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Research Article

Positive Solution for the Nonlinear Hadamard Type Fractional Differential Equation with p-Laplacian

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We study the following nonlinear fractional differential equation involving the p-Laplacian operator $D^{\beta}(\varphi_p(D^{\alpha}u(t))) = f(t,u(t))$, $1 < t < e$, $u(1) = u'(1) = u'(e) = 0$, $D^{\alpha}u(1) = D^{\alpha}u(e) = 0$, where the continuous function $f : [1,e] \times [0,+\infty) \to [0,+\infty)$, $2 < \alpha \le 3$, $1 < \beta \le 2$. D^{α} denotes the standard Hadamard fractional derivative of the order α, the constant $p > 1$, and the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s$. We show some results about the existence and the uniqueness of the positive solution by using fixed point theorems and the properties of Green's function and the p-Laplacian operator.

1. Introduction

Fractional differential equations have attracted more and more attention for their useful applications in various fields, such as economics, science, and engineering; see [1–4]. In the last few decades, much attention has been focused on the study of the existence and uniqueness of solutions for boundary value problems of Riemann-Liouville type or Caputo type fractional differential equations; see [5–19]. There are few papers devoted to the research of the p-Laplacian fractional differential equations; see [20–25].

By the use of the fixed point theorem on cones, Chai in [20] obtained the existence and multiplicity of positive solutions for a class of boundary value problem of fractional differential equation with p-Laplacian operator:

$$D^{\beta}_{0+}\left(\phi_p\left(D^{\alpha}_{0+}u(t)\right)\right) = f(t,u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) + \sigma D^{\gamma}_{0+}u(1) = 0, \qquad D^{\alpha}_{0+}u(0) = 0, \tag{1}$$

where $1 < \alpha \le 2$, $0 < \beta \le 1$, $0 < \gamma \le 1$, $0 \le \alpha - \gamma - 1$, σ is a positive constant number, and $D^{\alpha}_{0+}, D^{\beta}_{0+}, D^{\gamma}_{0+}$ are the standard Riemann-Liouville derivatives. $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$.

Han et al. in [22] studied the following boundary value problem of nonlinear fractional differential equation with p-Laplacian operator:

$$\begin{aligned} &D^{\beta}_{0+}\left(\phi_p\left(D^{\alpha}_{0+}u(t)\right)\right) + a(t)f(u) = 0, \quad 0 < t < 1,\\ &u(0) = \gamma u(\xi) + \lambda,\\ &\phi_p\left(D^{\alpha}_{0+}u(0)\right) = \left(\phi_p(D^{\alpha}_{0+}u(1))\right)'\\ &\quad = \left(\phi_p\left(D^{\alpha}_{0+}u(1)\right)\right)'' = 0, \end{aligned} \tag{2}$$

where $0 < \alpha \le 1$, $2 < \beta \le 3$, $0 \le \gamma < 1$, $0 \le \xi \le 1$, $\lambda > 0$ is a parameter, and $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Caputo fractional derivatives. By the properties of Green function and Schauder fixed point theorem, several existence and nonexistence results and the uniqueness of positive solutions are acquired.

Liu et al. in [23] investigated the solvability of a fractional differential equation model involving the p-Laplacian operator with boundary value conditions as follows:

$$\left(\varphi_p\left({}^{c}D^{\alpha}x(t)\right)\right)' = f(t,x(t)), \quad t \in (0,1),$$

$$x(0) = r_0 x(1), \qquad x'(0) = r_1 x'(1),$$
$$x^{(i)} = 0, \quad i = 2, 3, \ldots, [\alpha] - 1, \tag{3}$$

where $1 < \alpha \in \mathbb{R}$, $r_0, r_1 \neq 1$, and ${}^cD^\alpha$ is the standard Caputo derivative. By the means of the Banach contraction mapping principle, they obtained the existence and uniqueness of a solution for the model.

Lu et al. in [24] considered the following fractional boundary value problem with p-Laplacian operator:

$$D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = u'(0) = u'(1) = 0, \qquad D_{0+}^{\alpha}u(0) = D_{0+}^{\beta}u(1) = 0, \tag{4}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives. By the properties of Green's function, the Guo-Krasnosel'skii fixed point theorem, the Leggett-Williams fixed point theorem, and the upper and lower solutions method, some new results on the existence of positive solutions are gained.

Motivated by the mentioned papers, we will consider the Hadamard fractional boundary value with p-Laplacian operator as below:

$$D^{\beta}\left(\varphi_p\left(D^{\alpha}u(t)\right)\right) = f(t, u(t)), \quad 1 < t < e,$$
$$u(1) = u'(1) = u'(e) = 0, \quad D^{\alpha}u(1) = D^{\alpha}u(e) = 0, \tag{5}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $\varphi_p(s) = |s|^{p-2}s$, and $f : [1, e] \times [0, +\infty) \to [0, +\infty)$ is a positive continuous function. Evidently, for any $p > 1$, $\varphi_p^{-1} = \varphi_q$, here $1/p + 1/q = 1$. Here D^α is the standard Hadamard fractional derivative of order α which is described as follows.

Definition 1 (see [1, Page 111]). The αth Hadamard fractional order derivative of a function $u : [1, +\infty) \mapsto \mathbb{R}$ is defined by

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{u(s)}{s}ds, \tag{6}$$

where $\alpha > 0$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the largest integer which is less than or equal to α. Correspondingly, the αth Hadamard fractional order integral of $u : [1, +\infty) \mapsto \mathbb{R}$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds, \tag{7}$$

where Γ is the gamma function.

To the best of our knowledge, there are few contributions to the Hadamard type with p-Laplacian operator; we fill the gap in this paper. In fact, we will discuss the existence and the uniqueness of the positive solutions of (5). The structure of this paper goes on as follows. In Section 2, we will introduce some basic lemmas that will be used. In Section 3, we first give some existence results including Theorems 10 and 11, Corollary 12, and Theorem 13. Then, we will prove Theorems 14 and 15 which reveal the uniqueness of the solution. In Section 4, we give two examples to illustrate our results.

2. Preliminary Results

In this section, we will first recall the following preliminary facts that will be used in our main results.

Lemma 2 (see [1, Theorem 2.3]). *Let $\alpha > 0$, $n = [\alpha] + 1$; then*

$$I^{\alpha}D^{\alpha}u(t) = u(t) + \sum_{i=1}^{n} c_i (\log t)^{\alpha-i}, \tag{8}$$

where c_i, $i = 1, 2, \ldots, n$, are some constants in $\mathbb{R}$.

The following lemma is the Schauder fixed point theorem which is well known; see Theorem 2.10 in [22].

Lemma 3. *If U is a nonempty closed, bounded, and convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.*

Lemma 4 (see [24, Lemma 2.7]). *Let X be a Banach space, let $P \subseteq X$ be a cone, and let Ω_1, Ω_2 be two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$,

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The following conclusion is the nonlinear alternative of Leray-Schauder type; see Lemma 2.6 in [10].

Lemma 5. *Let X be a Banach space with $C \subseteq X$ being closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous, compact map. Then either*

(1) *A has a fixed point in $\overline{U}$, or*

(2) *there exists $u \in \partial U$ and $\lambda \in (0, 1)$, with $u = \lambda A u$.*

Next, we give several lemmas which will be applied in the proofs of our main results.

Lemma 6. *Let $u(t)$ be the solution of the problem (5); then it can be described as below:*

$$u(t) = \int_1^e G(t, s)\varphi_q\left(\int_1^e H(s, \tau) f(\tau, u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} = Y(t), \tag{9}$$

where

$$G(t,s)=\begin{cases}\dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}-(\log(t/s))^{\alpha-1}}{\Gamma(\alpha)}, & s\le t,\\[2mm] \dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}}{\Gamma(\alpha)}, & t\le s,\end{cases}$$

$$H(t,s)=\begin{cases}\dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}-(\log(t/s))^{\beta-1}}{\Gamma(\beta)}, & s\le t,\\[2mm] \dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}}{\Gamma(\beta)}, & t\le s.\end{cases} \tag{10}$$

Proof. Putting $y(t)=f(t,u(t))$, we have $D^{\beta}(\varphi_p(D^{\alpha}u(t)))=y(t)$. By Lemma 2 and the fact that $1<\beta\le 2$,

$$\varphi_p(D^{\alpha}u(t))=I^{\beta}y(t)+c_1(\log t)^{\beta-1}+c_2(\log t)^{\beta-2}. \tag{11}$$

The boundary value hypotheses give $D^{\alpha}u(1)=D^{\alpha}u(e)=0$. So we can get that

$$c_2=0,\qquad c_1=-\frac{1}{\Gamma(\beta)}\int_1^e(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}. \tag{12}$$

Therefore,

$$\begin{aligned}\varphi_p(D^{\alpha}u(t))&=I^{\beta}y(t)-\frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_1^e(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\\&=\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\\&\quad-\frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_1^t(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\\&\quad-\frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_t^e(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\\&=-\int_1^t\frac{(\log t)^{\beta-1}(1-\log\tau)^{\beta-1}-(\log t-\log\tau)^{\beta-1}}{\Gamma(\beta)}\\&\quad\times y(\tau)\frac{d\tau}{\tau}-\int_t^e\frac{(\log t)^{\beta-1}(1-\log\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\frac{d\tau}{\tau}\\&=-\int_1^e H(t,\tau)y(\tau)\frac{d\tau}{\tau}.\end{aligned} \tag{13}$$

Notice the fact that $\varphi_p^{-1}=\varphi_q$, $1/p+1/q=1$; we have

$$D^{\alpha}u(t)+\varphi_q\left(\int_1^e H(t,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)=0. \tag{14}$$

Putting $x(t)=\varphi_q(\int_1^e H(t,\tau)f(\tau,u(\tau))(d\tau/\tau))$, it follows from Lemma 2 that

$$u(t)=-I^{\alpha}x(t)+C_1(\log t)^{\alpha-1}+C_2(\log t)^{\alpha-2}+C_3(\log t)^{\alpha-3}. \tag{15}$$

This, combined with the fact that $u(1)=u'(1)=u'(e)=0$, yields

$$C_3=0,\qquad C_2=0,\qquad C_1=\frac{1}{\Gamma(\alpha)}\int_1^e(1-\log s)^{\alpha-2}x(s)\frac{ds}{s}. \tag{16}$$

Thus,

$$\begin{aligned}u(t)&=\int_1^e G(t,s)x(s)\frac{ds}{s}\\&=\int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},\end{aligned} \tag{17}$$

where

$$G(t,s)=\begin{cases}\dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}-(\log(t/s))^{\alpha-1}}{\Gamma(\alpha)}, & s\le t,\\[2mm] \dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}}{\Gamma(\alpha)}, & t\le s,\end{cases}$$

$$H(t,s)=\begin{cases}\dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}-(\log(t/s))^{\beta-1}}{\Gamma(\beta)}, & s\le t,\\[2mm] \dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}}{\Gamma(\beta)}, & t\le s.\end{cases} \tag{18}$$

This completes the proof of Lemma 6. □

Lemma 7. *Suppose that $2<\alpha\le 3$, $1<\beta\le 2$. Then the functions $G(t,s)$ and $H(t,s)$ defined in* (10) *have the following properties:*

(1) *$G(t,s)$, $H(t,s)$ are continuous on $[1,e]\times[1,e]$;*

(2) *for any $t,s\in[1,e]$, $G(t,s)\ge 0$, $H(t,s)\ge 0$;*

(3) *for any $t,s\in[1,e]$, $G(t,s)\le G(e,s)$, $H(t,s)\le H(s,s)$;*

(4) *there exist two positive functions $\gamma_1,\gamma_2\in C(1,e)$ such that*

$$\begin{aligned}\min_{e^{1/4}\le t\le e^{3/4}}G(t,s)&\ge\gamma_1(s)\max_{1\le t\le e}G(t,s)\\&=\gamma_1(s)G(e,s),\quad \textit{for } s\in(1,e),\end{aligned} \tag{19}$$

$$\begin{aligned}\min_{e^{1/4}\le t\le e^{3/4}}H(t,s)&\ge\gamma_2(s)\max_{1\le t\le e}H(t,s)\\&=\gamma_2(s)H(s,s),\quad \textit{for } s\in(1,e).\end{aligned} \tag{20}$$

Proof. (1) and (2) are evident from the expression of $G(t,s)$ and $H(t,s)$. Since, for any fixed number $s\in[1,e]$, $G(t,s)$ is an increasing function on $[1,e]$ and $H(t,s)$ is a decreasing function on $[s,e]$ and increasing on $[1,s]$, we get (3). To prove (4), suppose that

$$\gamma_1(s) = \begin{cases} \dfrac{(1/4)^{\alpha-1}(1-\log s)^{\alpha-2}-(1/4-\log s)^{\alpha-1}}{(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}}, & s\in\left(1,e^{1/4}\right], \\ \dfrac{(1/4)^{\alpha-1}(1-\log s)^{\alpha-2}}{(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}}, & s\in\left[e^{1/4},e\right), \end{cases} \quad (21)$$

and put

$$p_1(t,s)=\frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}-(\log(t/s))^{\alpha-1}}{\Gamma(\alpha)},$$
$$p_2(t,s)=\frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}}{\Gamma(\alpha)}. \quad (22)$$

The monotonicity of $G(t,s)$ gives

$$\min_{e^{1/4}\le t\le e^{3/4}} G(t,s) = \begin{cases} p_1\left(e^{1/4},s\right), & s\in\left(1,e^{1/4}\right], \\ p_2\left(e^{1/4},s\right), & s\in\left[e^{1/4},e\right), \end{cases}$$
$$= \begin{cases} \dfrac{1}{\Gamma(\alpha)}\left[\left(\dfrac{1}{4}\right)^{\alpha-1}(1-\log s)^{\alpha-2} -\left(\dfrac{1}{4}-\log s\right)^{\alpha-1}\right], & s\in\left(1,e^{1/4}\right], \\ \dfrac{1}{\Gamma(\alpha)}\left(\dfrac{1}{4}\right)^{\alpha-1}(1-\log s)^{\alpha-2}, & s\in\left[e^{1/4},e\right), \end{cases}$$
$$\max_{1\le t\le e} G(t,s) = G(e,s) = \frac{1}{\Gamma(\alpha)}\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right], \quad s\in(1,e). \quad (23)$$

Which implies that (19) holds.

Similarly, by writing

$$q_1(t,s)=\frac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}-(\log(t/s))^{\beta-1}}{\Gamma(\beta)},$$
$$q_2(t,s)=\frac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}}{\Gamma(\beta)} \quad (24)$$

and applying the monotonicity of $H(t,s)$, we have

$$\min_{e^{1/4}\le t\le e^{3/4}} H(t,s) = \begin{cases} q_1\left(e^{3/4},s\right), & s\in\left(1,e^{1/4}\right], \\ \min\left\{q_1\left(e^{3/4},s\right), q_2\left(e^{1/4},s\right)\right\}, & s\in\left[e^{1/4},e^{3/4}\right], \\ q_2\left(e^{1/4},s\right), & s\in\left[e^{3/4},e\right), \end{cases}$$
$$= \begin{cases} q_1\left(e^{3/4},s\right), & s\in(1,r], \\ q_2\left(e^{1/4},s\right), & s\in[r,e), \end{cases} \quad (25)$$
$$= \begin{cases} \dfrac{1}{\Gamma(\beta)}\left[\left(\dfrac{3}{4}\right)^{\beta-1}(1-\log s)^{\beta-1} -\left(\dfrac{3}{4}-\log s\right)^{\beta-1}\right], & s\in(1,r], \\ \dfrac{1}{\Gamma(\beta)}\left(\dfrac{1}{4}\right)^{\beta-1}(1-\log s)^{\beta-1}, & s\in[r,e), \end{cases}$$
$$\max_{1\le t\le e} H(t,s) = H(s,s) = \frac{1}{\Gamma(\beta)}\left[\log s(1-\log s)\right]^{\beta-1}, \quad s\in(1,e),$$

where $e^{1/4}<r<e^{3/4}$ is the unique solution of the equation

$$\left[\frac{3}{4}(1-\log s)\right]^{\beta-1}-\left(\frac{3}{4}-\log s\right)^{\beta-1}=\left[\frac{1}{4}(1-\log s)\right]^{\beta-1}. \quad (26)$$

Hence, setting

$$\gamma_2(s) = \begin{cases} \dfrac{[(3/4)(1-\log s)]^{\beta-1}-(3/4-\log s)^{\beta-1}}{[\log s(1-\log s)]^{\beta-1}}, & s\in(1,r], \\ \dfrac{1}{(4\log s)^{\beta-1}}, & s\in[r,e), \end{cases} \quad (27)$$

we obtain (20). This completes the proof of Lemma 7. □

Let $X=C[1,e]$, $\|u\|=\max_{1\le t\le e}|u(t)|$. we define the cone $P=\{u\in X\mid u(t)\ge 0\}$ and the bounded closed set $U=\{u\in X\mid 0\le u(t)\le K\}$.

The operator $T: X\to X$ is defined as the following form:

$$Tu(t)=Y(t) = \int_1^e G(t,s)\,\varphi_q\left(\int_1^e H(s,\tau)\,f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}. \quad (28)$$

Evidently, the solutions of boundary value problem (5) are the corresponding fixed points of the operator T.

Lemma 8. *Suppose that $T : P \to X$ is an operator as above; then $T : P \to P$ is completely continuous.*

Proof. It is easy to see that $T : P \to P$ is continuous. Let $\Omega \subset P$ be a bounded set; then there is a positive constant $A > 0$ such that $\|u\| \le A$ for any $u \in \Omega$. Write $B = \max_{1\le t\le e, 0\le u\le A} f(t, u(t)) + 1$. For any $u \in \Omega$, we have

$$|Tu(t)| = \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \le B^{q-1}\int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\right)\frac{d\tau}{\tau} < +\infty, \tag{29}$$

which shows that $T\Omega$ is uniformly bounded.

Next, the continuity of $G(t,s)$ implies that, for any $\varepsilon > 0$, there exists a constant δ such that, for any $t_1, t_2 \in [1,e]$, if $|t_1 - t_2| < \delta$, then

$$\left|G(t_1,s) - G(t_2,s)\right| < \frac{\varepsilon}{B^{q-1}\varphi_q\left(\int_1^e H(\tau,\tau)(d\tau/\tau)\right)}. \tag{30}$$

Therefore, for any $u \in \Omega$,

$$\begin{aligned}\left|Tu(t_1) - Tu(t_2)\right| &= \left|\int_1^e (G(t_1,s) - G(t_2,s)) \times \varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\right| \\ &\le B^{q-1}\int_1^e \left|G(t_1,s) - G(t_2,s)\right| \times \varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} < \varepsilon.\end{aligned} \tag{31}$$

That is, $T\Omega$ is equicontinuity. By the means of Arzela-Ascoli theorem [26], we have that $T : P \to P$ is completely continuous. This completes the proof of Lemma 8. □

In the final part of this section, we list the following basic properties of the p-Laplacian operator.

Lemma 9. (1) *If $1 < p < 2$, $xy > 0$, and $|x|, |y| \ge m > 0$, then*

$$\left|\varphi_p(x) - \varphi_p(y)\right| \le (p-1)m^{p-2}\left|x - y\right|. \tag{32}$$

(2) *If $p > 2$, $|x|, |y| \le M$, then*

$$\left|\varphi_p(x) - \varphi_p(y)\right| \le (p-1)M^{p-2}\left|x - y\right|. \tag{33}$$

3. Proofs of the Main Results

In this section, first, we consider the existence of the solutions of problem (5).

Theorem 10. *If $[\max_{1\le t\le e, 0\le u\le K} f(t,u)]^{q-1}\int_1^e G(e,s)(ds/s)\varphi_q(\int_1^e H(\tau,\tau)(d\tau/\tau)) \le K$, then the boundary value problem (5) has at least one positive solution.*

Proof. For any $u \in U$, by the assumption as above and the nonnegativeness of $G(t,s)$, $H(t,s)$, and $f(t,u)$, we have

$$\begin{aligned}0 \le Tu(t) &= \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \left[\max_{1\le t\le e, 0\le u\le K} f(t,u)\right]^{q-1} \times \int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \\ &\le K.\end{aligned} \tag{34}$$

Therefore, T is a mapping from U to U. This, combined with the continuity of $G(t,s)$, $H(t,s)$, and $f(t,u)$, implies that $T : U \to U$ is continuous.

Let $\Omega \subset U$ be a bounded set; then there exists a positive constant A such that $\|u\| \le A$ for any $u \in \Omega$. So we have, for any $u \in \Omega$,

$$\begin{aligned}|Tu(t)| &= \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \left[\max_{1\le t\le e, 0\le u\le A} f(t,u)\right]^{q-1} \times \int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \\ &\le \left[\max_{1\le t\le e, 0\le u\le K} f(t,u)\right]^{q-1} \times \int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \le K \le +\infty.\end{aligned} \tag{35}$$

Therefore, $T\Omega$ is uniformly bounded.

Since $G(t,s)$ is continuous, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ satisfying that, for any $t_1, t_2 \in [1,e]$ and $|t_1 - t_2| < \delta$,

$$\left|G(t_1,s) - G(t_2,s)\right| < \frac{\varepsilon}{\left[\max_{1\le t\le e, 0\le u\le A} f(t,u) + 1\right]^{q-1}\varphi_q\left(\int_1^e H(\tau,\tau)(d\tau/\tau)\right)}. \tag{36}$$

Then, for any $u \in \Omega$,

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| &\le \int_1^e |G(t_1,s) - G(t_2,s)| \\ &\quad\times \varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \left[\max_{1\le t\le e,\, 0\le u\le A} f(t,u) + 1\right]^{q-1} \\ &\quad\times \varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \\ &\quad\times \int_1^e |G(t_1,s) - G(t_2,s)|\frac{ds}{s} < \varepsilon,\end{aligned} \tag{37}$$

which shows that $T\Omega$ is equicontinuous. By Arzela-Ascoli theorem [26], $T : U \rightarrow U$ is a completely continuous operator. It follows from Lemma 3 that T has a fixed point u in U. That is, problem (5) has at least one positive solution. This completes the proof of Theorem 10. □

Let us denote

$$\begin{aligned}\Lambda_1 &= \left(\int_{e^{1/4}}^{e^{3/4}} \gamma_1(s) G(e,s)\frac{ds}{s}\right. \\ &\quad\left.\times \varphi_q\left(\int_{e^{1/4}}^{e^{3/4}} \gamma_2(\tau) H(\tau,\tau)\frac{d\tau}{\tau}\right)\right)^{-1}, \\ \Lambda_2 &= \left(\int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right)\right)^{-1}.\end{aligned} \tag{38}$$

Theorem 11. *Suppose that $f(t,u) : [1,e] \times [0,+\infty)$ is a continuous function and there exist two constants $r_2 > r_1 > 0$ satisfying that*

(i) $f(t,u) \ge \varphi_p(\Lambda_1 r_1)$, *for* $(t,u) \in [1,e] \times [0,r_1]$;

(ii) $f(t,u) \le \varphi_p(\Lambda_2 r_2)$, *for* $(t,u) \in [1,e] \times [0,r_2]$.

Then the boundary value problem (5) *has at least one positive solution u which satisfies that $r_1 \le \|u\| \le r_2$.*

Proof. Let $\Omega_1 = \{u \in P \mid \|u\| < r_1\}$. For any $u \in \partial\Omega_1$, we have $0 \le u(t) \le r_1$ for $t \in [1,e]$. By the assumption (i), for any $t \in [e^{1/4}, e^{3/4}]$,

$$\begin{aligned}|Tu(t)| &= \int_1^e G(t,s) \\ &\quad\times \varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \Lambda_1 r_1 \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \Lambda_1 r_1 \int_{e^{1/4}}^{e^{3/4}} G(t,s)\varphi_q\left(\int_{e^{1/4}}^{e^{3/4}} H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \Lambda_1 r_1 \int_{e^{1/4}}^{e^{3/4}} \gamma_1(s) G(e,s) \\ &\quad\times \varphi_q\left(\int_{e^{1/4}}^{e^{3/4}} \gamma_2(\tau) H(\tau,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} = r_1 = \|u\|.\end{aligned} \tag{39}$$

Hence,

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in \partial\Omega_1. \tag{40}$$

Similarly, let $\Omega_2 := \{u \in P \mid \|u\| < r_2\}$. For any $u \in \partial\Omega_2$, we get $0 \le u(t) \le r_2$, $t \in [1,e]$. It follows from (ii) that for any $t \in [1,e]$,

$$\begin{aligned}|Tu(t)| &= \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \Lambda_2 r_2 \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \Lambda_2 r_2 \int_1^e G(e,s)\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= r_2 = \|u\|.\end{aligned} \tag{41}$$

Therefore,

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \partial\Omega_2. \tag{42}$$

By Lemmas 4 and 8, T has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$. Therefore, the boundary value problem (5) has one positive solution in $\overline{\Omega}_2 \setminus \Omega_1$. This completes the proof of Theorem 11. □

Corollary 12. *Suppose that $f(t,u) : [1,e] \times [0,+\infty)$ is a continuous function and there exist two constants $r_2 > r_1 > 0$ satisfying that*

(i) $f(t,u) \le \varphi_p(\Lambda_2 r_1)$, *for* $(t,u) \in [1,e] \times [0,r_1]$;

(ii) $f(t,u) \ge \varphi_p(\Lambda_1 r_2)$, *for* $(t,u) \in [1,e] \times [0,r_2]$.

Then the boundary value problem (5) *has at least one positive solution u which satisfies that $r_1 \le \|u\| \le r_2$.*

Proof. The proof of Corollary 12 is similar to the one of Theorem 11. So we omit the detail. □

Theorem 13. *Suppose that $f(t,u):[1,e]\times[0,+\infty)$ is a positive continuous function and there exists a constant $r>0$ such that*

$$r>\left[\max_{1\le t\le e,0\le u\le r} f(t,u)+1\right]^{q-1} \times\int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right). \quad (43)$$

Then the boundary value problem (5) *has at least one positive solution.*

Proof. Let

$$E=\{u\in P:\|u\|<r\}. \quad (44)$$

From Lemma 8, we know $T:\overline{E}\to P$ is completely continuous. Assume that there exist $u\in\overline{E}$, $\lambda\in(0,1)$ such that $u=\lambda Tu$. Then we have

$$\begin{aligned}|u(t)|&=|\lambda Tu|\\ &\le\left|\int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\right|\\ &\le\left[\max_{1\le t\le e,0\le u\le r} f(t,u)+1\right]^{q-1}\\ &\quad\times\int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right).\end{aligned} \quad (45)$$

Thus,

$$\|u\|\le\left[\max_{1\le t\le e,0\le u\le r} f(t,u)+1\right]^{q-1} \times\int_1^e G(e,s)\frac{ds}{s}\varphi_q\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right). \quad (46)$$

By (43), we can imply that $\|u\|<r$, which means that $u\notin\partial E$. That is to say, there is no $u\in\partial E$ such that $u=\lambda Tu$ for some $\lambda\in(0,1)$. Therefore, by Lemma 5, we conclude that the problem (5) has at least one positive solution. This completes the proof of Theorem 13. □

Now we turn to the uniqueness of solution for boundary value problem (5).

Theorem 14. *Suppose that $p>2$. If there exists a nonnegative function g satisfying that*

(1) *for any $(t,u)\in[1,e]\times[0,+\infty)$, $f(t,u)\ge g(t)$;*

(2) $N=\int_{e^{1/4}}^{e^{3/4}}\gamma_2(\tau)H(\tau,\tau)g(\tau)(d\tau/\tau)>0$;

(3) *for any $t\in[1,e]$, $u,v\in[0,+\infty)$, $|f(t,u)-f(t,v)|\le L|u-v|$, where*

$$0<L<\left(\int_1^e G(e,s)\frac{ds}{s}\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}(q-1)N^{q-2}\right)^{-1}, \quad (47)$$

then the boundary value problem (5) *has a unique solution.*

Proof. Assume that u, v are two positive solutions of problem (5). It is easy to see that

$$\begin{aligned}&\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\\ &\quad\ge\int_{e^{1/4}}^{e^{3/4}}\gamma_2(\tau)H(\tau,\tau)g(\tau)\frac{d\tau}{\tau}=N>0;\end{aligned} \quad (48)$$

then by the fact $p>2$ (i.e., its dual exponent $1<q<2$) and Lemma 9, we have

$$\begin{aligned}|u(t)-v(t)|&\le\int_1^e G(t,s)(q-1)N^{q-2}\\ &\quad\times\left|\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}-\int_1^e H(s,\tau)f(\tau,v(\tau))\frac{d\tau}{\tau}\right|\frac{ds}{s}\\ &\le\int_1^e G(t,s)(q-1)N^{q-2}\\ &\quad\times\int_1^e H(s,\tau)\left|f(\tau,u(\tau))-f(\tau,v(\tau))\right|\frac{d\tau}{\tau}\frac{ds}{s}\\ &\le\int_1^e G(e,s)(q-1)N^{q-2}\\ &\quad\times\int_1^e H(\tau,\tau)L\|u-v\|\frac{d\tau}{\tau}\frac{ds}{s}\\ &=\int_1^e G(e,s)\frac{ds}{s}\\ &\quad\times\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}(q-1)N^{q-2}L\|u-v\|\\ &=\delta_1\|u-v\|,\end{aligned} \quad (49)$$

where $\delta_1=\int_1^e G(e,s)(ds/s)\int_1^e H(\tau,\tau)(d\tau/\tau)(q-1)N^{q-2}L$. So we can get

$$\|u-v\|\le\delta_1\|u-v\|. \quad (50)$$

By the third hypothesis, $0<\delta_1<1$, which implies that $u(t)=v(t)$. And this completes the proof of Theorem 14. □

By using the same way, we can prove the last one of our main uniqueness results.

Theorem 15. *Suppose that $1<p<2$. If there exists a nonnegative function h satisfying that*

(1) *for any $(t,u)\in[1,e]\times[0,+\infty)$, $f(t,u)\le h(t)$;*

(2) $M = \int_1^e H(\tau,\tau)h(\tau)(d\tau/\tau) > 0$;

(3) *for any* $t \in [1,e]$, $u, v \in [0,+\infty)$, $|f(t,u) - f(t,v)| \le L|u - v|$, *where*

$$0 < L < \left(\int_1^e G(e,s)\frac{ds}{s}\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}(q-1)M^{q-2}\right)^{-1}, \tag{51}$$

then the boundary value problem (5) *has a unique solution.*

4. Examples

In this section we give several examples to illustrate our main results.

Example 16. Consider the boundary value problem:

$$\begin{gathered} D^{3/2}\left(\varphi_{5/2}\left(D^{5/2}u(t)\right)\right) \\ = (1-\log t)^{1/2}\left(1+\sin^2 u\right), \quad 1 < t < e, \\ u(1) = u'(1) = u'(e) = 0, \\ D^{5/2}u(1) = D^{5/2}u(e) = 0. \end{gathered} \tag{52}$$

Then the boundary value problem has a unique positive solution.

Proof. Since $\alpha = 5/2$, $\beta = 3/2$, a straightforward calculation gives

$$\begin{aligned} \int_1^e G(e,s)\frac{ds}{s} &= \int_1^e \frac{(1-\log s)^{\alpha-2} - (1-\log s)^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)}\int_1^e (1-\log s)^{\alpha-2}\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_1^e (1-\log s)^{\alpha-1}\frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-2}dt \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}dt \\ &= \frac{1}{(\alpha-1)\Gamma(\alpha+1)} \\ &= \frac{1}{(3/2)\Gamma(7/2)} = \frac{16}{45\sqrt{\pi}}, \end{aligned}$$

$$\begin{aligned} \int_1^e H(\tau,\tau)\frac{d\tau}{\tau} &= \int_1^e \frac{(\log\tau)^{\beta-1}(1-\log\tau)^{\beta-1}}{\Gamma(\beta)}\frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma(\beta)}\int_0^1 t^{\beta-1}(1-t)^{\beta-1}dt \\ &= \frac{\Gamma(\beta)}{\Gamma(2\beta)} \\ &= \frac{\Gamma(3/2)}{\Gamma(3)} \\ &= \frac{\sqrt{\pi}}{4}. \end{aligned} \tag{53}$$

Taking $K = 1$, $f(t,u) = (1-\log t)^{1/2}(1+\sin^2 u)$, and $p = 5/2 > 2$ (its dual exponent $q = 5/3$), we have

$$\begin{gathered} \max_{1\le t\le e, 0\le u\le 1} f(t,u) = 1 + \sin^2 1 < 2, \\ \left[\max_{1\le t\le e, 0\le u\le 1} f(t,u)\right]^{(5/3)-1}\int_1^e G(e,s)\frac{ds}{s}\varphi_{5/3}\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \\ < 2^{2/3}\cdot\frac{16}{45\sqrt{\pi}}\cdot\left(\frac{\sqrt{\pi}}{4}\right)^{2/3} \approx 0.185 < 1. \end{gathered} \tag{54}$$

By Theorem 10, the boundary value problem (52) has at least one positive solution.

Choosing the nonnegative function $g(t) = (1-\log t)^{1/2}$, for any $(t,u) \in [1,e]\times[0,+\infty)$, we gain that $f(t,u) = (1-\log t)^{1/2}(1+\sin^2 u) \ge g(t)$. Then

$$\begin{aligned} N &= \int_{e^{1/4}}^{e^{3/4}} \gamma_2(\tau)H(\tau,\tau)g(\tau)\frac{d\tau}{\tau} \\ &= \int_{e^{1/4}}^{r}\frac{1}{\Gamma(3/2)}\left\{\left[\frac{3}{4}(1-\log\tau)\right]^{3/2-1}\right. \\ &\qquad \left. - \left(\frac{3}{4}-\log\tau\right)^{3/2-1}\right\}\cdot(1-\log\tau)^{1/2}\frac{d\tau}{\tau} \\ &\quad + \int_r^{e^{3/4}}\frac{1}{\Gamma(3/2)}\left(\frac{1}{4}\right)^{3/2-1}(1-\log\tau)^{3/2-1} \\ &\quad \cdot(1-\log\tau)^{1/2}\frac{d\tau}{\tau} \\ &\ge \int_r^{e^{3/4}}\frac{1}{\sqrt{\pi}}(1-\log\tau)\frac{d\tau}{\tau} \\ &= \frac{1}{2\sqrt{\pi}}(1-\log r)^2 - \frac{1}{32\sqrt{\pi}} \end{aligned}$$

$$= \frac{1}{2\sqrt{\pi}}\left(1 - 1 + \frac{\sqrt{3}}{6}\right)^2 - \frac{1}{32\sqrt{\pi}} = \frac{1}{96\sqrt{\pi}}, \tag{55}$$

where $r = e^{1-(\sqrt{3}/6)}$ is the solution of (26) when $\beta = 3/2$. For any $t \in [1, e]$, $u, v \in [0, +\infty)$, taking $L = 2$, we obtain

$$\begin{aligned} \left|f(t,u) - f(t,v)\right| &= (1 - \log t)^{1/2}\left|\sin^2 u - \sin^2 v\right| \\ &\leq 2|u - v| = L|u - v|. \end{aligned} \tag{56}$$

Thus,

$$\begin{aligned} &\left(\int_1^e G(e,s)\frac{ds}{s}\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}(q-1)N^{q-2}\right)^{-1} \\ &\quad = \left(\frac{16}{45\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{4}\cdot\frac{2}{3}N^{-1/3}\right)^{-1} \\ &\quad \geq \frac{135}{8}\left(\frac{1}{96\sqrt{\pi}}\right)^{1/3} \\ &\quad \approx 3.045 > 2 = L. \end{aligned} \tag{57}$$

By Theorem 14, the boundary value problem (52) has a unique solution. □

Example 17. Consider the following nonlinear boundary value problem:

$$\begin{gathered} D^{3/2}\left(\varphi_{3/2}\left(D^{5/2}u(t)\right)\right) = \log^{1/2} t \sin^2 u, \quad 1 < t < e, \\ u(1) = u'(1) = u'(e) = 0, \qquad D^{5/2}u(1) = D^{5/2}u(e) = 0. \end{gathered} \tag{58}$$

Then the boundary value problem has a unique positive solution.

Proof. Taking $r = 1$, since $\alpha = 5/2$, $\beta = 3/2$, $p = 3/2 < 2$ (its dual exponent $q = 3$) and $f(t,u) = \log^{1/2} t \sin^2 u$, we obtain

$$\max_{1\leq t\leq e, 0\leq u\leq 1} f(t,u) = \sin^2 1 < 1, \tag{59}$$

$$\begin{aligned} &\left[\max_{1\leq t\leq e, 0\leq u\leq 1} f(t,u) + 1\right]^{3-1}\int_1^e G(e,s)\frac{ds}{s}\varphi_3\left(\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}\right) \\ &\quad < 4\cdot\frac{16}{45\sqrt{\pi}}\cdot\frac{\pi}{16} \approx 0.158 < 1. \end{aligned} \tag{60}$$

By means of Theorem 13, the boundary value problem (58) has at least one positive solution.

Taking the nonnegative function $h(t) = \log^{1/2} t$, for $(t,u) \in [1,e] \times [0,+\infty)$, it is easy to obtain $f(t,u) = \log^{1/2} t \sin^2 u \leq h(t)$ and

$$\begin{aligned} M &= \int_1^e H(\tau,\tau)h(\tau)\frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma(3/2)}\int_1^e \log\tau(1-\log\tau)^{1/2}\frac{d\tau}{\tau} = \frac{8}{15\sqrt{\pi}}. \end{aligned} \tag{61}$$

Choosing $L = 2$, for any $t \in [1,e]$, $u, v \in [0,+\infty)$, we have

$$\begin{aligned} &\left|f(t,u) - f(t,v)\right| = \log^{1/2} t\left|\sin^2 u - \sin^2 v\right| \\ &\qquad \leq 2|u - v| = L|u - v|, \\ &\left(\int_1^e G(e,s)\frac{ds}{s}\int_1^e H(\tau,\tau)\frac{d\tau}{\tau}(q-1)M^{q-2}\right)^{-1} \\ &\qquad = \left(\frac{16}{45\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{4}\cdot 2\cdot\frac{8}{15\sqrt{\pi}}\right)^{-1} \\ &\qquad \approx 18.694 > 2 = L. \end{aligned} \tag{62}$$

From Theorem 15, the boundary value problem (58) has a unique solution. □

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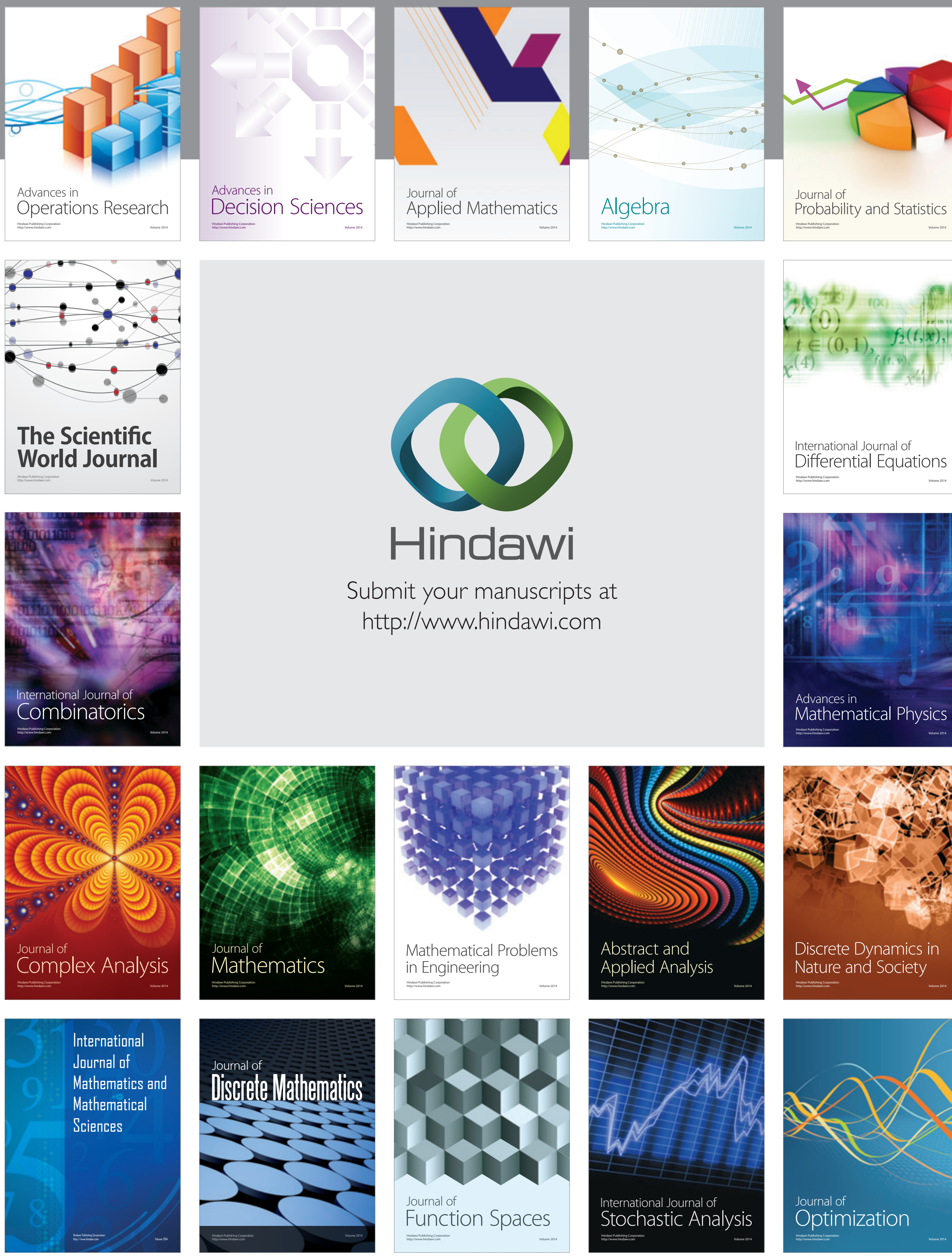
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