

## DERIVATIONS IN BANACH ALGEBRAS

KYOO-HONG PARK, YONG-SOO JUNG, and JAE-HYEONG BAE

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We present some conditions which imply that a derivation on a Banach algebra maps the algebra into its Jacobson radical.

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**1. Introduction.** Throughout this paper,  $A$  represents an associative algebra over the complex field  $\mathbb{C}$ , and the *Jacobson radical* of  $A$  and the *center* of  $A$  are denoted by  $\text{rad}(A)$  and  $Z(A)$ , respectively. Let  $I$  be any closed (2-sided) ideal of the Banach algebra  $A$ . Then let  $Q_I$  denote the canonical quotient map from  $A$  onto  $A/I$ . Recall that an algebra  $A$  is *prime* if  $aAb = \{0\}$  implies that either  $a = 0$  or  $b = 0$ . A mapping  $f : A \rightarrow A$  is called *commuting* (resp., *centralizing*) if  $[f(x), x] = 0$  (resp.,  $[f(x), x] \in Z(A)$ ) for all  $x \in A$ . More generally, for a positive integer  $n$ , we define a mapping  $f$  to be *n-commuting* (resp., *n-centralizing*) if  $[f(x), x^n] = 0$  (resp.,  $[f(x), x^n] \in Z(A)$ ) for all  $x \in A$ . A linear mapping  $d : A \rightarrow A$  is called a *derivation* if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in A$ .

The Singer-Wermer theorem, which is a classical theorem of Banach algebra theory, states that every continuous derivation on a commutative Banach algebra maps into its Jacobson radical [9], and Thomas [10] proved that the Singer-Wermer theorem remains true without assuming the continuity of the derivation. (This generalization is called the Singer-Wermer conjecture.) On the other hand, Posner [6] obtained two fundamental results in 1957: (i) the first result (the so-called Posner's first theorem) asserts that if  $d$  and  $g$  are derivations on a 2-torsion free prime ring such that the product  $dg$  is also a derivation, then either  $d = 0$  or  $g = 0$ . (ii) The second result (the so-called Posner's second theorem) states that if  $d$  is a centralizing derivation on a noncommutative prime ring, then  $d = 0$ . As an analytic analogue of Posner's second theorem, Mathieu and Runde [5, Theorem 1] generalized the Singer-Wermer conjecture by proving that every centralizing derivation on a Banach algebra maps into its Jacobson radical. The main objective of this paper is to obtain a generalization (Theorem 2.3) of the above Singer-Wermer conjecture which is inspired by Posner's first theorem.

**2. Results.** To prove our main result we need the following two lemmas.

**LEMMA 2.1.** *Let  $d$  and  $g$  be derivations on a noncommutative prime algebra  $A$ . If there exist a positive integer  $n$  and  $\alpha \in \mathbb{C}$  such that  $\alpha d^2 + g$  is  $n$ -commuting on  $A$ , then both  $d = 0$  and  $g = 0$  on  $A$ .*

**PROOF.** For the convenience, we write  $f$  instead of  $\alpha d^2 + g$ . Then the assumption of the lemma can be written in the form

$$[f(x), x^n] = 0 \quad (2.1)$$

for all  $x \in A$ . For  $\alpha = 0$ , the result is obtained from [3, Corollary, page 3713]. Let  $\alpha \neq 0$ . Substituting  $x + \lambda y$  ( $\lambda \in \mathbb{C}$ ) for  $x$  in (2.1), we obtain

$$\lambda Q_1(x, y) + \lambda^2 Q_2(x, y) + \cdots + \lambda^n Q_n(x, y) = 0, \quad x, y \in A, \quad (2.2)$$

where  $Q_i(x, y)$  denotes the sum of terms involving  $i$  factors of  $y$  in the expansion of  $[f(x + \lambda y), (x + \lambda y)^n] = 0$ . Since  $\lambda$  is arbitrary, we have

$$\begin{aligned} Q_1(x, y) &= [f(y), x^n] + [f(x), x^{n-1}y] \\ &\quad + [f(x), x^{n-2}yx] + \cdots + [f(x), yx^{n-1}] = 0, \quad x, y \in A. \end{aligned} \quad (2.3)$$

Substituting  $xy$  for  $y$  in (2.3), we get

$$\begin{aligned} 0 &= x[f(x), x^{n-1}y] + [f(x), x]x^{n-1}y \\ &\quad + x[f(x), x^{n-2}yx] + [f(x), x]x^{n-2}yx \\ &\quad + \cdots + x[f(x), yx^{n-1}] + [f(x), x]yx^{n-1} \\ &\quad + f(x)[y, x^n] + 2\alpha[d(x)d(y), x^n] + x[f(y), x^n], \quad x, y \in A; \end{aligned} \quad (2.4)$$

and left multiplying (2.3) by  $x$  and subtracting the result from (2.4), we have

$$\begin{aligned} 0 &= [f(x), x]x^{n-1}y + [f(x), x]x^{n-2}yx + \cdots + [f(x), x]yx^{n-1} \\ &\quad + f(x)[y, x^n] + 2\alpha[d(x)d(y), x^n], \quad x, y \in A. \end{aligned} \quad (2.5)$$

In (2.5), replace  $y$  by  $yx$  to obtain

$$\begin{aligned} 0 &= [f(x), x]x^{n-1}yx + [f(x), x]x^{n-2}yx^2 \\ &\quad + \cdots + [f(x), x]yx^n + f(x)[y, x^n]x \\ &\quad + 2\alpha[d(x)d(y), x^n]x + 2\alpha[d(x)y d(x), x^n], \quad x, y \in A; \end{aligned} \quad (2.6)$$

and multiply by  $x$  on the right in (2.5) to obtain

$$\begin{aligned} 0 &= [f(x), x]x^{n-1}yx + [f(x), x]x^{n-2}yx^2 + \cdots + [f(x), x]yx^n \\ &\quad + f(x)[y, x^n]x + 2\alpha[d(x)d(y), x^n]x, \quad x, y \in A. \end{aligned} \quad (2.7)$$

We now subtract (2.7) from (2.6) to get

$$d(x)y d(x)x^n - x^n d(x)y d(x) = 0, \quad x, y \in A. \quad (2.8)$$

Replacing  $y$  by  $y d(x)z$  in (2.8), we obtain

$$d(x)y d(x)z d(x)x^n - x^n d(x)y d(x)z d(x) = 0, \quad x, y, z \in A. \quad (2.9)$$

According to (2.8), we can write, in relation (2.9),  $x^n d(x)z d(x)$  for  $d(x)z d(x)x^n$  and  $d(x)y d(x)x^n$  instead of  $x^n d(x)y d(x)$ , which gives

$$d(x)y[d(x), x^n]z d(x) = 0, \quad x, y, z \in A. \tag{2.10}$$

From (2.10) and primeness of  $A$ , it follows that, for any  $x \in A$  we have either  $[d(x), x^n] = 0$  or  $d(x) = 0$ . In any case  $[d(x), x^n] = 0$  for all  $x \in A$ , which yields  $d = 0$  on  $A$  by [3, Corollary, page 3713]. Now the initial hypothesis yields that  $[g(x), x^n] = 0, x \in A$ , so  $g = 0$  on  $A$ , which completes the proof of the lemma.  $\square$

**LEMMA 2.2.** *Let  $d$  be a derivation on a Banach algebra  $A$  and  $J$  a primitive ideal of  $A$ . If there exists a real constant  $K > 0$  such that  $\|Q_J d^n\| \leq K^n$  for all  $n \in \mathbb{N}$ , then  $d(J) \subseteq J$ .*

**PROOF.** See [11, Lemma 1.2].  $\square$

Now we prove our main result.

**THEOREM 2.3.** *Let  $d$  and  $g$  be derivations on a Banach algebra  $A$ . If there exist a positive integer  $n$  and  $\alpha \in \mathbb{C}$  such that  $\alpha d^2 + g$  is  $n$ -commuting on  $A$ , then both  $d$  and  $g$  map  $A$  into  $\text{rad}(A)$ .*

**PROOF.** Let  $J$  be any primitive ideal of  $A$ . Using Zorn’s lemma, we find a minimal prime ideal  $P$  contained in  $J$ , and hence  $d(P) \subseteq P$  and  $g(P) \subseteq P$  (see [5, Lemma]). Suppose first that  $P$  is closed. Then the derivations  $d$  and  $g$  on  $A$  induce the derivations  $\bar{d}$  and  $\bar{g}$  on the Banach algebra  $A/P$ , defined by  $\bar{d}(x + P) = d(x) + P$  and  $\bar{g}(x + P) = g(x) + P$  ( $x \in A$ ). In case  $A/P$  is commutative, both  $\bar{d}(A/P)$  and  $\bar{g}(A/P)$  are contained in the Jacobson radical of  $A/P$  by [10]. We consider the case when  $A/P$  is noncommutative. The assumption that  $\alpha d^2 + g$  is  $n$ -commuting on  $A$  gives that the mapping  $\alpha \bar{d}^2 + \bar{g}$  is  $n$ -commuting on  $A/P$ . Since  $A/P$  is a prime algebra, it follows from Lemma 2.1 that both  $\bar{d} = 0$  and  $\bar{g} = 0$  on  $A/P$ . Consequently, we see that both  $d(A) \subseteq J$  and  $g(A) \subseteq J$ . If  $P$  is not closed, then we see that  $\mathcal{S}(d) \subseteq P$  by [2, Lemma 2.3], where  $\mathcal{S}(T)$  is the separating space of a linear operator  $T$ . Then we have, by [8, Lemma 1.3],  $\mathcal{S}(Q_{\bar{P}}d) = \overline{Q_{\bar{P}}(\mathcal{S}(d))} = \{0\}$  whence  $Q_{\bar{P}}d$  is continuous on  $A$ . This means that  $Q_{\bar{P}}d(\bar{P}) = \{0\}$ , that is,  $d(\bar{P}) \subseteq \bar{P}$ . Hence, we see that  $d$  induces a derivation  $\tilde{d}$  on the Banach algebra  $A/\bar{P}$ , defined by  $\tilde{d}(x + \bar{P}) = d(x) + \bar{P}$  ( $x \in A$ ). This shows that we can define a map

$$\Psi \tilde{d}^n Q_{\bar{P}} : A \rightarrow A/\bar{P} \rightarrow A/\bar{P} \rightarrow A/J \tag{2.11}$$

by  $\Psi \tilde{d}^n Q_{\bar{P}}(x) = Q_J d^n(x)$  ( $x \in A, n \in \mathbb{N}$ ), where  $\Psi$  is the canonical induced map from  $A/\bar{P}$  onto  $A/J$  (the relation  $\bar{P} \subseteq J$  guarantees its existence). The continuity of  $\tilde{d}$  is clear from [8, Lemma 1.4], and hence yields that  $\|Q_J d^n\| \leq \|\tilde{d}\|^n$  for all  $n \in \mathbb{N}$ . Now, according to Lemma 2.2, we obtain that  $d(J) \subseteq J$ . Following the same argument with  $g$ , we see that  $g(J) \subseteq J$ . Then the derivations  $d$  and  $g$  on  $A$  induce the derivations  $\hat{d}$  and  $\hat{g}$  on the Banach algebra  $A/J$ , defined by  $\hat{d}(x + J) = d(x) + J$  and  $\hat{g}(x + J) = g(x) + J$  ( $x \in A$ ). The rest follows as when  $P$  is closed since the primitive algebra  $A/J$  is prime. So we also obtain that  $d(A) \subseteq J$  and  $g(A) \subseteq J$ . Since  $J$  was arbitrary, we arrive at the conclusion that  $d(A) \subseteq \text{rad}(A)$  and  $g(A) \subseteq \text{rad}(A)$ .  $\square$

A mapping  $f : A \rightarrow A$  is said to be *skew-centralizing* if  $\langle f(x), x \rangle \in Z(A)$  for all  $x \in A$ , where  $\langle a, b \rangle$  denotes the Jordan product  $ab + ba$ .

**COROLLARY 2.4.** *Let  $d$  and  $g$  be derivations on a Banach algebra  $A$ . If there exists  $\alpha \in \mathbb{C}$  such that  $\alpha d^2 + g$  is skew-centralizing on  $A$ , then both  $d$  and  $g$  map  $A$  into  $\text{rad}(A)$ .*

**PROOF.** Since  $\langle \alpha d^2(x) + g(x), x \rangle \in Z(A)$  for all  $x \in A$ , we obtain that  $[\langle \alpha d^2(x) + g(x), x \rangle, x] = 0$  for all  $x \in A$ . From the relation

$$\begin{aligned} 0 &= [\langle \alpha d^2(x) + g(x), x \rangle, x] \\ &= \langle [\alpha d^2(x) + g(x), x], x \rangle \\ &= [\alpha d^2(x) + g(x), x^2], \end{aligned} \tag{2.12}$$

we see that  $\alpha d^2 + g$  is 2-commuting, and hence [Theorem 2.3](#) guarantees the conclusion.  $\square$

As a noncommutative version of the Singer-Wermer theorem, we also obtain the next result by using [Lemma 2.1](#).

**THEOREM 2.5.** *Let  $d$  and  $g$  be continuous derivations on a Banach algebra  $A$ . If there exist a positive integer  $n$  and  $\alpha \in \mathbb{C}$  such that the mapping  $\alpha d^2 + g$  is  $n$ -centralizing on  $A$ , then both  $d$  and  $g$  map  $A$  into  $\text{rad}(A)$ .*

**PROOF.** Given any primitive ideal  $J$  of  $A$ , we have  $d(J) \subseteq J$  and  $g(J) \subseteq J$  by [[7](#), Theorem 2.2]. Thus we can suppose that  $A$  is primitive. From  $[\alpha d^2(x) + g(x), x^n] \in Z(A)$  for all  $x \in A$ , we obtain  $[[\alpha d^2(x) + g(x), x^n], x^n] = 0$ , and hence  $[\alpha d^2(x) + g(x), x^n]$  is quasinilpotent by the Kleinecke-Shirokov theorem [[1](#), Proposition 18.13]. Since  $Z(A)$  is trivial, it follows that  $[\alpha d^2(x) + g(x), x^n]$  is a scalar multiple of 1, and so  $[\alpha d^2(x) + g(x), x^n] = 0$  for all  $x \in A$ . Note that a commutative primitive Banach algebra is isomorphic to the complex field  $\mathbb{C}$ . Hence we also can assume that  $A$  is noncommutative. Now, the primeness of  $A$  and [Lemma 2.1](#) allows that both  $d = 0$  and  $g = 0$  on  $A$ , which gives the result.  $\square$

We do not know whether [Theorem 2.5](#) can be proved without the continuity assumption. However, in the special case when the Banach algebra is semisimple, we obtain the following result.

**COROLLARY 2.6.** *Let  $d$  and  $g$  be derivations on a semisimple Banach algebra  $A$ . If there exist a positive integer  $n$  and  $\alpha \in \mathbb{C}$  such that  $\alpha d^2 + g$  is  $n$ -centralizing on  $A$ , then both  $d = 0$  and  $g = 0$  on  $A$ .*

**PROOF.** The fact that every derivation on a semisimple Banach algebra is continuous [[4](#), Remark 4.3] guarantees the conclusion.  $\square$

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KYOO-HONG PARK: DEPARTMENT OF MATHEMATICS EDUCATION, SEOWON UNIVERSITY, CHONGJU, CHUNGBUK 361-742, KOREA

*E-mail address:* [parkkh@seowon.ac.kr](mailto:parkkh@seowon.ac.kr)

YONG-SOO JUNG: DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, TAEJON 305-764, KOREA

*E-mail address:* [ysjung@math.cnu.ac.kr](mailto:ysjung@math.cnu.ac.kr)

JAE-HYEONG BAE: DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, TAEJON 305-764, KOREA

*E-mail address:* [jhbae@math.cnu.ac.kr](mailto:jhbae@math.cnu.ac.kr)



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