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SEPARABLE INJECTIVITY AND C^* TENSOR PRODUCTS

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ABSTRACT. Let A and B be C*-algebras and let D be a C*-subalgebra of B. We show that if D is separably injective then the triple (A, B, D) verifies the slice map conjecture. As an application, we prove that the minimal C*-tensor product $A \otimes B$ is separably injective if and only if both A and B are separably injective and either A or B is finite-dimensional.

KEY WORDS AND PHRASES. C-algebra, C*-tensor product, injective C*-algebra, separably injective C*-algebra, slice map.*

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1. INTRODUCTION.

Smith and Williams introduced the notion of separable injectivity in connection with the study of completely bounded maps ([**7**], [**8**]). As stated in the introduction of [**8**], it is weaker than the related concept of injectivity and yet is appropriate for certain desirable extension problems. From this point of view, we study C*-tensor products.

Let A and B be C*-algebras and D be a C*-subalgebra of B. If D is injective, then the triple (A, B, D) verifies the slice map conjecture in the sense of Wassermann ([**11**], [**12**]). We first show that the separable injectivity is enough for (A, B, D) to verify the slice map conjecture. Also the minimal C*-tensor product $A \otimes B$ is injective if and only if A and B are injective and either A or B is finite-dimensional (see the proof of [**9**, Theorem]). Using the above result and [**5**], we give a separably injective version of this theorem.

2. PRELIMINARIES AND NOTATION.

Let A and B be C*-algebras and let $A \otimes B$ denote their minimal (i.e., spatial) tensor product. Let M_n denote the C*-algebra of all $n \times n$ complex matrices for a positive integer n. If $\phi : A \to B$ is a linear map then $\phi \otimes id_n : A \otimes M_n \to B \otimes M_n$ is defined by $(\phi \otimes id_n)(a_{ij}) = (\phi(a_{ij}))$. ϕ is said to be completely positive if each $\phi \otimes id_n$ is positive.

A C*-algebra D is said to be injective if given C*-algebras $E \subseteq F$, any contractive completely positive map $\phi : E \to D$ has a contractive completely positive extension $\psi : F \to D$. We say that a C*-algebra D is separably injective if given separable C*-algebras $E \subseteq F$, any contractive completely positive map $\phi : E \to D$ has a contractive completely positive extension $\psi : F \to D$. The separable injectivity in this paper is weaker than one in ([**7**], [**8**]) and both coincide for commutative C*-algebras. A compact Hausdorff space is said to be substonean if every two disjoint co-zero sets have disjoint closures. A compact Hausdorff space X is substonean if and only if $C(X)$ is separably injective [**7**, Theorem 4.6].

A C*-algebra D is said to be subhomogeneous if every irreducible representation is finite-dimensional with bounded dimension. In particulr, it is said to be n-homogeneous if every irreducible representation is n-dimensional. If D is subhomogeneous then we identify the spectrum $\widehat{D}$ with the primitive ideal space [**2**, Chapters 3 and 4].

For $i = 1, 2$ let D_i be a C*-algebra and let $h_i \in D_i^*$. The right slice map $R_{h_1} : D_1 \otimes D_2 \to D_2$ and the left slice map $L_{h_2} : D_1 \otimes D_2 \to D_1$ are unique bounded linear maps satisfying $R_{h_1}(x_1 \otimes x_2) = h_1(x_1)x_2$ and $L_{h_2}(x_1 \otimes x_2) = h_2(x_2)x_1$ [**10**]. For C*-subalgebras A_i of D_i, we define the Fubini product $F(A_1, A_2, D_1 \otimes D_2)$ of A_1 and A_2 with respect to $D_1 \otimes D_2$ [**11**] by

$$F(A_1, A_2, D_1 \otimes D_2)$$
$$= \{x \in D_1 \otimes D_2 : R_{h_1}(x) \in A_2 \text{ and } L_{h_2}(x) \in A_1 \text{ for all } h_1 \in D_1^* \text{ and } h_2 \in D_2^*\}.$$

For fixed C*-algebras A_1 and A_2, $F(A_1, A_2, D_1 \otimes D_2)$ depends on $D_1 \otimes D_2$. But they are all isomorhic and are the largest among them if D_1 and D_2 are injective. We denote by $A_1 \otimes_F A_2$ any one of these isomorphic Fubini products of A_1 and A_2 [**4**]. Let A and B be C*-algebras and let D be a C*-subalgebra. The triple (A, B, D) is said to verify the slice map conjecture if $F(A, D, A \otimes B) = A \otimes D$ [**12**].

3. THE SLICE MAP PROBLEM.

A C*-algebra A is said to have property (S) if (A, B, D) verifies the slice map conjecture for every C*-algebra B and every C*-subalgebra D of B [**12**]. We now consider a property (S') as follows. A C*-algebra D is said to have property (S') if (A, B, D) verifies the slice map conjecture for every C*-algebra A and every C*-algebra B containing

D. Subhomogeneous or injective C*-algebras have property (S') [11].

THEOREM 1. *Let D be a C*-algebra. If D is serarably injective, then D has property (S').*

PROOF: Let A be a C*-algebra and B a C*-algebra containing D. Let $x \in F(A, D, A \otimes B)$. Then there exists a sequence $\{x_n\}$ such that $x_n = \sum_{i=1}^{m(n)} a(i,n) \otimes b(i,n)$, $\lim_n x_n = x$, where each $a(i,n) \in A$ and each $b(i,n) \in B$. Let B_0 be the C*-subalgebra generated by $\{b(i,n) : i = 1, \ldots, m(n), n = 1, 2, \ldots\}$ and let D_0 be the C*-subalgebra of D generated by $\{R_h(x) : h \in A^*\}$. Then we have

$$D_0 \subseteq B_0, \quad x \in F(A, D_0, A \otimes B_0)$$

by a similar argument of [4, Lemma 5]. By hypothesis, there exists a contractive completely positive map $\phi : B_0 \to D$ which extends the identity embedding of D_0 into D. Then

$$R_h((I_A \otimes \phi)(x)) = \phi(R_h(x)) = R_h(x) \qquad (h \in A^*).$$

Since $\{R_h : h \in A^*\}$ is total [10, Theorem 1], $x = (I_A \otimes \phi)(x) \in A \otimes D$ and so $F(A, D, A \otimes B) \subseteq A \otimes D$.

The opposite inclusion is immediate.

It is known that the direct sum of two C*-algebras having property (S') has property (S'). In order to show that Theorem 1 gives a new example having property (S'), two results will be needed. In the proof of [7, Theorem 4.6], Smith and Williams obtained the following lemma.

LEMMA 2. *Let B and D be C*-algebras. Then there exists a one to one correspondence θ between completely positive maps $\phi : B \to D \otimes M_n$ and completely positive maps $\psi : B \otimes M_n \to D$ for any positive integer n.*

We remark that θ is not necessarily norm preserving and that θ satisfies that $\theta(\phi)_{|A \otimes M_n} = \theta(\phi_{|A})$ for a C*-subalgebra A of B, where $\theta(\phi)_{|A \otimes M_n}$ and $\theta(\phi_{|A})$ denote the restrictions of $\theta(\phi)$ and ϕ to $A \otimes M_n$ and A, respectively.

The proof of the following proposition is based on an idea of [6, Theorem 2.1].

PROPOSITION 3. *Let D be a C*-algebra. If D is separably injective, then $D \otimes M_n$ is separably injective for any positive integer n.*

PROOF: Let A be a separable C*-algebra and let $\phi : A \to D \otimes M_n$ be a contractive completely positive map. Let B be a separable C*-algebra containing A. We will show that ϕ has a norm preserving, completely positive extension $\psi : B \to D \otimes M_n$.

Since the image $\phi(A)$ is separable, there exists a separable C*-subalgebra D_0 such that $\phi(A) \subseteq D_0 \otimes M_n$. Let A_1 and D_1 denote the C*-algebras obtained by adjoining identities to A and D_0, respectively. Then the unital map $\phi : A_1 \to D_1$ defined by $\phi_1(a + \alpha I) = \phi(a) + \alpha I$ is completely positive by [**1**, Lemma 3.9]. By hypothesis, there exists a contractive completely positive map $\pi : D_1 \to D$ which extends the identity embedding of D_0 into D. Define the map $\phi_2 : A_1 \to D \otimes M_n$ by $\phi_2(a) = \pi(\phi_1(a))$. Then ϕ_2 is a contractive completely positive map from A_1 to $D \otimes M_n$ which extends ϕ. Hence we may assume that A has the identity u.

Using the same notations as in Lemma 2, we have the map $\theta(\phi) : A \otimes M_n \to D$ associated with ϕ. Since D is separably injective, there exists a completely positive extension $\psi_1 : B \otimes M_n \to D$ of $\theta(\phi)$. Again by Lemma 2 we have the completely positive map $\phi_3 : B \to D \otimes M_n$ such that $\theta(\phi_3) = \psi_1$. By the remark about Lemma 2, ϕ_3 extends ϕ. Define the completely positive map $\psi : B \to D \otimes M_n$ by $\psi(b) = \phi_3(ubu)$. Then ψ is an extension of ϕ. If $b \in B$ with $\| b \| \leq 1$, then

$$\| \psi(b) \| = \| \phi_{3|uBu}(ubu) \| \leq \| \phi_{3|uBu} \| \| ubu \| \leq \| \phi_{3|uBu}(u) \| = \| \phi(u) \| = \| \phi \| .$$

This completes the proof.

EXAMPLE 4. *Let $\beta\mathbf{N}$ be the Stone-Čech compactification of the set $\mathbf{N}$ of positive integers and $\mathbf{N}^*$ the corona set $\beta\mathbf{N} - \mathbf{N}$. For each positive integer n put $D_n = C(\mathbf{N}^*) \otimes M_n$. Let D_∞ denote the C*-algebra of bounded sequences $\{x_n\}$ such that $x_n \in D_n$ for each n. Then D_∞ is separably injective and has no decomposition $D_\infty = D_s \oplus D_i$ such that D_s is subhomogeneous and D_i is injective.*

PROOF: For each n there exists a projection of norm one from D_n onto $C(\mathbf{N}^*) \otimes 1_n$, where 1_n denotes the identity of M_n. The algebra $C(\mathbf{N}^*)$ is separably injective by [**7**, Theorem 4.6], but is not injective. Then D_n is separably injective by Proposition 3, but is not injective. Hence D_∞ is separably injective, but is not injective.

Suppose that D_∞ has a decomposition $D_\infty = D_s \oplus D_i$. Then there exist central projections p and q of D_∞ such that $p \oplus q = 1$, where 1 denotes the identity of D_∞. We have the sequence $\{p_n\}$ of projections of $C(\mathbf{N}^*)$ such that $p = \{p_n \otimes 1_n\}$. Hence $D_s = \{x \in D_\infty : x = \{x_n\}$ with $x_n \in (C(\mathbf{N}^*)p_n) \otimes M_n$ for all $n\}$. If $p_n \neq 0$, there exists an irreducible representation of D_s with dimension n. Since D_s is subhomogeneous, we have an integer n_0 such that $p_n = 0$ for all $n \geq n_0$. Put $D_{i,n_0} = \{x \in D_\infty : x = \{x_n\}$ with $x_n = 0$ for all $n \leq n_0\}$. Then D_{i,n_0} is not injective. On the other hand, there exists a projection of norm one from D_i onto D_{i,n_0} and hence D_{i,n_0} is injective. This is a contradiction and completes the proof.

4. C*-TENSOR PRODUCTS OF SEPARABLY INJECTIVE C*-ALGEBRAS.

In this section we prove the following theorem.

THEOREM 5. *Let A and B be C^*-algebras. The following two statements are equivalent:*

(i) *$A \otimes B$ is separably injective.*

(ii) *Both A and B are separably injective and either A or B is finite-dimensional.*

We need several lemmas.

LEMMA 6. *Let A_i be a C^*-subalgebra of a C^*-algebra D_i for $i = 1, 2, 3$. Then, under the obvious identification, we have*

$$F(F(A_1 \otimes A_2, D_1 \otimes D_2), A_3, (D_1 \otimes D_2) \otimes D_3)$$
$$= F(A_1, F(A_2, A_3, D_2 \otimes D_3), D_1 \otimes (D_2 \otimes D_3)).$$

PROOF: Let $z \in F(F(A_1, A_2, D_1 \otimes D_2), A_3, (D_1 \otimes D_2) \otimes D_3)$ and $h_i \in D_i^*$ for $i = 1, 2, 3$. Then, we have

$$R_{h_2}(R_{h_1}(z)) = R_{h_1 \otimes h_2}(z) \in A_3,$$
$$L_{h_3}(R_{h_1}(z)) = R_{h_1}(L_{h_3}(z)) \in A_2,$$

because $L_{h_3}(z) \in F(A_1, A_2, D_1 \otimes D_2)$ by the assumption. These imply that

$$R_{h_1}(z) \in F(A_2, A_3, D_2 \otimes D_3).$$

Now, we have

$$L_{h_2 \otimes h_3}(z) = L_{h_2}(L_{h_3}(z)) \in A_1,$$

because $L_{h_3}(z) \in F(A_1, A_2, D_1 \otimes D_2)$ by the assumption. Since the family of all product functionals $h_2 \otimes h_3$ on $D_2 \otimes D_3$ is total, we obtain

$$L_h(z) \in A_1 \qquad \text{for all} \quad h \in (D_2 \otimes D_3)^*$$

by a standard approximation argument (see, for example, [**11**, Lemma 2.1]). Hence we have

$$z \in F(A_1, F(A_2, A_3, D_2 \otimes D_3), D_1 \otimes (D_2 \otimes D_3)).$$

The reverse inclusion can be shown similarly.

LEMMA 7. *Let A be an infinite-dimensional C^*-algebra and D be a non-subhomogeneous C^*-algebra. Then $D \otimes A$ is not separably injective.*

PROOF: Let $B(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H such that $B(H) \supseteq D \otimes A$. Since A is infinite-dimensional, there exists an orthogonal sequence $\{A_n\}$ of commutative C*-subalgebras of A. The C*-subalgebra generated by $\{A_n\}$ may be identitied with the c_0-sum $\oplus_n A_n$ of $\{A_n\}$.

Suppose that $F(B(H), D \otimes A, B(H) \otimes B(H)) = B(H) \otimes (D \otimes A)$. Then, we have

$$
\begin{aligned}
&B(H)\otimes_F(D\otimes(\oplus_n A_n))\\
&\quad = F(B(H), D\otimes(\oplus_n A_n), B(H)\otimes B(H))\\
&\quad \subseteq F(B(H), D\otimes A, B(H)\otimes B(H)) = B(H)\otimes(D\otimes A).
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
&B(H)\otimes_F(D\otimes(\oplus_n A_n))\\
&\quad = F(B(H), D\otimes(\oplus_n A_n), B(H)\otimes(D\otimes A))\\
&\quad = F(B(H), F(D, \oplus_n A_n, D\otimes A), B(H)\otimes(D\otimes A)) \qquad \text{(by [12, Theorem 4])}\\
&\quad = F(F(B(H), D, B(H)\otimes D), \oplus_n A_n, (B(H)\otimes D)\otimes A) \qquad \text{(by Lemma 6)}\\
&\quad = F(B(H)\otimes D, \oplus_n A_n, (B(H)\otimes D)\otimes A)\\
&\quad = (B(H)\otimes D)\otimes(\oplus_n A_n) \qquad \text{(by [12,Theorem 4])}\\
&\quad = B(H)\otimes(D\otimes(\oplus_n A_n)).
\end{aligned}
$$

Since $D\otimes(\oplus_n A_n)$ is canonically *-isomorphic to $\oplus_n(D\otimes A_n)$, this contradicts [**5**, Theorem 3.2]. Hence $F(B(H), (D\otimes A), B(H)\otimes B(H))$ contains properly $B(H)\otimes(D\otimes A)$, and so $D\otimes A$ is not separably injective by Theorem 1.

Lemma 8. *Let A and B be C*-algebras. If $A\otimes B$ is separably injective, then both A and B are separably injective.*

Proof: Let $E\subseteq F$ be separable C*-algebras. Let $\phi: E\to A$ be a contractive completely positive map. Let b be a positive element of B with $\| b \|= 1$. Define $\psi: E\to A\otimes B$ by $\psi(x)=\phi(x)\otimes b$. Then ψ has a contractive completely positive extension $\psi_1: F\to A\otimes B$. Let h be a state of B such that $h(b)=1$. Define $\phi_1: F\to A$ by $\phi_1(x)=L_h(\psi_1(x))$. Then ϕ_1 is the desired extension of ϕ. This implies that A is separably injective. A similar argument shows that B is separably injective.

The following lemma is a slight modification of the proof of [**8**, Propositon 2.6].

Lemma 9. *Let A and B be C*-algebras and let A^1 and B^1 denote the C*-algebras obtained by adjoining identities to A and B, respectively. If $A\otimes B$ is separably injective then $A^1\otimes B^1$ is separably injective.*

Proof: Let $E\subseteq F$ be separable C*-algebras and let $\phi: E\to A^1\otimes B$ be a contractive completely positive map. Choose A_0 and B_0 be separable C*-subalgebras such that $\phi(E)\subseteq A_0\otimes B_0+\mathbf{C}I\otimes B_0$. By [**8**, Proposition 2.5] there exist positive elements $a\in A, b\in B$ and $c\in A\otimes B$ of unit norm such that a, b, and c act as identities of A_0, B_0 and the C*-subalgebra generated by $A_0\otimes B_0$ and $a\otimes b$, respectively. We note that $(a\otimes b)(1\otimes d)=a\otimes bd=(1\otimes d)(a\otimes b)$ for each $d\in B_0$. Let h be the state of A^1 which annihilates A. Define $\psi: E\to A\otimes B$ by $\psi(x)=c\phi(x)c$ and $\theta: E\to B$ by $\theta(x)=R_h(\phi(x))$. By Lemma 8, B is separably injective. Then θ has a contractive completely positive extension $\theta_1 : F \to B$. Define $\theta_2 : F \to \mathbf{C}I \otimes B$ by $\theta_2(x) = I \otimes \theta_1(x)$. By hypothesis ψ has a contractive completely positive extension $\psi_1 : F \to A \otimes B$. Define $\phi_1 : F \to A^1 \otimes B$ by

$$\phi_1(x) = (a \otimes b)\psi_1(x)(a \otimes b) + (I - (a \otimes b)^2)^{\frac{1}{2}}\theta_2(x)(I - (a \otimes b)^2)^{\frac{1}{2}}.$$

Since $\phi_1(x)$ may be written

$$(a \otimes b, (I - (a \otimes b)^2)^{\frac{1}{2}}) \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \theta_2(x) \end{pmatrix} \begin{pmatrix} a \otimes b \\ (I - (a \otimes b)^2)^{\frac{1}{2}} \end{pmatrix},$$

ϕ_1 is a contractive completely positive map. To see ϕ_1 extends ϕ, let $x \in E$ and write $\phi(x) = y + z, y \in A_0 \otimes B_0, z \in \mathbf{C}I \otimes B_0$. Then $\psi_1(x) = y + czc = y + zc^2$ and $\theta_2(x) = z$. Thus

$$\begin{aligned}\phi_1(x) &= (a \otimes b)(y + zc^2)(a \otimes b) + (I - (a \otimes b)^2)^{\frac{1}{2}} z (I - (a \otimes b)^2)^{\frac{1}{2}} \\ &= y + z(a \otimes b)^2 + z(I - (a \otimes b)^2) = \phi(x).\end{aligned}$$

Hence $A^1 \otimes B$ is separably injective.

A symmetric argument shows that $A^1 \otimes B^1$ is separably injective.

LEMMA 10. *Let A be a unital infinite-dimensional subhomogeneous C^*-algebra. Then there exist a *-homomorphism π of A and a norm one projection ϕ such that the image $\phi(\pi(A))$ is *-isomorhpic to the C^*-algebra of all continuous functions on some infinite compact Hausdorff space X.*

PROOF: By [**2**, 3.6.3 Proposition] and the proof of [**8**, Theorem 3.2], we may assume that there exists a closed two-sided ideal J such that A/J is finite-dimensional, J is n-homogeneous and $\widehat{J}$ is an infinite set.

Suppose first that $\widehat{J}$ has a limit point. By [**2**, 3.6.4 Proposition] $\widehat{J}$ is a locally compact Hausdorff space. Thus there exists a closed two-sided ideal J_0 such that $(\widehat{J/J_0})$ is an infinite compact Hausdorff space. Let $(\widehat{J/J_0}) = X$. Then $C(X)$ may be identified with the center of J/J_0. Let $\pi : A \to A/J_0$ be the quotient map. From [**2**, 3.6.4 Proposition] for $a \in A$ the map $\lambda \to tr_n(\lambda(a))$ is continuous on $\widehat{J}$, where tr_n denotes the normalized trace on M_n. Note that $\ker \lambda \supseteq \ker \pi$ for each $\lambda \in X$. Define $\phi : \pi(A) \to C(X)$ by

$$\phi(\pi(a))(\lambda) = tr_n(\lambda(a)) \qquad (a \in A, \lambda \in X).$$

It is easy to see that π and ϕ are desired maps.

Suppose now that $\widehat{J}$ has no limit point. Let T be a non-empty set. Let $\ell_T^\infty(M_n)$ be the C*-algebra of $(x_\lambda) = (x_\lambda)_{\lambda \in T}$ such that $x_\lambda \in M_n$ for all $\lambda \in T$ and $\sup_\lambda \| x_\lambda \| < \infty$ and let $c_T^0(M_n)$ be the ideal of $\ell_T^\infty(M_n)$ such that for each $\varepsilon > 0$ $\| x_\lambda \| \leq \varepsilon$ for all but a finite number of indices λ.

Let $\widehat{J} = Y$. Define $\rho : A \to \ell^\infty_Y(M_n)$ by

$$\rho(a)(\lambda) = \lambda(a) \qquad (a \in A, \lambda \in Y).$$

Since Y is discrete, by [**2**, 10.10.1] we have $\rho(J) = c^0_Y(M_n)$. Let $\mu : \ell^\infty_Y(M_n) \to \ell^\infty_Y(M_n)/c^0_Y(M_n)$ denote the quotient map. Since $(\mu\rho)^{-1}(0) \supseteq J$, $\mu\rho(A)$ is finite-dimensional. Hence there exists a finite set $\{a_1, \cdots a_k\}$ of A such that $\{\mu\rho(a_1), \cdots, \mu\rho(a_k)\}$ spans $\mu\rho(A)$. Then we have

$$\rho(A) = c^0_Y(M_n) + \mathbf{C}\rho(a_1) + \cdots + \mathbf{C}\rho(a_k).$$

Let X be the one-point compactification of the set $\mathbf{N}$ of positive integers. Then $C(X) \otimes M_n$ may be identified with the C*-algebra of convergent sequences of elements of M_n. Let $\rho(a_i) = (m^i_\lambda) \in \ell^\infty_Y(M_n)$. Passing to convergent subsequences, there exists a sequence $\{\lambda_n\}$ of Y such that $(m^i_{\lambda_n}) \in C(X) \otimes M_n$ for each i. Define $\nu : \ell^\infty_Y(M_n) \to \ell^\infty_{\{\lambda_n\}}(M_n)$ by

$$\nu((a_\lambda)) = (a_{\lambda_n}).$$

Then $\nu\rho(a) \in C(X) \otimes M_n$ for $a \in A$. Let $\pi = \nu\rho$. Then $\pi(A) = C(X) \otimes M_n$. Define $\phi : \pi(A) \to C(X)$ by

$$\phi(\pi(a))(\lambda_n) = tr_n(\rho(a)_{\lambda_n}).$$

It is well known that ϕ has the desired property.

Using Choi-Effros lifting theorem [**1**], Smith and Williams showed in the proof of [**8**, Lemma 3.3] that every quotient algebra of a nuclear separably injective C*-algebra is separably injective. By this useful result we have the following lemma.

LEMMA 11. *Let A be a nuclear separably injective C*-algebra. Let π be a *-homomorphism of A and let B be a commutative C*-subalgebra of $\pi(A)$. If there exists a norm one projection $\phi : \pi(A) \to B$ such that $\phi(\pi(A)) = B$, then B is separably injective.*

PROOF: Let $E \subseteq F$ be separable C*-algebras and let $\psi : E \to B$ be a contractive completely positive map. By the above remark, $\pi(A)$ is separably injective. Then ψ has a contractive completely positive extension $\psi_1 : F \to \pi(A)$. Then $\psi_2 = \phi\psi_1 : F \to B$ is a contractive completely positive extension of ψ. Hence B is separably injective.

PROOF OF THEOREM 5: (i) $\Rightarrow$ (ii). By Lemma 8, it suffices to show that either A or B is finite-dimensional. To do this, we may assume that A and B are unital by Lemma 9.

Suppose that A and B are infinite-dimensional. If A or B is non-subhomogeneous, it follows from Lemma 7 that $A \otimes B$ is not separably injective. This is a contradiction. Now if A and B are subhomogeneous, by Lemma 10 there exist *-homomorphisms

π_1, π_2, infinite compact Housdorff spaces X_1, X_2 and norm one projections $\phi_1 : \pi_1(A) \to C(X_1)$, $\phi_2 : \pi_2(B) \to C(X_2)$. By Lemma 11 and [**7**, Theorem 4.6] X_1 and X_2 are substonean. We may identify $C(X_1) \otimes C(X_2)$ with $C(X_1 \times X_2)$. Then $\phi_1 \odot \phi_2 : \pi_1 \odot \pi_2(A \otimes B) \to C(X_1 \times X_2)$ is a norm one projection such that $\phi_1 \otimes \phi_2(\pi_1 \otimes \pi_2(A \otimes B)) = C(X_1 \times X_2)$. Again by Lemma 11 and [**7**, Theorem 4.6] $X_1 \times X_2$ is substonean. But this contradicts [**3**, Proposition 1.7].

(ii) $\Rightarrow$ (i). Since a finite-dimensional C*-algebra is a finite direct sum of matrix algebras, Propositon 3 implies that $A \otimes B$ is separably injective.

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