ON THE SOLVABILITY OF A CLASS OF REACTION-DIFFUSION SYSTEMS

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We deal with a class of parabolic reaction-diffusion systems. We use an iterative process based on results obtained for a linearized problem, then we derive some a priori estimates to establish the existence, uniqueness, and continuous dependence of the weak solution for a class of quasilinear systems.

1. Introduction

Reaction-diffusion systems of PDEs furnish valuable mathematical models for a great number of phenomena in engineering and biology. For instance, the following system describes the dynamics of a simple isothermal chemical reaction system [25]:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= uv, \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= uv - hv, \end{aligned} \tag{1.1}$$

where h is a positive parameter. Moreover, the next system is a model for the description of the patchy distributions of microscopic aquatic organisms known as plankton (see, [17]):

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f_1(u,v), \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= f_2(u,v), \end{aligned} \tag{1.2}$$

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where

$$f_1(u,v) = a(a_1 + b_1 u + u^2)u - uv, \qquad f_2(u,v) = uv - (a_2 v + b_2 v^2), \tag{1.3}$$

and (a, a_1, a_2, b_1, b_2) are positive parameters. Likewise, if $u(x,t)$ represents the population density, $v(x,t)$ the concentration of the attractant, $F(u)$ and $H(u,v)$ describe the local kinetics of the population and the attractant respectively, t is the time, and x is the one-dimensional spatial variable, then the system

$$\begin{aligned} \frac{\partial u}{\partial t} - d_1 \frac{\partial^2 u}{\partial x^2} &= F(u) - T(u,v)\frac{\partial u}{\partial x}, \\ \frac{\partial v}{\partial t} - d_2 \frac{\partial^2 v}{\partial x^2} &= H(u,v) \end{aligned} \tag{1.4}$$

represents a model for a population with attractant and has growth-diffusion-chemotaxis type. Some versions of this model were investigated in [18, 19, 23]. For other models, we refer the reader, for instance, to [1, 5, 7–9, 13, 20, 24].

The purpose of this paper is to study the following quasilinear reaction-diffusion parabolic system:

$$\frac{\partial u}{\partial t} - d_1 \Delta u = f_1(x,t,u,v,\nabla u, \nabla v) + F(x) \quad \text{in } Q_T, \tag{1.5}$$

$$\frac{\partial v}{\partial t} - d_2 \Delta v = f_2(x,t,u,v,\nabla u, \nabla v) + G(x) \quad \text{in } Q_T, \tag{1.6}$$

$$\begin{gathered} u(x,0) = u_0, \quad v(x,0) = v_0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \Sigma_T, \end{gathered} \tag{1.7}$$

where d_1, d_2 are positive constants, Ω is an open bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $Q_T = \Omega \times I$, $T > 0$, and $\Sigma_T = \partial\Omega \times I$, $T > 0$; $f_1, f_2 : \Omega \times I \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are measurable functions.

For the semilinear case of (1.5)–(1.7) (when the functions f_1 and f_2 do not depend on the gradient), the existence of positive solutions has been established in [10–12, 16], under the following assumptions.

Assumption 1.1. The total mass of the components u, v is controlled with time, which is ensured by

$$\begin{gathered} (f_1 + f_2)(x,t,u,v,p,q) \le L_1(u + v + 1), \\ \forall (u,v,p,q) \in \mathbb{R}_+^2 \times \mathbb{R}^{2N} \quad \text{a.e. } (x,t) \in Q_T,\ L_1 \ge 0. \end{gathered} \tag{1.8}$$

Assumption 1.2. The function f_1 verifies

$$\begin{gathered} f_1(x,t,u,v,s,r) \le L_2(u + v + 1), \\ \forall (u,v,p,q) \in \mathbb{R}_+^2 \times \mathbb{R}^{2N} \quad \text{a.e. } (x,t) \in Q_T,\ L_2 \ge 0. \end{gathered} \tag{1.9}$$

Note that if Assumption 1.1 or 1.2 does not hold, the authors in [21] have been proved blowup in finite time of the solutions to some semilinear reaction-diffusion systems.

As for the quasilinear case, it is showed in [4] the existence of positive weak solutions when the initial data are in L^1 under Assumptions 1.1 and 1.2.

Assumption 1.3. The positivity of the solution is preserved with time, which is ensured by

$$\begin{gathered} f_1(x,t,0,v,p,q), f_2(x,t,u,0,p,q) \geq 0, \\ \forall (u,v,p,q) \in \mathbb{R}_+^2 \times \mathbb{R}^{2N} \quad \text{a.e. } (x,t) \in Q_T. \end{gathered} \tag{1.10}$$

Assumption 1.4. The nonlinear term with respect to the gradient is subquadratic, namely,

$$\begin{gathered} \left| f_1(x,t,u,v,\nabla u,\nabla v)\right| + \left| f_2(x,t,u,v,\nabla u,\nabla v)\right| \\ \leq C(|u|,|v|)\left(\|\nabla u\|^m + \|\nabla z_1\|^m + 1\right), \end{gathered} \tag{1.11}$$

where $1 \leq m < 2$, $C : [0,\infty)^2 \to [0,\infty)$ is nondecreasing.

A more general result has been obtained later when the initial data are in L^2 (see [2]). The authors in [2] have investigated problems (1.5)–(1.7) under Assumptions 1.1, 1.3 together with the following assumptions.

Assumption 1.5. The functions f_1 and f_2 have critical growth with respect to $|\nabla u|$, $(m=2)$.

Assumption 1.6. The function f_1 satisfies the "sign condition"

$$u f_2(x,t,u,v,\nabla u,\nabla v) \leq 0 \quad \forall u,v \geq 0, \text{ a.e. } (x,t) \in Q_T. \tag{1.12}$$

Note that for a single equation ($d_1 = d_2$ and $f_1 = f_2$), existence results have been obtained by many authors; see for instance [1, 3, 6, 15]. Finally, we mention that in order to establish the existence, many authors have used some regularizations in time and some truncation based on the so-called natural truncation T_k defined by

$$T_k(s) = \max\left(-k, \min(k,s)\right), \tag{1.13}$$

where k is a positive real number.

The present paper can be considered as a continuation of works cited above, especially [2, 4]. Our main goal is to extend those results, in a certain sense. Namely, we will establish the existence, uniqueness, and continuous dependence of a weak solution of problems (1.5)–(1.7) without supposing Assumptions 1.1–1.6. We will consider only the following.

Assumption 1.7. The functions f_i $(i = 1,2)$ are bounded in L^2 and satisfy

$$\left| f_i(\cdot,\cdot,p_1,q_1) - f_i(\cdot,\cdot,p_2,q_2)\right| \leq L\left(\left|p_1 - p_2\right| + \|q_1 - q_2\|\right), \tag{1.14}$$

where L is a positive constant.

The paper is organized as follows. In Section 2 we transform problems (1.5)–(1.7) to an equivalent one which is easier to analyze, and we make precise in which sense we solve the reduced problem. Then, in Section 3, we formulate an approximate problem. In Section 4, we derive some useful a priori estimates. Section 5 is devoted to establish the existence of a weak solution, while the uniqueness and continuous dependence of the solution are given in Section 6.

2. An equivalent problem

In this section, we will consider the linearization of (1.5)–(1.7) obtained by assuming that $f_1 = -f_2$; $d_1 = d_2 = d$. The sum of the two components u and v satisfies the following linear parabolic equation:

$$\begin{gathered}\frac{\partial w}{\partial t} - d\Delta w = 0 \quad \text{in } Q_T,\\ w(x,0) = u_0(x) + v_0(x) \quad \text{in } \Omega,\\ w = 0 \quad \text{on } \Sigma_T.\end{gathered} \tag{2.1}$$

Consequently, the function u of problems (1.5)–(1.7) fulfills

$$\begin{gathered}\frac{\partial u}{\partial t} - d\Delta u = f_1(x,t,u,w-u,\nabla u,\nabla w - \nabla u) \quad \text{in } Q_T,\\ u(x,0) = u_0(x) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \Sigma_T.\end{gathered} \tag{2.2}$$

It is well known that (2.1) has a unique solution in $L^2(I;H_0^1(\Omega)) \cap C(I;L^2(\Omega))$ that satisfies $\partial w/\partial t \in L^2(I;L^2(\Omega))$. Then, if we show that u, solution of problem (2.2), exists and sets

$$v := w - u, \tag{2.3}$$

then problems (1.5)–(1.7) will be solved.

Consider now the following auxiliary problem:

$$\begin{gathered}\frac{\partial \sigma}{\partial t} - d\Delta \sigma = 0 \quad \text{in } Q_T,\\ \sigma(x,0) = u_0(x) \quad \text{in } \Omega,\\ \sigma = 0 \quad \text{on } \Sigma_T.\end{gathered} \tag{2.4}$$

Let $z = u - \sigma$, where u is the solution of problem (2.2). Therefore z verifies

$$\begin{gathered}\frac{\partial z}{\partial t} - d\Delta z = f(x,t,z,\nabla z) \quad \text{in } Q_T,\\ z(x,0) = 0 \quad \text{in } \Omega,\\ z = 0 \quad \text{on } \Sigma_T,\end{gathered} \tag{2.5}$$

where

$$f(x,t,z,\nabla z) = f_1(x,t,z+\sigma,w-z-\sigma,\nabla z+\nabla\sigma,\nabla w-\nabla z-\nabla\sigma). \tag{2.6}$$

Since problem (2.4) possesses a unique solution, our objective is to solve problem (2.5).

Let us now define the notion of the solution we are looking for.

Definition 2.1. Say that $z(x,t)$ is a weak solution of problem (2.5) if the following properties are verified:

(i) $z \in L^2(I;H_0^1(\Omega)) \cap C(I;L^2(\Omega))$;
(ii) $\partial z/\partial t \in L^2(I;L^2(\Omega))$;
(iii) z satisfies the initial condition $z(\cdot,0) = 0$ in $L^2(\Omega)$;
(iv) the integral identity

$$\left(\frac{\partial z(\cdot,t)}{\partial t},\theta(\cdot,t)\right) + d\big(\nabla z(\cdot,t),\nabla\theta(\cdot,t)\big) = \big(f(\cdot,t),\theta(\cdot,t)\big) \tag{2.7}$$

holds for all $\theta \in H_0^1(\Omega)$, and all $t \in I$.

As functions f_i $(i = 1,2)$, the function f verifies the following.

Assumption 2.2. The function f is bounded in L^2 and satisfies the Lipschitz condition: $\exists L > 0$ such

$$\left| f(\cdot,\cdot,p_1,q_1) - f(\cdot,\cdot,p_2,q_2)\right| \le L\big(\left|p_1-p_2\right| + \|q_1-q_2\|\big). \tag{2.8}$$

3. Formulation of an approximate problem

Let $\{z_n\}_n$ be a sequence constructed as follows.

For $n = 0$, we set $z_0(x,t) = 0$ for all $(x,t) \in Q_T$, the other terms of the sequence are obtained iteratively as solutions of the linear parabolic equation

$$\begin{gathered} \frac{\partial z_n}{\partial t} - d\Delta z_n = f_n(x,t) \quad \text{in } Q_T, \\ z_n(x,0) = 0 \quad \text{in } \Omega, \\ z_n = 0 \quad \text{on } \Sigma_T, \end{gathered} \tag{3.1}$$

where

$$f_n(x,t) = f\big(x,t,z_{n-1},\nabla z_{n-1}\big). \tag{3.2}$$

It is well known that for any fixed "n," problem (3.1) has a unique solution z_n in $L^2(I;H_0^1(\Omega)) \cap C(I;L^2(\Omega))$, verifying $\partial z_n/\partial t \in L^2(I;L^2(\Omega))$.

Set

$$y_n(x,t) = z_{n+1}(x,t) - z_n(x,t), \tag{3.3}$$

one can easily check that y_n verifies

$$\begin{gathered}\frac{\partial y_n}{\partial t} - d\Delta y_n = F_n(x,t) \quad \text{in } Q_T,\\ y_n(x,0) = 0 \quad \text{in } \Omega,\\ y_n = 0 \quad \text{on } \Sigma_T,\end{gathered} \tag{3.4}$$

where

$$F_n(x,t) = f_{n+1}(x,t) - f_n(x,t). \tag{3.5}$$

4. A priori estimates

In this section, we will establish useful estimates on y_n in some suitable spaces in order to prove the convergence of the sequence $\{z_n\}_n$ to the solution of problem (3.1). To this end, we consider the weak formulation of problem (3.4),

$$\left(\frac{\partial y_n(\cdot,t)}{\partial t},\theta(\cdot,t)\right) + d(\nabla y_n(\cdot,t),\nabla\theta(\cdot,t)) = (F_n(\cdot,t),\theta(\cdot,t)), \tag{4.1}$$

in which we set $\theta = y_n$ and integrate over $(0,\tau)$ to obtain

$$||y_n(\cdot,\tau)||^2 + 2d\int_0^\tau ||\nabla y_n(\cdot,t)||^2 dt = 2\int_0^\tau (F_n(\cdot,t),y_n(\cdot,t))dt. \tag{4.2}$$

According to the Cauchy inequality, it follows

$$2d\int_0^\tau ||\nabla y_n(\cdot,t)||^2 dt + ||y_n(\cdot,\tau)||^2 = \int_0^\tau ||F_n(\cdot,t)||^2 dt + \int_0^\tau ||y_n(\cdot,t)||^2 dt. \tag{4.3}$$

The application of a lemma of Gronwall's type leads to

$$2d\int_0^\tau ||\nabla y_n(\cdot,t)||^2 dt + ||y_n(\cdot,\tau)||^2 \le e^T \int_0^T ||F_n(\cdot,t)||^2_{H_0^1(\Omega)} dt. \tag{4.4}$$

Therefore, by omitting the second term on the left-hand side of (4.4) and applying Assumption 2.2 to the right-hand side, we get

$$d\int_0^\tau ||\nabla y_n(\cdot,t)||^2 dt \le 2e^T L^2 \int_0^T ||y_{n-1}(\cdot,t)||^2_{H_0^1(\Omega)} dt. \tag{4.5}$$

In light of the Friedrichs inequality [22], we have

$$\int_0^\tau ||y_n(\cdot,t)||^2_{H_0^1(\Omega)} dt \le \frac{2e^T L^2 C_\Omega}{d} \int_0^T ||y_{n-1}(\cdot,t)||^2_{H_0^1(\Omega)} dt. \tag{4.6}$$

It follows that

$$||y_n||_{L^2(I,H_0^1(\Omega))} \leq Lc_1||y_{n-1}||_{L^2(I,H_0^1(\Omega))}, \tag{4.7}$$

where

$$c_1 = \frac{2^{1/2}e^{T/2}C_\Omega^{1/2}}{d^{1/2}}. \tag{4.8}$$

On the other hand, by virtue of (4.1) in which we set $\theta = \partial y_n/\partial t$, it yields

$$2\int_0^\tau \left|\left|\frac{\partial y_n(\cdot,t)}{\partial t}\right|\right|^2 dt + d||\nabla y_n(\cdot,t)||^2 dt = 2\int_0^\tau \left(F_n(\cdot,t), \frac{\partial y_n(\cdot,t)}{\partial t}\right)dt. \tag{4.9}$$

In light of the Cauchy inequality and Assumption 2.2, the right-hand side of (4.9) is then dominated by

$$\int_0^\tau \left|\left|\frac{\partial y_n(\cdot,t)}{\partial t}\right|\right|^2 dt + 2L^2\int_0^\tau ||y_{n-1}(\cdot,t)||_{H_0^1(\Omega)}^2 dt, \tag{4.10}$$

where the integral over $\partial y_n/\partial t$ will be absorbed in the left-hand side of (4.9). Thanks to the Friedrichs' inequality the second term on the left-hand side of (4.9) is controlled from below by

$$dC_\Omega^{-1}||y_n(\cdot,\tau)||_{H_0^1(\Omega)}^2. \tag{4.11}$$

Therefore, we have

$$\begin{aligned}
&\int_0^\tau \left|\left|\frac{\partial y_n(\cdot,t)}{\partial t}\right|\right|^2 dt + dC_\Omega^{-1}||y_n(\cdot,\tau)||_{H_0^1(\Omega)}^2 \\
&\quad\leq 2L^2\int_0^T ||y_{n-1}(\cdot,t)||_{H_0^1(\Omega)}^2 dt \\
&\quad\leq 2TL^2||y_{n-1}||_{C(I;H_0^1(\Omega))}^2 \\
&\quad\leq 2TL^2\left(||y_{n-1}||_{C(I;H_0^1(\Omega))}^2 + \left|\left|\frac{\partial y_{n-1}}{\partial t}\right|\right|_{L^2(I;L^2(\Omega))}^2\right).
\end{aligned} \tag{4.12}$$

The right-hand side here is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, thus we obtain

$$\begin{aligned}
&\left|\left|\frac{\partial y_n}{\partial t}\right|\right|_{L^2(I;L^2(\Omega))}^2 + ||y_n||_{C(I;H_0^1(\Omega))}^2 \\
&\quad\leq \frac{2L^2T}{\min(1,dC_\Omega^{-1})}\left(\left|\left|\frac{\partial y_{n-1}}{\partial t}\right|\right|_{L^2(I;L^2(\Omega))}^2 + ||y_{n-1}||_{C(I;H_0^1(\Omega))}^2\right),
\end{aligned} \tag{4.13}$$

implying finally

$$||y_n||_B \leq Lc_2||y_{n-1}||_B, \tag{4.14}$$

where

$$c_2 = \sqrt{\frac{2T}{\min(1, dC_\Omega^{-1})}}. \tag{4.15}$$

Hence we can present the following theorem.

Theorem 4.1. *Suppose that Assumption 2.2 is fulfilled. Then the following estimates hold, for $n = 1, 2, \ldots$:*

$$||y_n||_{L^2(I,H_0^1(\Omega))} \le Lc_1 ||y_{n-1}||_{L^2(I,H_0^1(\Omega))}, \qquad ||y_n||_B \le Lc_2 ||y_{n-1}||_B, \tag{4.16}$$

where B is the Banach space endowed with the finite norm

$$\|y\|_B = \left(\left\| \frac{\partial y}{\partial t} \right\|^2_{L^2(I;L^2(\Omega))} + \|y\|^2_{C(I;H_0^1(\Omega))} \right)^{1/2}, \tag{4.17}$$

c_1 and c_2 are positive constants defined by (4.8) and (4.15).

5. Convergence and existence result

Theorem 5.1. *Take the assumption of Theorem 4.1. If*

$$Lc_3 < 1, \tag{5.1}$$

then there exists a pair

$$\left(z, \frac{\partial z}{\partial t} \right) \in L^2(I, H_0^1(\Omega)) \cap C(I; L^2(\Omega)) \times L^2(I; L^2(\Omega)), \tag{5.2}$$

verifying

$$\left(z_n, \frac{\partial z_n}{\partial t} \right) \xrightarrow[n \to \infty]{} \left(z, \frac{\partial z}{\partial t} \right), \tag{5.3}$$

where $c_3 = \max(c_1, c_2)$.

Proof. In inequality (4.7), if $Lc_1 < 1$, then the series $\sum_{n=0}^{\infty} y_n$ converges in $L^2(I, H_0^1(\Omega))$. Observe that

$$z_n = S_n = \sum_{k=0}^{n-1} (z_{k+1} - z_k), \tag{5.4}$$

hence by passing to the limit, we have

$$z_n \longrightarrow \widetilde{z} \quad \text{in } L^2(I, H_0^1(\Omega)). \tag{5.5}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we have also

$$z_n \longrightarrow \widetilde{z} \quad \text{in } L^2(I, L^2(\Omega)). \tag{5.6}$$

On the other hand, in estimate (4.14), if $Lc_2 < 1$, we deduce that the series $\sum_{n=0}^{\infty} y_n$ and thus the sequence $\{z_n\}_n$ converge in the space B. According to the definition of space B (see Theorem 4.1), we deduce that

$$\frac{\partial z_n}{\partial t} \xrightarrow[n\to\infty]{} \varphi \quad \text{in } L^2(I;L^2(\Omega)). \tag{5.7}$$

We have to prove that φ equals $\partial\widetilde{z}/\partial t$ in $L^2(I;L^2(\Omega))$. To this end, we consider the identity

$$z_n(\cdot,t) = \int_0^t \frac{\partial z_n}{\partial \tau} d\tau \quad \forall t \in I. \tag{5.8}$$

Then, by passing to the limit in (5.8), when n tends to infinity by taking (5.6) and (5.7) into account, it yields

$$\widetilde{z}(\cdot,t) = \int_0^t \varphi\, d\tau \quad \forall t \in I, \tag{5.9}$$

from which we conclude, see for instance [14, Lemmas 1.3.2 and 1.3.6], that

$$\widetilde{z} \in C(I;L^2(\Omega)), \tag{5.10}$$

differentiable for almost everywhere $t \in I$ and $\partial\widetilde{z}/\partial t = \varphi$ in $L^2(I;L^2(\Omega))$, namely,

$$\frac{\partial z_n}{\partial t} \xrightarrow[n\to\infty]{} \frac{\partial \widetilde{z}}{\partial t} \quad \text{in } L^2(I;L^2(\Omega)). \tag{5.11}$$

Consequently, for $Lc_3 < 1$ the limit relation (5.3) is satisfied. □

Theorem 5.2. *Suppose that assumption of Theorem 4.1 is satisfied, moreover assume that $f(x,t,0,0) \in L^2(I;L^2(\Omega))$. Then the limit function $\widetilde{z} = \widetilde{z}(x,t)$ is the weak solution of problem (2.5) in the sense of Definition 2.1.*

Proof. According to Theorem 5.1, assertions (i) and (ii) of Definition 2.1 are fulfilled. Moreover, from (5.9) we conclude that $\widetilde{z}(\cdot,0) = 0$ holds in $L^2(\Omega)$, and so assertion (iii) is verified. It remains to prove that $\widetilde{z}$ satisfies the integral identity (iv). Since z_n is the solution of (3.1), we have for all $\theta \in L^2(I;H_0^1(\Omega))$,

$$\left(\frac{\partial z_n}{\partial t},\theta\right)_{L^2(I;L^2(\Omega))} + d(\nabla z_n,\nabla\theta)_{L^2(I;L^2(\Omega))} = (f_n,\theta)_{L^2(I;L^2(\Omega))}, \tag{5.12}$$

which can be written

$$\begin{aligned}\left(\frac{\partial z_n}{\partial t} - \frac{\partial \widetilde{z}}{\partial t},\theta\right)_{L^2(I;L^2(\Omega))} &+ \left(\frac{\partial \widetilde{z}}{\partial t},\theta\right)_{L^2(I;L^2(\Omega))} + d(\nabla z_n - \nabla\widetilde{z},\nabla\theta)_{L^2(I;L^2(\Omega))} \\ &+ d(\nabla\widetilde{z},\nabla\theta)_{L^2(I;L^2(\Omega))} = (f_n - f,\theta)_{L^2(I;L^2(\Omega))} + (f,\theta)_{L^2(I;L^2(\Omega))}.\end{aligned} \tag{5.13}$$

If we show that

$$\begin{gathered} I_1 = \left(\frac{\partial z_n}{\partial t} - \frac{\partial \widetilde{z}}{\partial t}, \theta\right)_{L^2(I;L^2(\Omega))} + d(\nabla z_n - \nabla \widetilde{z}, \nabla \theta)_{L^2(I;L^2(\Omega))} \xrightarrow[n\to\infty]{} 0, \\ I_2 = (f_n - f, \theta)_{L^2(I;L^2(\Omega))} \xrightarrow[n\to\infty]{} 0, \end{gathered} \tag{5.14}$$

$\widetilde{z}$ will be a solution of (2.5) in the sense of Definition 2.1.

For I_1, we have

$$\begin{aligned} I_1 &= \left(\frac{\partial z_n}{\partial t} - \frac{\partial \widetilde{z}}{\partial t}, \nabla \theta\right)_{L^2(I;L^2(\Omega))} + (\nabla z_n - \nabla \widetilde{z}, \nabla \theta)_{L^2(I;L^2(\Omega))} \\ &\le \left\|\frac{\partial z_n}{\partial t} - \frac{\partial \widetilde{z}}{\partial t}\right\|_{L^2(I;L^2(\Omega))} \|\theta\|_{L^2(I;L^2(\Omega))} \\ &\quad + \|\nabla z_n - \nabla \widetilde{z}\|_{L^2(I;L^2(\Omega))} \|\nabla \theta\|_{L^2(I;L^2(\Omega))} \\ &\le \left(\left\|\frac{\partial z_n}{\partial t} - \frac{\partial \widetilde{z}}{\partial t}\right\|_{L^2(I;L^2(\Omega))} + \|z_n - \widetilde{z}\|_{L^2(I;H_0^1(\Omega))}\right) \|\theta\|_{L^2(I;H_0^1(\Omega))}. \end{aligned} \tag{5.15}$$

Thanks to (5.3) we obtain $\lim_{n\to\infty} I_1 = 0$. For the remaining term I_2, we use the Schwarz inequality and Assumption 2.2 to get

$$\begin{aligned} I_2 &= (f_n - f, \theta)_{L^2(I;L^2(\Omega))} \\ &\le \|f_n - f\|_{L^2(I;L^2(\Omega))} \|\theta\|_{L^2(I;L^2(\Omega))} \\ &= \|f_n(\cdot,\cdot,z_{n-1},\nabla z_{n-1}) - f(\cdot,\cdot,\widetilde{z},\nabla \widetilde{z})\|_{L^2(I;L^2(\Omega))} \|\theta\|_{L^2(I;L^2(\Omega))} \\ &\le L\left[\|z_{n-1} - \widetilde{z}\|_{L^2(I;L^2(\Omega))} + \|\nabla z_{n-1} - \nabla \widetilde{z}\|_{L^2(I;L^2(\Omega))}\right] \|\theta\|_{L^2(I;L^2(\Omega))} \\ &\le L\|z_{n-1} - \widetilde{z}\|_{L^2(I;H_0^1(\Omega))} \|\theta\|_{L^2(I;L^2(\Omega))}. \end{aligned} \tag{5.16}$$

Therefore by passing to the limit, we obtain $\lim_{n\to\infty} I_2 = 0$. This completes the proof of Theorem 5.2. □

6. Uniqueness and continuous dependence

THEOREM 6.1. *Suppose that assumptions of Theorem 5.2 are fulfilled. Let z_1 and z_2 be two weak solutions of (2.5) in $L^2(I;H_0^1(\Omega))$. Then*

$$\|z_1 - z_2\|_{L^2(I;H_0^1(\Omega))} \le Lc_1 \|z_1 - z_2\|_{L^2(I;H_0^1(\Omega))}, \tag{6.1}$$

where c_1 is defined by (4.8).

Proof. Let $y = z_1 - z_2$, it is clear that y satisfies $t \in I$,

$$\begin{aligned} \frac{\partial y}{\partial t} - d\Delta y = f(x,t,z_1,\nabla z_1) - f(x,t,z_2,\nabla z_2) \quad \text{in } Q_T, \\ y(x,0) = 0 \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \Sigma_T. \end{aligned} \tag{6.2}$$

Considering the weak formulation of problem (6.2) and performing a similar calculation to that for the establishment of estimate (4.7), we derive the desired result. □

As a consequence of Theorem 6.1, we obtain the following.

COROLLARY 6.2. *Under assumptions of Theorem 6.1, the solution of problem (2.5) is unique.*

Proof. The proof is obvious, it suffices to observe that by Theorem 6.1,

$$(1 - Lc_1)\|y\|_{L^2(I;H_0^1(\Omega))} \leq 0, \tag{6.3}$$

and by assumption (5.1) $1 - Lc_1 > 0$. Consequently, $y = 0$, which means $z_1 = z_2$, which achieves the proof. □

THEOREM 6.3. *Suppose that $u = u(x,t)$ and $u^* = u^*(x,t)$ are two solutions corresponding to (u_0, f) and (u_0^*, f^*). Moreover, assume that*

$$|f_1(\cdot,t,p_1,q_1) - f_1^*(\cdot,t,p_2,q_2)| \leq K(t) + L(||p_1 - p_2|| + ||q_1 - q_2||), \tag{6.4}$$

for some continuous nonnegative function $K(t)$ and certain constant L. Then

$$||u(\cdot,t) - u^*(\cdot,t)||^2 \leq \left(||u_0 - u_0^*||^2 + \int_0^t K^2(\tau)d\tau\right)e^{c_4 t}, \tag{6.5}$$

for all $t \in I$, where

$$c_4 = \max\left(\frac{2}{L^2}, 1\right)\exp\left(\left(\frac{3L^2}{2} + 2\right)\tau\right). \tag{6.6}$$

Proof. Considering the weak formulation of problem (2.2) written for u, subtracting from it the same integral identity written for u^* and putting $\theta = u - u^*$, and performing an integration by parts, we get

$$\begin{aligned} &\frac{\partial}{\partial t}||u(\cdot,t) - u^*(\cdot,t)||^2 + 2||\nabla u(\cdot,t) - \nabla u^*(\cdot,t)||^2 \\ &\quad = 2(f_1(\cdot,t,u(\cdot,t),\nabla u(\cdot,t)) - f_1^*(\cdot,t,u^*(\cdot,t),\nabla u^*(\cdot,t)),u(\cdot,t) - u^*(\cdot,t)). \end{aligned} \tag{6.7}$$

Integrating (6.7) over $(0,t)$, applying the Cauchy inequality, and using assumption (6.4), we obtain

$$\begin{aligned}
&\|u(\cdot,t)-u^*(\cdot,t)\|^2+2\int_0^t\|\nabla u(\cdot,\tau)-\nabla u^*(\cdot,\tau)\|^2\\
&\quad\le\|u_0-u_0^*\|^2+3\varepsilon\int_0^t K^2(\tau)d\tau\\
&\qquad+\left(\frac{1}{\varepsilon}+3\varepsilon L^2\right)\int_0^t\|u(\cdot,\tau)-u^*(\cdot,\tau)\|^2d\tau\\
&\qquad+3\varepsilon L^2\int_0^t\|\nabla u(\cdot,\tau)-\nabla u^*(\cdot,\tau)\|^2d\tau.
\end{aligned}\tag{6.8}$$

Choosing $\varepsilon=2/3L^2$ and applying Gronwall's lemma to the obtained inequality, we get estimate (6.5), which completes the proof. □

7. Generalizations

(i) The above results can be extended for more general equations of the type of (1.5), (1.6) with higher-order derivatives and coefficients depending on x and t, namely, for the following problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\sum_{|i|,|j|\le m}(-1)^{|i|}D^i\left(a_{ij}^{(1)}(x)D^ju\right)\\
&\qquad=f_1(x,t,u,v,\nabla u,\nabla v,\dots,\nabla^{m-1}u,\nabla^{m-1}v)+F(x)\quad\text{in }Q_T,\\
&\frac{\partial v}{\partial t}-\sum_{|i|,|j|\le m}(-1)^{|i|}D^i\left(a_{ij}^{(2)}(x)D^jv\right)\\
&\qquad=f_2(x,t,u,v,\nabla u,\nabla v,\dots,\nabla^{m-1}u,\nabla^{m-1}v)+G(x)\quad\text{in }Q_T,
\end{aligned}\tag{7.1}$$

since the compactness $H_0^m(\Omega)\hookrightarrow H^{m-1}(\Omega)$ can be used, where $a_{ij}^{(k)}(x)$ $(k=1,2)$ are such that $\sum_{|i|,|j|\le m}a_{ij}^{(k)}\xi_i\xi_j\ge C\sum_{|i|=m}\xi_i^2$, $a_{ij}^{(k)}=a_{ji}^{(k)}$ for all couples $|i|\ge 1,|j|\ge 1$, $a_{ij}^{(k)}$ a Lipschitz continuous in t on $\overline{Q}$ $(|i|,|j|\le m)$. Here, homogeneous Dirichlet boundary conditions

$$\frac{\partial^s u}{\partial\vartheta^s}=\frac{\partial^s v}{\partial\vartheta^s}=0\quad\text{on }\partial\Omega,\tag{7.2}$$

for $s=0,1,\dots,m-1$, are considered instead of (1.1_4).

(ii) Our results make it possible to study the boundary value problem for a quasilinear pluriparabolic system, that is a system with the form

$$
\begin{gathered}
\sum_{i=1}^{n} \frac{\partial u}{\partial t_i} - d_1 \Delta u = f_1(x,t,u,v,\nabla u,\nabla v) + F(x) \quad \text{in } Q_T, \\
\sum_{i=1}^{n} \frac{\partial v}{\partial t_i} - d_2 \Delta v = f_2(x,t,u,v,\nabla u,\nabla v) + G(x) \quad \text{in } Q_T, \\
u(x,t^{i,0}) = \Phi_i(x,t^i), \quad v(x,t^{i,0}) = \Psi_i(x,t^i) \quad \text{for } (x,t) \in Q_0^i, \\
u = v = 0, \quad \text{on } \Sigma_T,
\end{gathered}
\tag{7.3}
$$

or more generally for a quasilinear pluriparabolic system with nonlocal initial conditions

$$
\begin{gathered}
\sum_{i=1}^{n} \frac{\partial u}{\partial t_i} - d_1 \operatorname{sing} \prod_{i=1}^{n} (1 - \|\lambda_i\|^2) \Delta u = f_1(x,t,u,v,\nabla u,\nabla v) + F(x) \quad \text{in } Q_T, \\
\sum_{i=1}^{n} \frac{\partial v}{\partial t_i} - d_2 \operatorname{sing} \prod_{i=1}^{n} (1 - \|\lambda_i\|^2) \Delta v = f_2(x,t,u,v,\nabla u,\nabla v) + G(x) \quad \text{in } Q_T, \\
(\ell_i u)(x,t) := u(x,t^{i,0}) - \lambda_i u(x,t^{i,T}) = \Phi_i(x,t^i) \quad \text{in } Q_0^i, \\
(\ell_i v)(x,t) := v(x,t^{i,0}) - \lambda_i v(x,t^{i,T}) = \Psi_i(x,t^i) \quad \text{in } Q_0^i, \\
u = v = 0, \quad \text{on } \Sigma_T,
\end{gathered}
\tag{7.4}
$$

where

$$
\begin{gathered}
t = (t_1,\dots,t_n), \qquad t^i = (t_1,\dots,t_{i-1},t_{i+1},\dots,t_n), \\
t^{i,0} = (t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_n), \\
t^{i,T} = (t_1,\dots,t_{i-1},T_i,t_{i+1},\dots,t_n), \\
(i = 1,\dots,n) \quad I_i = (0,T_i), \quad T_i < \infty \ (i = 1,\dots,n),\ I = \prod_{i=1}^{n} I_i, \\
Q_0^i = I_1 \times \cdots \times I_{i-1} \times \{0\} \times I_{i+1} \times \cdots \times I_n \quad (i = 1,\dots,n).
\end{gathered}
\tag{7.5}
$$

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