(f) \setminus \Gamma(f)$  (resp.,  $\overline{R(f)} \setminus \Gamma(f)$ ) is infinite and, hence,  $\Lambda(f) \setminus \Gamma(f)$  (resp.,  $\overline{R(f)} \setminus \Gamma(f)$ ) is countably infinite.  $\square$

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