Internat. J. Math. & Math. Sci.
Vol. 7 No. 2 (1984) 339-350

CONTACT CO-ISOTROPIC CR SUBMANIFOLDS OF A PSEUDO-SASAKIAN MANIFOLD

VLADISLAV V. GOLDBERG AND RADU ROSCA

Department of Mathematics
N.J. Institute of Technology
323 M.L. King Jr. Boulevard
Newark, N.J. 07102 U.S.A.

(Received April 5, 1984 and in revised form May 31, 1984)

ABSTRACT. It is proved that any co-isotropic submanifold M of a pseudo-Sasakian manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ is a CR submanifold (such submanifolds are called CICR submanifolds) with involutive vertical distribution $D^{\perp}$. The leaves $M^{\perp}$ of $D^{\perp}$ are isotropic and M is $D^{\perp}$-totally geodesic. If M is foliate, then M is almost minimal. If M is Ricci $D^{\perp}$-exterior recurrent, then M receives two contact Lagrangian foliations. The necessary and sufficient conditions for M to be totally minimal is that M be contact $D^{\perp}$-exterior recurrent.

KEY WORDS AND PHRASES. *CR submanifold, CICR submanifold, pseudo-Sasakian manifold, para f-structure, transversal quadratic vectorial form, mixed isotropic manifold, index of relative nullity, contact Lagrangian distribution, almost mean curvature vector field, Ricci $D^{\perp}$-exterior recurrent submanifold, totally minimal submanifold, contact $D^{\perp}$-exterior recurrent submanifold.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 53C25, 53C40, 53B25.

1. INTRODUCTION.

Many papers have been recently concerned with *Sasakian* manifolds $\tilde{M}(\phi,\xi,\tilde{\eta},\tilde{g})$ and contact CR submanifolds of M (see for example Yano and Kon [1]; Kobayashi [2]. *Pseudo-Sasakian* manifolds $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ were developed by Rosca [3].

The purpose of the present paper is to study *co-isotropic* submanifolds M of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ (since $\tilde{g}$ is pseudo-Riemannian, M are real). If $T_p(M)$ and $T_p^{\perp}(M)$ are the tangent and normal spaces of M at a point $p \in M$, M is co-isotropic if and only if $T_p^{\perp}(M) \subset T_p(M)$. It is proved that any co-istropoic submanifold M is a contact CR submanifold and such kind of CR submanifolds is called *CICR submanifolds*. If M is a horizontal CICR submanifold, then the canonical vector field ξ belongs to the *horizontal* distribution

$D: p \to D_p \subset T_p(M)$ (see Kobayashi [2]), and the *vertical* distribution $D^\perp_p(D^\perp: p \to D^\perp_p \subset T_p(M))$ coincides with $T^\perp_p(M)$.

The following basic properties are proved: $D^\perp$ is always *involutive* (as in the case of a proper immersion), the leaves $M^\perp$ of $D^\perp$ are isotropic, and M is both $D^\perp$-*totally geodesic* and *mixed totally geodesic* (Bejancu [4]).

In addition, the *almost mean curvature vector* $\Gamma^\perp$ (which is defined) of $M^\perp$ is a geodesic section on $M^\perp$ and $M^\perp$ is of constant almost mean curvature.

In Section 3 we study *foliate* (Kobayashi [2]) CICR submanifolds. For this purpose we define a *transversal quadratic vectorial form* II_t associated with $x: M \to \tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$. The following results are proved:

(i) the necessary and sufficient conditions for M to be foliate is that $II_t(X,UY) = II_t(UX,Y)$ for any $X,Y \in D$;

(ii) any foliate CICR submanifold is *almost minimal*.

There is a class of foliate CICR submanifolds for which the simple unit form which corresponds to $D^\perp$ is exterior recurrent (Datta [5]). Such submanifolds are said to be *Ricci $D^\perp$-exterior recurrent* and, if the recurrence 1-form is conformal to η, then M is said to be *contact $D^\perp$-exterior recurrent*. The following result is proved: the necessary and sufficient condition for a CICR submanifold M to be *minimal* is that M be contact $D^\perp$-exterior recurrent.

Finally in Section 4 we discuss the case when M is a contact CICR submanifold of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ which is ξ-*vertical* (Kobayashi [2]). In this case the leaves $M^\perp$ of $D^\perp$ are *mixed isotropic* (Rosca [6]; Goldberg and Rosca [7]) submanifolds and such submanifolds M can not be foliate.

2. PRELIMINAIRES.

Let M be a $(2m+1)$-dimensional pseudo-Riemannian manifold of index $m+1$, i.e. of signature $(m+1,m)$. At each point $\tilde{p} \in \tilde{M}$ one has the standard decomposition (see Rosca [3]; Libermann [9]):

$$T_{\tilde{p}}(\tilde{M}) = H_{\tilde{p}}(\tilde{M}) \oplus \mathcal{T}_{\tilde{p}}(\tilde{M}) \qquad (2.1)$$

where $T_{\tilde{p}}$, $H_{\tilde{p}}$, and $\mathcal{T}_{\tilde{p}}$ are the tangent space at p, a $(2m)$-dimensional *neutral* vector space, and a *time-like* line orthogonal to $H_{\tilde{p}}$ respectively.

Let $S_{\tilde{p}}, S^*_{\tilde{p}} \subset H_{\tilde{p}}$ be two *self-orthogonal* subspaces (both of dimension m) which define an involutive automorphism U of square $+1$ (U is the para-complex operator defined by Sinha [10]). Let $\xi \in \mathcal{T}_{\tilde{p}}$ and $\eta \in \Lambda^1(\tilde{M})$ be the pairng which defines a contact structure σ_c on $\tilde{M}$ and $\tilde{\nabla}$ be the covariant differentiation operator defined by the metric tensor $\tilde{g}$. Then if for any vector fields $\tilde{Z}$, $\tilde{Z}'$ on $\tilde{M}$ the structure tensors $(U,\xi,\tilde{\eta},\tilde{g})$ satisfy

$$\begin{aligned} &U^2(\tilde{Z}) = \tilde{Z}-\tilde{\eta}(\tilde{Z})\xi, && \tilde{g}(U\tilde{Z},U\tilde{Z}') = -\tilde{g}(\tilde{Z},\tilde{Z}')+\tilde{\eta}(\tilde{Z})\tilde{\eta}(\tilde{Z}'), \\ &\tilde{g}(\tilde{Z},\xi) = \tilde{\eta}(\tilde{Z}), && \tilde{\nabla}_{\tilde{Z}}\xi = U\tilde{Z}, \\ &d\tilde{\eta}(\tilde{Z},\tilde{Z}') = -2\tilde{g}(U\tilde{Z},\tilde{Z}'), && \tilde{\eta}(\xi) = 1, \end{aligned} \qquad (2.2)$$

the manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ has been called by Rosca in [3] *pseudo-Sasakian manifold*.

Since the (1,1)-tensor field U satisfies $U^3-U = 0$, one may say that any pseudo-Sasakian manifold is a *para-f-manifold* (Goldberg and Rosca [11]; Vranceanu and Rosca [12]) (U defines a para f-structure).

In order to study improper immersions in $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$, we consider on $\tilde{M}$ adapted Witt frames (Morvan and Rosca [8]) $W = \{h_A: A,B = 0,1,\dots,2m\}$ where $\{h_a: a = 1,\dots,m\} = S_{\tilde{p}}$ and $\{h_{a^*}: a^* = a+m\} = S^*_p$ are null vector fields and $h_0 = \xi$ is the *anisotropic* vector field of W.

As is known (Libermann [9]), one has

$$\begin{aligned} \tilde{g}(h_a,h^*_b) &= \delta_{ab}, & \tilde{g}(\xi,h_a) &= 0 \\ \tilde{g}(\xi,h^*_a) &= 0, & \tilde{g}(\xi,\xi) &= 1 \end{aligned} \tag{2.3}$$

and

$$Uh_a = h_a,\ Uh_{a^*} = -h_{a^*},\ U\xi = 0. \tag{2.4}$$

If $\{\tilde{\omega}^A\}$ is the dual basis of W, we set $\tilde{\omega}^0 = \tilde{\eta}$ and the line element $d\tilde{p}$ ($d\tilde{p}$ is a canonical vector 1-form on $\tilde{M}$) is expressed by

$$d\tilde{p} = \tilde{\omega}^A \otimes h_A . \tag{2.5}$$

It follows from (2.3) that the metric tensor $\tilde{g}$ is expressed by

$$\tilde{g} = 2\sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*} + \tilde{\eta}\otimes\tilde{\eta} \tag{2.6}$$

If $\tilde{\theta}^A_B = \tilde{\gamma}^A_{BC}\tilde{\omega}^C$ ($\tilde{\gamma}^A_{BC} \in C^\infty(M)$) and $\tilde{\Theta}^A_B$ are the connection forms and curvature 2-forms on the bundle $W(\tilde{M})$ respectively, then the structure equations (E. Cartan) may be written in indexless form as

$$\tilde{\nabla}h = \tilde{\theta}\otimes h, \tag{2.7}$$

$$d\tilde{\omega} = -\tilde{\theta}\wedge\tilde{\omega}, \tag{2.8}$$

$$d\tilde{\theta} = -\tilde{\theta}\wedge\tilde{\theta} + \tilde{\Theta} . \tag{2.9}$$

Referring to (2.3) and (2.7), one finds

$$\begin{aligned} &\tilde{\theta}^a_b + \tilde{\theta}^{b^*}_{a^*} = 0, && \tilde{\theta}^{a^*}_b = 0, && \tilde{\theta}^a_{b^*} = 0, \\ &\tilde{\theta}^0_a + \tilde{\theta}^{a^*}_0 = 0, && \tilde{\theta}^a_0 + \tilde{\theta}^0_{a^*} = 0 \end{aligned} \tag{2.10}$$

and

$$\tilde{\theta}^0_a = \tilde{\omega}^{a^*}, \qquad \tilde{\theta}^0_{a^*} = -\tilde{\omega}^a . \tag{2.11}$$

The 1-form

$$\gamma = \sum_a \tilde{\theta}^a_a \tag{2.12}$$

is called the *Ricci* 1-form (Rosca [13]). By virtue of (2.7), (2.8), and (2.11) one has

$$d\tilde{\eta} = 2\sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*}, \tag{2.13}$$

$$\tilde{\nabla}\xi = Ud\tilde{p} \Rightarrow \langle\tilde{\nabla}_{\tilde{X}}\xi,\tilde{Y}\rangle + \langle\tilde{\nabla}_{\tilde{Y}}\xi,\tilde{X}\rangle = 0 \tag{2.14}$$

where $\tilde{X}$, $\tilde{Y}$ are any vector fields on $\tilde{M}$ ((2.14) proves in intrinsic manner that ξ is a Killing vector field as in the case of Sasakian manifolds).

Further we recall (see Yano and Kon [1]) that a submanifold M of $\tilde{M}$ is called a *contact CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $D: p \to D_p \subset T_p(M)$ (one denotes the induced elements on M by supressing $\sim$) satisfying:

(i) D is *paraholomorphic* i.e. $UD_p^{\perp} \subset D_p$ for each $p \in M$, and

(ii) the complementary orthogonal distribution $D^{\perp}: p \to D_p^{\perp} \subset T_p(M)$ is *anti-invariant* i.e. $UD_p^{\perp} \subset T_p^{\perp}(M)$ for each $p \in M$ ($T_p^{\perp}(M)$ is the normal space to M at p).

The distribution D (respectively $D^{\perp}$) is called the *horizontal* (respectively *vertical*) distribution.

Further, according to Kobayashi [2], we say that M is a *contact ξ-horizontal* (respectively *ξ-vertical*) CR submanifold if $\xi \in D_p$ (respectively $\xi \in D_p^{\perp}$) for each $p \in M$.

If the immersion $x: M \to \tilde{M}$ is improper and d is the *defect* of M($d = \dim M$ - rank of the mapping x), then according to Rosca [6] and Goldberg and Rosca [7], M is *mixed isotropic* if one has

$$T_p(M) \cap T_p^{\perp}(M) \neq 0,\ T_p(M) \not\subset T_p^{\perp}(M),\ T_p^{\perp}(M) \not\subset T_p(M) \iff d \neq 0,\ d \neq \dim M,\ d \neq \operatorname{codim} M. \tag{2.15}$$

3. CICR SUBMANIFOLDS.

Let $x: M \to \tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ be the improper immersion of a co-isotropic submanifold M in $\tilde{M}$. Then by definition one has $T_p^{\perp}(M) \subset T_p(M)$ and without loss of generality one may suppose $T_p^{\perp}(M) \subset S_p$. We agree to call S_p the *normal self-orthogonal* (abr. n.s.o.) space associated with x and assume that $\dim T_p^{\perp}(M) = \ell$ ($\ell < m$).

Consider now the two complementary differentiable distributions

$$D: p \to D_p = T_p(M) \setminus T_p^{\perp}(M);\ D^{\perp}: p \to D_p^{\perp} = T_p^{\perp}(M) \subset T_p(M).$$

Referring to (2.4), one has

$$UD_p \subset D_p,\ UD_p^{\perp} = T_p^{\perp}(M). \tag{3.1}$$

Therefore, one may say that any co-isotropic submanifold M of a pseudo-Sasakian manifold $\tilde{M}$ is a contact ξ-horizontal CR submanifold.

A CR submanifold which is co-isotropic will be called in the following a *CICR submanifold.*

Suppose that the manifold M under consideration is defined by

$$\omega^{r^*} = 0;\ r^*, s^* = 2m+1-\ell, \ldots, 2m\ . \tag{3.2}$$

Then one has $D_p = \{h_i, h_{i^*}, \xi\}$ and $D_p^{\perp} = \{h_r; r = m+1-\ell, \ldots, m\}$. Further, according to Rosca [13], we agree to call $D_p^{*\perp} = C_{S^*}T_p(M) \cap S_p^*$ the *transversal* vector space associated with the co-isotropic immersion x. Hence one may write $T_p\tilde{}(\tilde{M})|_M = D_p \oplus D_p^{\perp} \oplus D_p^{*\perp}$. On the other hand, referring to (2.13), one has

$$d\eta = d\tilde{\eta}|_M = 2 \sum_i \omega^i \wedge \omega^{i^*}\ ,$$

and one may say that D_p is a contact vector subspace $(\dim D_p = 2(m-\ell)+1)$ of $T_p(M)$.

If we denote by ψ the simple unit form (see Rosca [14]) which corresponds to D_p, one may write

$$\psi = (\wedge d\eta)^{m-\ell} \wedge \eta / 2^{m-\ell}(m-\ell)! \tag{3.3}$$

Clearly one has $d\psi = 0$. Therefore the ideal $J(D^{\perp}) = \{\psi \in \Lambda(M);\ \psi$ annihilates $D^{\perp}$, $dJ \subset J\}$ is a *differentiable ideal,* and we conclude that the distribution $D^{\perp}$ is always involutive (as in the case of proper CR submanifold of $\tilde{M}(U, \xi, \tilde{\eta}, \tilde{g})$ — see Yano and Kon' [1]; Kobayashi [2] — and in the case of CICR submanifolds of a para Kaehlerian manifold — see Rosca [14].)

Consider the vector bundle $L^* = S^* \backslash D^{*\perp} \subset D$ over M. Then, as is known, the elements of $A^1(M, L^*) = SL^*\mathrm{Hom}(\Lambda^1 TM)$ (SL^*: space of sections) are 1-forms of M with values in L^*. Set $\alpha \in \{i, i^*, r\}$. Then by virtue of (3.2) the 1-forms $\theta_\alpha^{r^*}$ represent the *mixed* connection forms (Rosca [13]) associated with α. Then taking the exterior derivative of (3.2), one finds by (2.8), (2.10), and (2.11) that

$$\theta_\alpha^{r^*} = \theta_{i^*}^{r^*} \in A^1(M, L^*)\ . \tag{3.4}$$

Next denote by $\ell_r = -\langle dp, \nabla h_r \rangle$ the second fundamental quadratic forms associated with x (coefficients of ℓ_r are symmetric covariant 2-tensors and depend only on the normal connection $\nabla^{\perp}$). It follows from (2.5), (2.7), and Cartan's lemma that

$$\ell_r = \gamma_{i^*j^*}^{r^*}\, \omega^{i^*} \otimes \omega^{j^*}\ . \tag{3.5}$$

Then the second fundamental quadratic vectorial form on M, i.e. $II = \sum_r \ell_r h_r$ (II is an M-morphism of $T_0^2(M)$ in $\tilde{M}$ and does not depend on $\nabla^{\perp}$) is expressed by

$$II = \gamma_{i^*j^*}^{r^*}\, \omega^{i^*} \otimes \omega^{j^*} \otimes h_r\ . \tag{3.6}$$

If $\nabla^{\perp}$ defines the connection in the normal bundle $T^{\perp}(M)$, then for any $X \subset T_p(M)$ and any $N \subset T_p^{\perp}(M)$ a basic formula for submanifolds is the Weingarten formula

$$\nabla_X N = -A_N(X) + \nabla_X^{\perp} N \tag{3.7}$$

In (3.7) $A_N(X)$ and $\nabla_X^{\perp} N$ are the tangential and the normal part of $\nabla_X N$ respectively.

Setting $N = N^r h_r$, one finds by (3.4) that

$$A_N(x) = \sum_i N^r \theta^{r^*}_{i^*}(X) h_i \in S\backslash D^\perp . \tag{3.8}$$

Then, as is known, the vector space

$$T_p = \{X \in T_p(M): A_N(X) = 0\}$$

is called the *space of relative nullity*. It follows from (2.8) that $T_p = S_p \oplus \{\xi\}_p$ and one may write

$$T_p(M) = T_p \oplus (S^*\backslash D^{*\perp})_p . \tag{3.9}$$

Since $\dim T_p = m+1$, this integer represents the *index of relative nullity* (Gardner [15]).

We may also consider the following basic invariants of II. Setting $N = T^\perp_p(M)$, $T = T_p(M)$, one has according to Gardner [15]:

1) The *target rank*, $\dim_N \mathrm{II}$, is the integer $r(p)$ which in the case under discussion is defined by

$$r(p) = \dim_N \mathrm{II}(p) = \dim\{\sum_r \gamma^{r^*}_{i^*j^*} h_r\} = \ell(\ell+1)/2. \tag{3.10}$$

2) The *source rank*, $\dim_T \mathrm{II}$, is the integer $s(p)$ defined by

$$\begin{aligned} s(p) &= \dim_T \mathrm{II}(p) = \dim\{\theta^{r^*}_{i^*}\} = \ell(m-\ell) \\ &= \operatorname{codim} M \cdot \dim L^*_p. \end{aligned} \tag{3.11}$$

Furthermore it follows from (3.6) that

$$\mathrm{II}(D^\perp, D^\perp) = 0, \tag{3.12}$$

$$\mathrm{II}(D, D^\perp) = 0, \tag{3.13}$$

and

$$\mathrm{II}(S,S) = 0. \tag{3.14}$$

Hence, from the above equations we may say that any CICR submanifold is

(i) *vertical totally geodesic,*

(ii) *mixed totally geodesic,*

(iii) *n.s.o. geodesic.*

Set $L = S\backslash D^\perp \subset D$ and consider the distributions

$$\begin{aligned} \Sigma_p &= L_p \oplus \{\xi\}_p \subset D_p , \\ \Sigma^*_p &= L^*_p \oplus \{\xi\}_p \subset D_p , \end{aligned} \tag{3.15}$$

each of dimension $m-\ell+1$. It follows from this that

$$\begin{aligned} U\Sigma_p &= L_p = \operatorname{orth} L_p; \quad d\eta|_{\Sigma_p} = 0 , \\ U\Sigma^*_p &= L^*_p = \operatorname{orth} L^*_p; \quad d\eta|_{\Sigma^*_p} = 0 \end{aligned} \tag{3.16}$$

and referring to Weinstein [16] and Rosca [13], we agree to call Σ_p and Σ^*_p the *principal contact Lagrangian distributions* of D_p.

Denote now by $M^{\perp}$ the maximal integral manifold of $D^{\perp}$ and by $T_p(M^{\perp})$ and $T_p^{\perp}(M^{\perp})$ the tangent and normal spaces of $M^{\perp}$ at any point $p \in M^{\perp}$. Obviously one has

$$T_p^{\perp}(M^{\perp}) = D_p^{\perp} \oplus D_p \tag{3.17}$$

and this implies $T_p(M^{\perp}) = D_p^{\perp} \subset T_p^{\perp}(M^{\perp})$ that is the submanifold $M^{\perp}$ is *isotropic* (dim $M^{\perp}$ = codim M = defect of $x = \ell$). Since $M^{\perp}$ is orientable, we choose an orientation on $M^{\perp}$ with the volume element τ and the star operator $*$.

Since the line element of $M^{\perp}$ is

$$dp = \omega^r \otimes h_r;\ r = m-\ell+1,\dots,m \tag{3.18}$$

(we denote the elements induced on $M^{\perp}$ by the same letters), one finds using the star isomorphism that

$$*dp = \sum_r (-1)^{r-(m-\ell+1)}\omega^{r-(m-\ell+1)}\wedge\dots\wedge\hat{\omega}^r\wedge\dots\wedge\omega^m \otimes h_{r^*} \tag{3.19}$$

(the roof ^ means omission). Hence, we may say that $*dp$ is a vectorial $(\ell-1)$-form on the transversal bundle $D^{*\perp}$.

Let $\Delta = d\circ\delta + \delta\circ d$ be the harmonic operator on $\wedge T^*M^{\perp}$. Since dp given by (3.18) is closed, one has $\Delta p = (\dim M^{\perp})\Gamma$ where Γ is an invariant vector field. Using (2.7), (2.8), (2.10), (2.1), (3.2), and (3.4), we infer from (3.19) that

$$d*dp = (\Gamma^T+\Gamma_t) \otimes \tau \tag{3.20}$$

where we have set

$$\Gamma^T = -(\ell\xi + \sum_{i,r} \gamma^r_{ir} h_{i^*}) \in D_p \tag{3.21}$$

and

$$\Gamma_t = -\sum_r \gamma^r_{rr} h_{r^*} \in D_p^{*\perp}\ . \tag{3.22}$$

Since by (3.17) the vector field Γ^T is normal to $M^{\perp}$ (this can be easily checked by a direct computation), we define Γ^T/ℓ as the *almost mean curvature vector* of $M^{\perp}$.

From (3.21) and (2.2) one easily finds $\langle\Gamma^T,\Gamma^T\rangle = \ell^2$. Hence, one may say that $M^{\perp}$ is of *constant almost mean curvature*.

Denote by $\ell_\Gamma = \langle dp,\nabla\Gamma^T\rangle$ the *mean quadratic differential* of $M^{\perp}$. By (3.18) and (3.21) an easy calculation gives

$$\ell_{\Gamma^T} = \sum_{i,r} (\sum_s \gamma^s_{is})\omega^r \otimes \theta^{r^*}_{i^*}\ . \tag{3.23}$$

Since $\theta^{r^*}_{i^*} \in A^1(M,L^*)$, it follows from this that on $M^{\perp}$ one has $\theta^{r^*}_{i^*} = 0$, and this implies $\ell_{\Gamma^T} = 0$.

This above fact together with $|\Gamma| = \text{const}$ proves that Γ^T is a *geodesic section* on $M^{\perp}$.

THEOREM 1. Let M be any co-isotropic submanifold of a pseudo-Sasakian manifold $\tilde{M}$. Then M is a CR submanifold of $\tilde{M}$ whose vertical distribution $D^{\perp}$ is involutive and the leaves $M^{\perp}$ of $D^{\perp}$ are isotropic.

Further M possesses the following properties:

(i) it is $D^{\perp}$-totally geodesic;

(ii) it is mixed totally geodesic;

(iii) it is n.s.o. geodesic.

If $\dim \tilde{M} = 2m+1$ and $\operatorname{codim} M = \ell$, then the source rank and the index of relative nullity at each point $p \in M$ are $\ell(m-\ell)$ and $m+1$ respectively.

Finally the maximal integral manifold $M^{\perp}$ of $D^{\perp}$ is of constant almost mean curvature, and the almost mean curvature vector field is a geodesic section on $M^{\perp}$.

4. FOLIATE CICR SUBMANIFOLDS.

We shall now consider the quadratic forms $\ell_{r^*} = -\langle dp, \nabla h_{r^*}\rangle$ and agree to call $II_t = \sum_r \ell_{r^*} h_{r^*}$ the *transversal quadratic vectorial form* on M.

Using (2.2), (2.7), (2.10), and (3.2), we obtain

$$II_t = (\theta^r_i \otimes \omega^i - \eta \otimes \omega^r) \otimes h_{r^*} . \tag{4.1}$$

Let now X and Y be any vector fields on the horizontal distribution D_p. Then the equation

$$II_t(X,UY) = II_t(UX,Y) \tag{4.2}$$

gives

$$\gamma^r_{ij} = \gamma^r_{ji},\ \gamma^r_{ij^*} = 0 . \tag{4.3}$$

It is easy to see by (2.6) that (4.3) is equivalent to $[X,Y] \in D_p$ that is the distribution D is *involutive*.

We shall say in this case according to Bejancu [4] and Kobayashi [2] that the CICR submanifold M is *foliate*.

If M^T are the leaves of D, then, as it has been proven by Rosca [3], M^T are *invariant* and *minimal* submanifolds of $\tilde{M}$.

Denote by ϕ the simple unit form corresponding to the vertical distribution $D^{\perp}$:

$$\phi = \omega^{m-\ell+1} \wedge \ldots \wedge \omega^m . \tag{4.4}$$

Let us express that ϕ is *exterior recurrent* with $u \in \Lambda(M)$ as a *recurrence 1-form*. Hence, according to Datta [5] we must write

$$d\phi = u \wedge \phi. \tag{4.5}$$

If u is given by

$$u = \ell\eta - \sum_r \theta^r_r , \tag{4.6}$$

we say that M is *Ricci $D^{\perp}$-exterior recurrent*.

Then by (2.7), (2.9), (2.10) and (3.2) one derives from (4.5) and (4.6) in addition to condition (4.3) (which proves that M is foliate) the following relations

$$\gamma^r_{i0} = 0, \tag{4.7}$$

$$\sum_r \gamma^r_{ir} = 0 . \tag{4.8}$$

In the following we shall set

$$\gamma^\perp = \sum_r \theta^r_r \tag{4.9}$$

and agree to call $\gamma^\perp$ the *vertical component* of the Ricci 1-form $\gamma = \tilde{\gamma}|_M$ on M.

Let now X and Y be any vector fields of the contact Lagrangian distributions $\Sigma \subset D$. Taking into account (4.3) and (4.7), one finds $[X,Y] \subset \Sigma$, that is Σ is involutive. It is easily deduced that the same property holds for the contact Lagrangian distribution $\Sigma^* \subset D$.

Therefore one may say that if M is a Ricci $D^\perp$-exterior recurrent CICR submanifold, then it receives two contact Lagrangian foliations. Moreover, since ξ is geodesic on $\tilde{M}$ (see Rosca [3]), then if X (resp. X^*) is any constant vector field of Σ_p (resp. Σ^*_p), one finds by (4.7) that

$$\nabla_\xi X = 0 \quad (\text{resp. } \nabla_\xi X^* = 0).$$

Hence, any constant vector field of Σ_p or Σ^*_p is ξ-parallel.

Let now M be any CICR submanifold with the line element

$$dp = \omega^i \otimes h_i + \omega^{i^*} \otimes h_{i^*} + \eta \otimes \xi + \omega^r \otimes h_r \tag{4.10}$$

and the volume element

$$\tau = \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \tag{4.11}$$

(M is defined by equations (3.2)). Taking the star isomorphism of (4.10), one has by (2.2) and (4.11)

$$\begin{aligned} {}^*dp = \textcircled{\Gamma} &= \sum_i (-1)^{i-1} \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \hat{\omega}^{i^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \otimes h_{i^*} \\ &+ \sum_i (-1)^{i^*-1} \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \hat{\omega}^{i^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \phi \otimes h_i \\ &+ (-1)^\ell \omega^1 \wedge \ldots \wedge \omega^{m-\ell} \wedge \omega^{1^*} \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \phi \otimes \xi \\ &+ \omega^1 \wedge \ldots \wedge \omega^{(m-\ell)^*} \wedge \eta \wedge \Big(\sum_r (-1)^{r-(m-\ell+1)} \omega^{m-\ell+1} \wedge \ldots \wedge \hat{\omega}^r \wedge \ldots \wedge \omega^m \otimes h_{r^*}\Big). \end{aligned} \tag{4.12}$$

We agree to define the vectorial $(2(m-\ell)+r)$-form $\textcircled{\Gamma}$ as the *improper* mean curvature form of M. Taking the exterior derivative of $\textcircled{\Gamma}$ and using (2.7) and (2.8), we obtain by a straight forward calculation that

$$d\textcircled{\Gamma} = (\Gamma^T + \Gamma^\perp + \Gamma_t) \otimes \tau . \tag{4.13}$$

In (4.13) we have set

$$\Gamma^T = -\sum_{i^*}\sum_r (\gamma^r_{ri}+\gamma^r_{ir})h_{i^*} - \sum_i (\sum_r \gamma^r_{ri^*})h_i - (\sum_r \gamma^r_{r0})\xi, \tag{4.14}$$

$$\Gamma^\perp = \sum_i \gamma^r_{ii^*} h_r, \tag{4.15}$$

$$\Gamma_t = -\sum_{s^*}(\sum_r \gamma^r_{rs})h_{s^*}, \quad r,s = m-\ell+1,\dots,m. \tag{4.16}$$

Putting

$$\Gamma = \Gamma^T + \Gamma^\perp + \Gamma_t, \tag{4.17}$$

we agree to say that the invariant vector field Γ is the *improper* mean curvature vector of M, and Γ^T, $\Gamma^\perp$, Γ_t are the horizontal, vertical and transversal components of Γ respectively.

On the other hand, if the vertical Ricci 1-form $\gamma^\perp$ vanishes, then the recurrence 1-form of equation (4.5) is $\ell\eta$. We shall say in this case that M is a *contact* $D^\perp$*-exterior recurrent CICR submanifold.*

We shall give now the following

DEFINITION. Let $x: M \to \tilde{M}$ be the improper immersion of a CICR submanifold M in a pseudo-Sasakian manifold $\tilde{M}$, and let Γ be the improper mean curvature vector associated with x. Then if the vertical component of Γ vanishes, we say that M is *almost minimal*, and if Γ vanishes, we say that M is *minimal*.

Referring now to (4.3) and (4.15), we see that if M is foliate, then it is almost minimal.

Furthermore, if M is Ricci $D^\perp$-exterior recurrent, then one readily derives that conditions (4.8) and $\sum_r \theta^r_r = 0$ imply $\Gamma = 0$, that is M is minimal.

It is easy to see that the converse is also valid.

THEOREM 2. Let M be a CICR submanifold and let II_t be the transversal quadratic vectorial form of M. Then the necessary and sufficient condition for M to be foliate is that for any vector fields X and Y of the horizontal distribution D one has $II_t(X,UY) = II_t(UX,Y)$, and in this case M is almost minimal. If M is Ricci $D^\perp$-exterior recurrent, then it receives two contact Lagrangian foliations and the necessary and sufficient condition for M to be minimal is that M be contact $D^\perp$-exterior recurrent.

5. CO-ISOTROPIC CONTACT ξ-VERTICAL CR SUBMANIFOLDS.

We shall consider now the improper immersion $x: M \to \tilde{M}$ where M is a contact *ξ-vertical* CICR submanifold of $\tilde{M}$, that is $\xi \in D^\perp_p$. As in Section 3, we suppose that M is defined by equations (3.2). Then the horizontal and vertical distributions at each point $p \in M$ are defined by $D_p = \{h_i, h_{i^*}, i = 1,\dots,m-\ell;\ i^* = i+m\}$ and $D^\perp_p = \{h_r, \xi;\ r = m+1-\ell,\dots,m\}$ respectively.

In this case D_p is of even dimension (Kobayashi [2]); in the case under discussion $\dim D_p = 2(m-\ell))$ and its corresponding simple unit form ψ is equal to $(\Lambda d\eta)^{m-\ell}/2^{m-\ell}(m-\ell)!$.

It is easily deduced, that, as in Section 3, $J(D^{\perp}) = \{\eta\psi \in \Lambda(M);\ \psi$ annihilates $D^{\perp}\}$ is a differentiable ideal and this proves that $D^{\perp}$ is involutive.

Denote by $M^{\perp}$ the maximal integral manifold of $D^{\perp}$. The normal space $T_p^{\perp}(M^{\perp})$ at each point $p \in M^{\perp}$ in the case under discussion is defined by $(D_p^{\perp}\backslash\xi) \oplus D_p$.

On the other hand, since the tangent space $T_p(M^{\perp})$ at each point p is defined by $D_p^{\perp}$, it follows from this that we are in the situation of conditions (2.13). Therefore according to the definition given in Section 2, it follows that $M^{\perp}$ is a mixed isotropic submanifold of $\tilde{M}$.

Denote now by $\phi = \omega^{m-\ell+1}\Lambda\ldots\Lambda\omega^{m}\Lambda\eta$ the simple unit form which corresponds to $D_p^{\perp}$. Taking into account (2.12), one readily finds that the ideal $J(D) = \{\phi \in \Lambda M;$ ϕ annihilates D, $dJ \subset J\}$, that is the ideal $J(D)$ is not a differentiable ideal.

Thus we conclude that the distribution D can not be involutive.

THEOREM 3. Let $x\colon M \to \tilde{M}$ be the improper immersion of a contact ξ-vertical CICR submanifold M in $\tilde{M}$. Then:

(i) the vertical distribution $D^{\perp}$ is always involutive, and leaves of $D^{\perp}$ are mixed isotropic;

(ii) there does not exist a foliate ξ-vertical CICR submanifold of $\tilde{M}$.

REFERENCES

1. YANO, K., and KON, M. CR Submanifolds of Kaehlerian and Sasakian Manifolds, Birkhauser, Boston, 1983.

2. KOBAYASHI, M. CR Submanifolds of a Sasakian Manifold, Tensor (N.S.) 35(1981), 297-307.

3. ROSCA, R. On Pseudo-Sasakian Manifolds, Rend. Mat., 1984 (to appear).

4. BEJANCU, A. CR Submanifolds of a Kaehler Manifold II, Trans. Amer. Math. Soc. 250(1979), 333-345.

5. DATTA, D.K. Exterior Recurrent Forms on a Manifold, Tensor (N.S.) 36(1982), 115-120.

6. ROSCA, R. Improper Immersions in a Lorentzian Manifold Admitting a Sasakian Structure, Libertas Math. 3(1983), 117-127.

7. GOLDBERG, V.V. and ROSCA, R. Mixed Isotropic Submanifolds and Isotropic Cosympletic Structures, Soochow J. Math. 9(1983), 25-38.

8. MORVAN, J.M. and ROSCA, R. Structures Presque Cosymplectiques a Couple de Champs Quasirecurrents Réciproques, Rend. Sem. Mat. Univ. Politec. Torino 369(1977-1978), 225-234.

9. LIBERMANN, P. Sur le Probleme d'Équivalence de Certaines Structures Infinitesimales, Ann. Mat. Pura Appl. 36(1951), 27-120.

10. SINHA, B.B. A Differentiable Manifold with Para f-Structure of Rank r, Ann. Fac. Sci. Univ. Nat. Zaire (Kinshasa) Sect. Math.-Phys. 6(1980), No. 1-2, 79-94.

12. VRANCEANU, GH. and ROSCA, R. Introduction in Relativity and Pseudo-Riemannian Geometry, Edit. Acad. Rep. Soc. Roumania, Bucuresti, 1976.

13. ROSCA, R. Codimension 2 CR Submanifolds with Null Covariant Decomposable Vertical Distribution of a Neutral Manifold $\tilde{M}$, Rend. Mat. (4) 2(1982), 787-796.

14. ROSCA, R. CR-Subvariétés Co-Isotropes d'une Variété Para Kahlerienne, C.R. Acad. Sci. Paris Sér. I Math., 1984 (to appear).

15. GARDNER, R. New Viewpoints in the Geometry of Submanifolds in R^N, Bull. Amer. Math. Soc. 83(1977), No. 1, 1-34.

16. WEINSTEIN, A. Lectures on Symplectic Manifolds, C.B.M.S. Regional Conference Univ. North Carolina, 1976, 1-44.