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Research Article

The Existence of Spanning Ended System on Claw-Free Graphs

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We prove that every connected claw-free graph G contains a spanning k-ended system if and only if $\mathrm{cl}(G)$ contains a spanning k-ended system, where $\mathrm{cl}(G)$ denotes Ryjáček closure of G.

1. Introduction

Graph theory focuses on graphs composed of vertices and edges. The vertices in a graph are considered as discrete points usually discussed in control problems. Then there are a lot of results using graph theory to solve control and other application problems [1–13].

We consider only finite and simple graphs. For notation and terminology not defined here we refer to [14]. For a subgraph H of a graph G, $G-H$ denotes the induced subgraph by $V(G)-V(H)$, and $G[S]$ denotes the induced subgraph by S for $S \subseteq V(G)$. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$. $P[a,b]$ denotes a path with end vertices a, b and a positive orientation from a to b. For a path $P[a,b]$, $x, y \in V(P)$, let xPy denote the subpath from x to y with positive orientation and $yP^{-}x$ denote the subpath from y to x with negative orientation. Similarly, for a cycle C with a given direction, we can define $C[a,b]$, $C^{-}[a,b]$ with $a, b \in V(C)$. In the paper, we define clockwise as the positive direction of a cycle. We use K_m to denote a complete graph with order m, and if $m = 1$, then it is trivial vertex. A tree with at most k leaves is called *k-ended tree*.

A graph is called *claw-free* if it does not contain $K_{1,3}$ induced subgraph. For a vertex $x \in V(G)$, let G_x denote the graph with $V(G_x) = V(G)$ and $E(G_x) = E(G) \cup \{uv : u, v \in N(x)\}$, and then G_x is called *the local completion* of G at x. For a graph G, $x \in V(G)$, if $G[N(x)]$ is connected, then x is *locally connected*; if $G[N(x)]$ is a complete induced subgraph of G, then x is *simplicial*; if x is locally connected, but not simplicial, then x is *eligible*.

Ryjáček [15] proposed a closure operation on a claw-free graph G by joining all nonadjacent pairs of vertices in the neighbourhood of every eligible vertex till there is no eligible vertex, and then we get the closure $\mathrm{cl}(G)$. Ryjáček also gave the following result, which is considered a useful tool to research on the Hamiltonian properties of claw-free graphs.

Theorem 1 (Ryjáček [15]). *If G is a connected claw-free graph, then $\mathrm{cl}(G)$ is Hamiltonian if and only if G is Hamiltonian.*

Actually, there are a lot of results which present that $\mathrm{cl}(G)$ and G have many common properties.

Theorem 2 (Brandt et al. [13]). *A claw-free graph G is traceable if and only if $\mathrm{cl}(G)$ is traceable.*

Theorem 3 (Ryjáček et al. [16]). *Let G be a claw-free graph. If $\mathrm{cl}(G)$ contains 2 factors with k components, then G contains 2 factors with at most k components.*

A tree with at most k leaves is called *k-ended tree*. Win [17] provided sufficient conditions for a graph to contain spanning k-ended trees by spanning k-ended system. A system of a graph which contains paths, cycles, and trivial vertices is defined by a function $f(\alpha)$ as follows:

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is } K_1, K_2, \text{or a cycle}, \\ 2 & \text{if } \alpha \text{ is a path of order at least 3}. \end{cases} \quad (1)$$

A system $\mathcal{S}$ is called *k-ended system* if $\sum_{\alpha \in \mathcal{S}} f(\alpha) \leq k$. Moreover, if $V(\mathcal{S}) = V(G)$, then $\mathcal{S}$ is called a spanning

k-ended system of G. Obviously, if G contains a spanning k-ended system, then G contains a spanning k-ended tree. It follows that if a graph contains no spanning k-ended tree, then it contains no spanning k-ended systems.

In this paper, we prove that $\mathrm{cl}(G)$ can preserve the existence of spanning k-ended system of G.

Theorem 4. *A claw-free graph G contains a spanning k-ended system if and only if $\mathrm{cl}(G)$ contains a spanning k-ended system.*

2. Proof of Theorem 4

We divide a k-ended system $\mathcal{S}$ of a graph G into two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and let

$$\mathcal{S}_1 = \{\alpha \in \mathcal{S} : f(\alpha) = 1\}, \quad \mathcal{S}_2 = \{\alpha \in \mathcal{S} : f(\alpha) = 2\}. \tag{2}$$

For every component $C \in \mathcal{S}_1$, we take one vertex $x_C \in V(C)$. For every path $P \in \mathcal{S}_2$, let x_P and y_P denote the two end vertices of P. We define

$$\mathrm{End}(\mathcal{S}_1) = \bigcup_{C\in\mathcal{S}_1} \{x_C\}, \quad \mathrm{End}(\mathcal{S}_2) = \bigcup_{P\in\mathcal{S}_2} \{x_P, y_P\}, \quad \mathrm{End}(\mathcal{S}) = \mathrm{End}(\mathcal{S}_1) \cup \mathrm{End}(\mathcal{S}_2). \tag{3}$$

For a spanning t-ended system $\mathcal{S}$ of a graph G, if there is no spanning s-ended system with $s < t$, then we call the system *minimum spanning t-ended system*. Obviously, for a minimum spanning t-ended system $\mathcal{S}$ of G, $\mathrm{End}(\mathcal{S})$ is an independent set of G with $|\mathrm{End}(\mathcal{S})| = t$.

In order to prove Theorem 4, we only need to prove that the following result holds.

Theorem 5. *Let G be a claw-free graph with $\delta(G) \geq 2$. Then G_x contains a spanning k-ended system for any vertex $x \in V(G)$ if and only if G contains a spanning k-ended system.*

Proof. Obviously, the sufficiency holds and we only need to prove the necessity. Assume G_x contains a spanning k-ended system $\mathcal{S}$ which satisfies the following properties.

(T1) $\mathcal{S}$ is a minimum spanning k-ended system of G_x.

(T2) $E(G_x) - E(G)$ is minimum, subject to (T1).

(T3) $|\mathcal{S}_2|$ is minimum, subject to (T1) and (T2).

(T4) If P contains x, then P contains as many vertices in $N(x)$ as possible, where $P \in \mathcal{S}_2$ subject to (T1), (T2), and (T3).

(T5) If C contains x, then C contains as many vertices in $N(x)$ as possible, where $C \in \mathcal{S}_1$ subject to (T1), (T2), and (T3).

If $E(\mathcal{S}) - E(G) = \emptyset$, then $\mathcal{S}$ is a spanning k-ended system in G, and we are done. Thus we assume $|E(\mathcal{S}) - E(G)| \geq 1$. □

Claim 1. For $P \in \mathcal{S}_2$, if $E(P) - E(G) \neq \emptyset$, then $x \in V(P)$.

Proof. To the contrary, suppose $x \notin V(P)$, $P = aPb$, and $uv \in E(P) - E(G)$ with $v = u^+$. Then $u, v \in N(x)$. Suppose $x \in cP_1d$, $P_1 \in \mathcal{S}_2$. Since $G[x, u, v, x^+(x^-)] \neq K_{1,3}$, $ux^+(ux^-) \in E(G)$ or $vx^+(vx^-) \in E(G)$. Without loss of generality, suppose $x \neq d$ and $ux^+ \in E(G)$. Then G_x contains two paths $P_2 = cP_1xvPb$ and $P_3 = aPux^+P_1d$. Replacing P and P_1 by P_2 and P_3, then G_x contains a spanning k-ended system with less edge than $E(G_x) - E(G)$, a contradiction to (T2).

Suppose $x \in V(C)$, $C \in \mathcal{S}_1$. If $C = \{u\}$, then G_x contains a path $P' = aPuxvPb$ with $V(P') = V(C) \cup V(P)$. Replacing P and C by P', G_x contains a spanning $(k-1)$-ended system, a contradiction to (T1). Thus $|V(C)| \geq 2$. Since $G[x, u, v, x^+] \neq K_{1,3}, ux^+ \in E(G)$ or $vx^+ \in E(G)$. Without loss of generality, suppose $ux^+ \in E(G)$. Then G_x contains a path $P' = aPuC[x^+, x]vPb$ with $V(P') = V(P) \cup V(C)$. Replacing P and C by P', then G_x contains a spanning $(k-1)$-ended system, a contradiction to (T1). □

Claim 2. For $C \in \mathcal{S}_1$, if $E(C) - E(G) \neq \emptyset$, then $x \in V(C)$.

Proof. Since $E(C) - E(G) \neq \emptyset$, $|V(C)| \geq 2$. Suppose $uv \in E(C) - E(G)$, $v = u^+$. Assume to the contrary $x \notin V(C)$. Suppose $x \in aPb$, $P \in \mathcal{S}_2$. Since $G[x, x^+(x^-), u, v] \neq K_{1,3}$, $x^+u(x^-u) \in E(G)$ or $x^+v(x^-v) \in E(G)$. Without loss of generality, assume $x \neq b$ and $x^+u \in E(G)$. Then G_x contains a path $P' = bP^-x^+C^-[u, v]xP^-a$ with $V(P') = V(P) \cup V(C)$. Replacing P, C by P', then G_x contains a spanning $(k-1)$-ended system, a contradiction.

Suppose $x \in V(C')$, $C' \in \mathcal{S}_1 - \{C\}$. If $V(C') = \{x\}$, then G_x contains a cycle $C'' = C[v, u]xv$ with $V(C'') = V(C) \cup V(C')$. Replacing C, C' by C'', then G_x contains a spanning $(k-1)$-ended system, a contradiction. If $|V(C')| \geq 2$, then by the preceding proof $x^+u \in E(G)$ or $x^+v \in E(G)$. Without loss of generality, assume $x^+u \in E(G)$. Then G_x contains a cycle $C'' = xC[v, u]C'[x^+, x]$ with $V(C'') = V(C) \cup V(C')$. Replacing C, C' by C'', then G_x contains a spanning $(k-1)$-ended system, a contradiction.

Since $\mathcal{S}$ is a disjoint system, we can get the following two results by Claims 1 and 2. □

Claim 3. For $P \in \mathcal{S}_2$, if $E(P) - E(G) \neq \emptyset$, then $E(\mathcal{S}) - E(G) \subseteq E(P)$.

Claim 4. For $C \in \mathcal{S}_1$, if $E(C) - E(G) \neq \emptyset$, then $E(\mathcal{S}) - E(G) \subseteq E(C)$.

Now we prove the case that $E(P) - E(G) \neq \emptyset$, for $P \in \mathcal{S}_2$. Then, by Claim 3, $E(\mathcal{S}) - E(G) \subseteq E(P)$. Suppose $P = aPb$, and then we can get the following results.

Claim 5. Consider the following: $|E(P) - E(G)| \leq 2$.

Proof. Suppose, to the contrary, $u_1v_1, u_2v_2, u_3v_3 \in E(P) - E(G)$, where u_1, v_1, u_2, v_2, u_3, and v_3 are labeled in order along the positive orientation of P. Since $G[x, u_1, v_1, v_2] \neq K_{1,3}$, $u_1v_2 \in E(G)$ or $v_1v_2 \in E(G)$. Without loss of generality, assume $u_1v_2 \in E(G)$. Then G_x contains a path $P' = aPu_1v_2Pu_3v_1Pu_2v_3Pb$ with $V(P') = V(P)$ and $|E(P') - E(G)| < |E(P) - E(G)|$, a contradiction to (T2). □

Claim 6. Consider the following: $|E(P) - E(G)| = 1$.

Proof. Suppose, to the contrary, $|E(P) - E(G)| = 2$ by Claim 5 and $u_1v_1, u_2v_2 \in E(P) - E(G)$, where u_1, v_1, u_2, and v_2 are labeled in order along the positive orientation of P. By the proof of Claim 5, $u_1v_2, v_1u_2 \in E(G)$, $v_1v_2 \notin E(G)$. By Claim 1, $x \in V(P)$. Assume $x \in P[a, u_1]$. Then $x^+v_1 \notin E(G)$; otherwise G_x contains a path $P' = aPxu_2P^-v_1x^+Pu_1v_2Pb$ with $V(P) = V(P')$ and $|E(P') - E(G)| = 1$, a contradiction to (T2) by Claim 3. If $x^+v_2 \in E(G)$, then G_x contains a path $P' = aPxu_2P^-v_1u_1P^-x^+v_2Pb$ with $V(P) = V(P')$ and $|E(P') - E(G)| < |E(P) - E(G)|$, a contradiction to (T2) by Claim 3. Thus $G[x, x^+, v_1, v_2] = K_{1,3}$, a contradiction. By similar proof, we can prove that Claim 6 holds if $x \in P[v_1, u_2] \cup P[v_2, b]$.

By Claim 6, we assume that $E(P) - E(G) = \{u_1v_1\}$, where u_1, v_1 are labeled in order along the positive orientation of P, and without loss of generality assume $x \in P[a, u_1]$. Since x is eligible, there exists at least one path in $N(x)$ connecting u_1 and v_1. Suppose P_0 is the shortest path in $N(x)$ connecting u_1 and v_1. Since G is claw-free, $3 \le |V(P_0)| \le 4$. Assume $y \in V(P_0)$, $u_1y \in E(G)$. □

Claim 7. Consider the following: $y \in V(P)$.

Proof. To the contrary, suppose $y \in V(P')$, where $P' = P'[c, d] \in \mathcal{S}_2 - \{P\}$. If $y = c$, then G_x contains a path $P_1 = aPu_1yP'd$ and a path $P_2 = v_1Pb$. Replacing P and P' by P_1 and P_2, then G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). Similarly, $y \ne d$. Thus $y \notin \{c, d\}$. If $y^-v_1 \in E(G)$, then G_x contains a path $P_1 = aPu_1yP'd$ and a path $P_2 = cP'y^-v_1Pb$. Replacing P and P' by P_1 and P_2, G_x contains a spanning k-ended system $\mathcal{S}'$ with $E(G_x) - E(G) = \emptyset$, a contradiction to (T2). Thus $y^-v_1 \notin E(G)$. Similarly, $v_1y^+ \notin E(G)$. If $y^-y^+ \in E(G)$, then G_x contains a path $P_1 = aPu_1yv_1Pb$ and a path $P_2 = cP'y^-y^+P'd$. Replacing P and P' by P_1 and P_2, G_x contains a spanning k-ended system $\mathcal{S}'$ with $E(G_x) - E(G) \subseteq E(P_1)$ such that P_1 contains more vertices in $N(x)$ than P, a contradiction to (T4). Thus $y^-y^+ \notin E(G)$. $yv_1 \notin E(G)$; otherwise $G[y, v_1, y^-, y^+] = K_{1,3}$, a contradiction. $x^+v_1 \notin E(G)$; otherwise G_x contains a path $P_1 = aPxu_1P^-x^+v_1Pb$ with $V(P) = V(P_1)$ and $E(P_1) - E(G) = \emptyset$, a contradiction to (T2). Then $x^+y \in E(G)$ by $G[x, x^+, y, v_1] \ne K_{1,3}$. $y^-x^+ \in E(G)$ or $y^+x^+ \in E(G)$ by $G[y, y^-, y^+, x^+] \ne K_{1,3}$ and $y^-y^+ \notin E(G)$. If $x^+y^- \in E(G)$, then G_x contains two paths $P_1 = aPxv_1Pb$ and $P_2 = cP'y^-x^+Pu_1yP'd$. Replacing P and P' by P_1 and P_2, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). If $x^+y^+ \in E(G)$, G_x contains two paths $P_1 = aPxv_1Pb$ and $P_2 = cP'yu_1P^-x^+y^+P'd$. Replacing P and P' by P_1 and P_2, then G_x contains a spanning $(k-1)$-ended system, a contradiction. Using a similar proof, we can get a contradiction if $y_1 \in V(C)$ with $C \in \mathcal{S}_1$. Thus $y \in V(P)$.

By Claim 1 without loss of generality, in the following proof, assume $x \in P[a, u_1)$. Then $x^+u_1, x^-v_1 \in E(G)$, $x^-x^+, x^-u_1, x^+v_1 \notin E(G)$ since G is claw-free and by (T2). □

Claim 8. Consider the following: $v_1y \notin E(G)$.

Proof. To the contrary, suppose $v_1y \in E(G)$. By Claim 7, without loss of generality, assume $y \in P[a, u_1]$. If $y = a$, then, replacing P by $P_1 = u_1P^-yv_1Pb$, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). Thus $y \ne a$. If $y^-u_1 \in E(G)$, then, replacing P by $P_1 = aPy^-u_1P^-yv_1Pb$, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). Similarly, $y^-y^+ \notin E(G)$. By $G[y, y^-, y^+, u_1] \ne K_{1,3}$, $y^+u_1 \in E(G)$. By $G[y, y^-, y^+, x] \ne K_{1,3}$, $y^-x \in E(G)$ or $y^+x \in E(G)$. If $y^-x \in E(G)$, then, replacing P by $P_1 = aPxy^-P^-x^+u_1P^-yv_1Pb$, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). Thus $y^+x \in E(G)$. Replacing P by path $P_1 = aPxy^+Pux^+PyvPb$, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). □

Claim 9. P can be transformed to a path P_1 such that $V(P_1) = V(P)$, $v_1y \in E(P_1)$, and $E(G_x) - E(G) = \{v_1y\}$.

Proof. By Claim 7, without loss of generality, assume $y \in P(x, u_1)$. Since $G[y, y^-, y^+, x] \ne K_{1,3}$, $y^-y^+, y^-x, y^+x \in E(G)$. If $y^-y^+ \in E(G)$, then $P_1 = aPy^-y^+Pu_1yv_1Pb$. If $y^-x \in E(G)$, then $P_1 = aPxy^-P^-x^+u_1P^-yv_1Pb$. If $y^+x \in E(G)$, then $P_1 = aPxy^+Pu_1x^+Pyv_1Pb$.

By Claim 8, $v_1y \notin E(G)$, and then $|V(P_0)| = 4$. Suppose $P_0 = u_1yzv_1$. By Claim 9, replace P by P_1. By the proof of Claim 7, $z \in P_1$. By the proof of Claim 9, G_x contains a spanning k-ended system with no edge in $E(G_x) - E(G)$, a contradiction to (T2). It follows that Theorem 4 holds and then Theorem 5 holds. □

Competing Interests

The authors declare that they have no competing interests.

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