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Research Article

New Gronwall-Bellman Type Inequalities and Applications in the Analysis for Solutions to Fractional Differential Equations

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Some new Gronwall-Bellman type inequalities are presented in this paper. Based on these inequalities, new explicit bounds for the related unknown functions are derived. The inequalities established can also be used as a handy tool in the research of qualitative as well as quantitative analysis for solutions to some fractional differential equations defined in the sense of the modified Riemann-Liouville fractional derivative. For illustrating the validity of the results established, we present some applications for them, in which the boundedness, uniqueness, and continuous dependence on the initial value for the solutions to some certain fractional differential and integral equations are investigated.

1. Introduction

It is well known that the Gronwall-Bellman inequality [1, 2] and their generalizations can provide explicit bounds for solutions to differential and integral equations as well as difference equations. Recently, many authors have researched various generalizations of the Gronwall-Bellman inequality (e.g., see [3–26] and the references therein). These Gronwall-Bellman type inequalities established have proved to be useful in the research of boundedness, global existence, uniqueness, stability, and continuous dependence of solutions to differential and integral equations as well as difference equations. However, in the research for the properties of solutions to some fractional differential and integral equations, the earlier inequalities established are inadequate to fulfill such analysis, and it is necessary to establish new Gronwall-Bellman type inequalities so as to obtain the desired result.

On the other hand, recently, Jumarie presented a new definition for the fractional derivative named the modified Riemann-Liouville fractional derivative (see [27, 28]). The modified Riemann-Liouville fractional derivative is defined by the following expression.

Definition 1. The modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\xi)^{-\alpha} \\ \quad \cdot \left(f(\xi) - f(0)\right) d\xi, & 0 < \alpha < 1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{1}$$

Definition 2. The Riemann-Liouville fractional integral of order α on the interval $[0, t]$ is defined by

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(1+\alpha)} \int_0^t f(s)(ds)^{\alpha} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \end{aligned} \tag{2}$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows (the interval concerned below is always defined by $[0, t]$):

(a) $D_t^{\alpha} t^r = (\Gamma(1+r)/\Gamma(1+r-\alpha)) t^{r-\alpha}$.

(b) $D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t)$.

(c) $D_t^{\alpha}f[g(t)] = f'_g[g(t)]D_t^{\alpha}g(t) = D_g^{\alpha}f[g(t)](g'(t))^{\alpha}$.

(d) $I^{\alpha}(D_t^{\alpha}f(t)) = f(t) - f(0)$.

(e) $I^{\alpha}(g(t)D_t^{\alpha}f(t)) = f(t)g(t) - f(0)g(0) - I^{\alpha}(f(t)D_t^{\alpha}g(t))$.

(f) The modified Riemann-Liouville derivative for a constant is zero.

The modified Riemann-Liouville derivative has many excellent characters in handling many fractional calculus problems. Many authors have investigated various applications of the modified Riemann-Liouville fractional derivative. For example, in [29–31], the authors seeked exact solutions for some types of fractional differential equations based on the modified Riemann-Liouville fractional derivative, and in [32], the modified Riemann-Liouville fractional derivative was used in fractional calculus of variations, where the authors considered the fractional basic problem with free boundary conditions as well as problems with isoperimetric and holonomic constraints in the calculus of variations. In [33], Khan et al. presented a fractional homotopy perturbation method (FHPM) for solving fractional differential equations of any fractional order based on the modified Riemann-Liouville fractional derivative. In [34–36], fractional variational iteration method based on the modified Riemann-Liouville fractional derivative was concerned. In [37], a fractional variational homotopy perturbation iteration method was proposed.

Motivated by the wide applications of the modified Riemann-Liouville fractional derivative, in this paper, we use this type of fractional derivative to establish some fractional Gronwall-Bellman type inequalities. Based on these inequalities and some basic properties of the modified Riemann-Liouville fractional derivative, we derive explicit bounds for unknown functions concerned in these inequalities. As for applications, we apply these inequalities to research qualitative properties such as the boundedness, uniqueness, and continuous dependence on initial data for solutions to some certain fractional differential and integral equations.

We organize the rest of this paper as follows. In Section 2, we present the main inequalities, and based on them derive explicit bounds for unknown functions in these inequalities. Then in Section 3, we apply the results established in Section 2 to research boundedness, uniqueness, and continuous dependence on initial data for the solution to some certain fractional differential and integral equations.

2. Main Results

Lemma 3 (see [38]). *Assume that $a \geq 0$, $p \geq q \geq 0$; and $p \neq 0$, then, for any $K > 0$, one has*

$$a^{q/p} \leq \frac{q}{p}K^{(q-p)/p}a + \frac{p-q}{p}K^{q/p}. \tag{3}$$

Lemma 4. *Let $\alpha > 0$, a, b, u be continuous functions defined on $t \geq 0$. Then for $t \geq 0$,*

$$D_t^{\alpha}u(t) \leq a(t) + b(t)u(t) \tag{4}$$

implies

$$\begin{aligned} u(t) &\leq u(0)\exp\left[\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}a(\tau) \\ &\quad \times \exp\left[-\int_{\tau^{\alpha}/\Gamma(1+\alpha)}^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau. \end{aligned} \tag{5}$$

Proof. By the properties (a), (b), and (c) we have the following observation:

$$\begin{aligned} &D_t^{\alpha}\left\{u(t)\exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]\right\} \\ &= \exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\quad \times D_t^{\alpha}u(t) + u(t)D_t^{\alpha} \\ &\quad \times \left\{\exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]\right\} \\ &= \exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\quad \times D_t^{\alpha}u(t) - b(t)u(t) \\ &\quad \times \exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\quad \times D_t^{\alpha}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\ &= \exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\quad \times \left[D_t^{\alpha}u(t) - b(t)u(t)\right] \\ &\leq a(t)\exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]. \end{aligned} \tag{6}$$

Substituting t with τ, fulfilling fractional integral of order α for (6) with respect to τ from 0 to t, we deduce that

$$\begin{aligned} &u(t)\exp\left[-\int_0^{t^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right] \\ &\leq u(0) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}a(\tau) \\ &\quad \times \exp\left[-\int_0^{\tau^{\alpha}/\Gamma(1+\alpha)} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau, \end{aligned} \tag{7}$$

which implies

$$u(t) \le \exp\left[\int_0^{t^\alpha/(\Gamma(1+\alpha))} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] \times \left\{u(0) + \frac{1}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} a(\tau) \times \exp\left[-\int_0^{\tau^\alpha/(\Gamma(1+\alpha))} b\left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] d\tau\right\}. \tag{8}$$

The desired result can be obtained subsequently. □

Lemma 5. *Suppose $\alpha > 0$, the functions u, a, b, g, and h are nonnegative continuous functions defined on $t \ge 0$, and a, b are nondecreasing, p, q, and r are constants with $p \ge q > 0$, $p \ge r > 0$. If the following inequality holds:*

$$u^p(t) \le a(t) + \frac{b(t)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} \times \left[g(s)u^q(s) + \int_0^s h(\xi)u^r(\xi)\,d\xi\right] ds, \quad t \ge 0, \tag{9}$$

then one has the following explicit estimate for u(t):

$$u(t) \le \left\{a(t)b(t) \times \exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)} H_2\left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] + \frac{b(t)}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} H_1(\tau) \times \exp\left[-b(t) \times \int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)} H_2 \times \left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] d\tau\right\}^{1/p}, \quad t \ge 0, \tag{10}$$

where

$$H_1(t) = g(t)\left(\frac{p-q}{p}K^{q/p}\right) + \int_0^t h(\xi)\left(\frac{p-r}{p}K^{r/p}\right) d\xi,$$
$$H_2(t) = g(t)\frac{q}{p}K^{(q-p)/p} + \int_0^t h(\xi)\frac{r}{p}K^{(r-p)/p} d\xi, \tag{11}$$

and $K > 0$ is an arbitrary constant.

Proof. Fix $T \ge 0$, and let $t \in [0, T]$. Denote

$$v(t) = a(T) + \frac{b(T)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1}\left[g(s)u^q(s) + \int_0^s h(\xi)u^r(\xi)\,d\xi\right] ds. \tag{12}$$

Then we have

$$u^p(t) \le v(t), \quad t \in [0, T]. \tag{13}$$

Since u, g, and h are continuous, then there exists a constant M such that $|g(t)u^q(t) + \int_0^t h(\xi)u^r(\xi)d\xi| \le M$ for $t \in [0, \varepsilon]$, where $\varepsilon > 0$. So for $t \in [0, \varepsilon]$, we have $|\int_0^t (t-s)^{\alpha-1}[g(s)u^q(s) + \int_0^s h(\xi)u^r(\xi)d\xi]ds| \le M\int_0^t (t-s)^{\alpha-1}ds = (M/\alpha)t^\alpha$. Then one can see $v(0) = a(T)$. Furthermore,

$$D_t^\alpha v(t) = b(T)\left[g(t)u^q(t) + \int_0^t h(\xi)u^r(\xi)\,d\xi\right] \le b(T)\left[g(t)v^{q/p}(t) + \int_0^t h(\xi)v^{r/p}(\xi)\,d\xi\right]. \tag{14}$$

By Lemma 3, we have

$$\begin{aligned} D_t^\alpha v(t) &\le b(T)\left[g(t)\left(\frac{q}{p}K^{(q-p)/p}v(t) + \frac{p-q}{p}K^{q/p}\right) + \int_0^t h(\xi)\left(\frac{r}{p}K^{(r-p)/p}v(\xi) + \frac{p-r}{p}K^{r/p}\right) d\xi\right] \\ &= b(T)\left[g(t)\left(\frac{p-q}{p}K^{q/p}\right) + \int_0^t h(\xi)\left(\frac{p-r}{p}K^{r/p}\right) d\xi\right] \\ &\quad + b(T)\left[g(t)\frac{q}{p}K^{(q-p)/p}v(t) + \int_0^t h(\xi)\frac{r}{p}K^{(r-p)/p}v(\xi)\,d\xi\right] \end{aligned}$$

$$\begin{aligned}&\leq b(T)\left[g(t)\left(\frac{p-q}{p}K^{q/p}\right)\right.\\&\qquad\left.+\int_0^t h(\xi)\left(\frac{p-r}{p}K^{r/p}\right)d\xi\right]\\&\quad+b(T)\left[g(t)\frac{q}{p}K^{(q-p)/p}\right.\\&\qquad\left.+\int_0^t h(\xi)\frac{r}{p}K^{(r-p)/p}d\xi\right]v(t)\\&=b(T)H_1(t)+b(T)H_2(t)v(t),\quad t\in[0,T],\end{aligned}\tag{15}$$

where $H_1(t)$, $H_2(t)$ are defined in (11). Applying Lemma 4 to (15), considering $v(0)=a(T)$, we get that

$$\begin{aligned}v(t)&\leq a(T)b(T)\\&\quad\times\exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)}H_2\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]+\frac{b(T)}{\Gamma(\alpha)}\\&\quad\times\int_0^t(t-\tau)^{\alpha-1}H_1(\tau)\\&\quad\times\exp\left[-b(T)\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)}H_2\right.\\&\quad\left.\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau,\\&\qquad t\in[0,T].\end{aligned}\tag{16}$$

Letting $t=T$ in (16) and considering $T\geq 0$ is arbitrary, after substituting T with t, we get that

$$\begin{aligned}v(t)&\leq a(t)b(t)\\&\quad\times\exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)}H_2\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]\\&\quad+\frac{b(t)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}H_1(\tau)\\&\quad\times\exp\left[-b(t)\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)}H_2\right.\\&\quad\left.\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau,\\&\qquad t\geq 0.\end{aligned}\tag{17}$$

Combining (13) and (17), we get (10). □

Remark 6. In Lemma 5, if a, b are not necessarily nondecreasing, then one can let $v(t)=(1/\Gamma(\alpha))\int_0^t(t-s)^{\alpha-1}[g(s)u^q(s)+\int_0^s h(\xi)u^r(\xi)d\xi]ds$ instead in the proof, and following in a similar process, obtain another explicit bound for $u(t)$:

$$\begin{aligned}u(t)&\leq\left\{a(t)+\frac{b(t)}{\Gamma(\alpha)}\right.\\&\quad\times\int_0^t(t-\tau)^{\alpha-1}\widehat{H}_1(\tau)\\&\quad\times\exp\left[-\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)}\widehat{H}_2\right.\\&\quad\left.\left.\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau\right\}^{1/p},\\&\qquad t\geq 0,\end{aligned}\tag{18}$$

where

$$\begin{aligned}\widehat{H}_1(t)&=g(t)\left(\frac{p-q}{p}K^{q/p}+\frac{q}{p}K^{(q-p)/p}a(t)\right)\\&\quad+\int_0^t h(\xi)\left(\frac{p-r}{p}K^{r/p}+\frac{r}{p}K^{(r-p)/p}a(\xi)\right)d\xi,\\\widehat{H}_2(t)&=b(t)\left[g(t)\frac{q}{p}K^{(q-p)/p}+\int_0^t h(\xi)\frac{r}{p}K^{(r-p)/p}d\xi\right].\end{aligned}\tag{19}$$

Remark 7. We note that if we take $g(t)\equiv 1$, $h(t)\equiv 0$, and $p=q=1$, then the inequality (9) in Lemma 5 reduces to the inequality in [39, Theorem 1]. So the present inequality is of more general form than that in [39]. Furthermore, the explicit bounds obtained for the function $u(t)$ above are essentially different from that in [39].

Theorem 8. *Suppose that $\alpha>0$, the functions u, a, b, g, p, q, and r are defined as in Lemma 5, and $p\geq 1$, m is a nonnegative continuous function defined on $t\geq 0$. If the following inequality holds:*

$$\begin{aligned}u^p(t)&\leq a(t)+\int_0^t m(s)u^p(s)ds\\&\quad+\frac{b(t)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times\left[g(s)u^q(s)\right.\\&\quad\left.+\int_0^s h(\xi)u^r(\xi)d\xi\right]ds,\quad t\geq 0,\end{aligned}\tag{20}$$

then we have

$$\begin{aligned}&u(t)\\&\leq\exp\left(\frac{1}{p}\int_0^t m(s)ds\right)\end{aligned}$$

$$\times\left\{a(t)b(t)\exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)}\widetilde{H}_2\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]+\frac{b(t)}{\Gamma(\alpha)}\times\int_0^t(t-\tau)^{\alpha-1}\widetilde{H}_1(\tau)\times\exp\left[-b(t)\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)}\widetilde{H}_2\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau\right\}^{1/p},\quad t\geq 0, \tag{21}$$

where

$$\widetilde{H}_1(t)=\widetilde{g}(t)\left(\frac{p-q}{p}K^{q/p}\right)+\int_0^t\widetilde{h}(\xi)\left(\frac{p-r}{p}K^{r/p}\right)d\xi, \tag{22}$$

$$\widetilde{H}_2(t)=\widetilde{g}(t)\frac{q}{p}K^{(q-p)/p}+\int_0^t\widetilde{h}(\xi)\frac{r}{p}K^{(r-p)/p}d\xi, \tag{23}$$

$$\widetilde{g}(t)=g(t)\exp\left(\frac{q}{p}\int_0^s m(\tau)\,d\tau\right),\qquad \widetilde{h}(t)=h(t)\exp\left(\frac{r}{p}\int_0^t m(\tau)\,d\tau\right). \tag{24}$$

Proof. Denote $z^p(t)$ by $a(t)+(b(t)/\Gamma(\alpha))\int_0^t(t-s)^{\alpha-1}[g(s)u^q(s)+\int_0^s h(\xi)u^r(\xi)d\xi]ds$. Then

$$u^p(t)\leq z^p(t)+\int_0^t m(s)u^p(s)\,ds. \tag{25}$$

Since $a(t)$, $b(t)$ are both nondecreasing, then $z(t)$ is also nondecreasing, and subsequently we can deduce that

$$u^p(t)\leq z^p(t)\exp\left(\int_0^t m(s)\,ds\right). \tag{26}$$

Furthermore,

$$\begin{aligned}z^p(t)&\leq a(t)+\frac{b(t)}{\Gamma(\alpha)}\times\int_0^t(t-s)^{\alpha-1}\times\left[g(s)z^q(s)\times\exp\left(\frac{q}{p}\int_0^s m(\tau)\,d\tau\right)+\int_0^s h(\xi)z^r(\xi)\exp\left(\frac{r}{p}\int_0^\xi m(\tau)\,d\tau\right)d\xi\right]ds\\&=a(t)+\frac{b(t)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\times\left[\widetilde{g}(s)z^q(s)+\int_0^s\widetilde{h}(\xi)z^r(\xi)\,d\xi\right]ds,\end{aligned} \tag{27}$$

where $\widetilde{g}(t)$, $\widetilde{h}(t)$ are defined in (24). Then applying Lemma 5 to (23) yields

$$z(t)\leq\left\{a(t)b(t)\times\exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)}\widetilde{H}_2\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]+\frac{b(t)}{\Gamma(\alpha)}\times\int_0^t(t-\tau)^{\alpha-1}\widetilde{H}_1(\tau)\times\exp\left[-b(t)\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)}\widetilde{H}_2\times\left((s\Gamma(1+\alpha))^{1/\alpha}\right)ds\right]d\tau\right\}^{1/p},\quad t\geq 0, \tag{28}$$

where $\widetilde{H}_1(t)$, $\widetilde{H}_2(t)$ are defined in (22) and (23). Combining (26) and (28), we obtain the desired result. □

Lemma 9. *Suppose $\alpha>0$, the functions u, a, b, and g are defined as in Lemma 5, and ω is a nonnegative continuous function defined on $t\geq 0$ being nondecreasing, and $\omega(r)>0$ for $r>0$. Define $G(v)=\int_0^v(1/\omega(r))dr$, and assume $G(v)<\infty$ for $v<\infty$. If the following inequality holds:*

$$u(t)\leq a(t)+\frac{b(t)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\times g(s)\,\omega(u(s))\,ds,\quad t\geq 0, \tag{29}$$

then we have the following explicit estimate for $u(t)$:

$$u(t) \leq G^{-1} \times \left[G(a(t)) + \frac{b(t)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau\right], \quad t \geq 0. \tag{30}$$

Proof. Fix $T \geq 0$, and let $t \in [0, T]$. Denote

$$v(t) = a(T) + \delta + \frac{b(T)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, \omega(u(s))\, ds, \tag{31}$$

where $\delta > 0$. Then we have

$$u(t) \leq v(t), \quad t \in [0, T]. \tag{32}$$

Since u, g, and ω are continuous, then there exists a constant M such that $|g(t)\omega(u(t))| \leq M$ for $t \in [0, \varepsilon]$, where $\varepsilon > 0$. So for $t \in [0, \varepsilon]$, we have $|\int_0^t (t-s)^{\alpha-1} g(s)\omega(u(s))ds| \leq M\int_0^t (t-s)^{\alpha-1} ds = (M/\alpha)t^{\alpha}$. Then one can see $v(0) = a(T) + \delta$, and

$$D_t^{\alpha} v(t) = b(T)\, g(t)\, \omega(u(t)) \leq b(T)\, g(t)\, \omega(v(t)), \tag{33}$$

which implies

$$\frac{D_t^{\alpha} v(t)}{\omega(v(t))} \leq b(T)\, g(t). \tag{34}$$

That is,

$$D_t^{\alpha} G(v(t)) \leq b(T)\, g(t). \tag{35}$$

Substituting t with τ, fulfilling fractional integral of order α for (35) with respect to τ from 0 to t, we deduce that

$$G(v(t)) - G(v(0)) \leq \frac{b(T)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau, \tag{36}$$

which implies

$$v(t) \leq G^{-1} \times \left[G(a(T)+\delta) + \frac{b(T)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau\right], \tag{37}$$

and furthermore,

$$u(t) \leq G^{-1}\left[G(a(T)+\delta) + \frac{b(T)}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau\right], \quad t \in [0, T]. \tag{38}$$

Letting $t = T$ in (38), we get that

$$u(T) \leq G^{-1}\left[G(a(T)+\delta) + \frac{b(T)}{\Gamma(\alpha)} \times \int_0^T (T-\tau)^{\alpha-1} g(\tau)\, d\tau\right]. \tag{39}$$

Since $T > 0$ is arbitrary, substituting T with t in (39) and after letting $\delta \to 0$, we can obtain the desired result. □

Theorem 10. *Suppose $\alpha > 0$, the functions u, a, b, and g are defined as in Lemma 5, m is a nonnegative continuous function defined on $t \geq 0$, and ω is defined as in Lemma 9, and furthermore, assume that ω is submultiplicative; that is, $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta)$, α, $\beta \geq 0$. Define $G(v) = \int_0^v (1/\omega(r))dr$, and assume $G(v) < \infty$ for $v < \infty$. If the following inequality holds:*

$$u(t) \leq a(t) + \int_0^t m(s)\, u(s)\, ds + \frac{b(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, \omega(u(s))\, ds, \quad t \geq 0, \tag{40}$$

then we have the following explicit estimate for $u(t)$:

$$u(t) \leq G^{-1}\left[G(a(t)) + b(t)\frac{1}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} g(\tau) \times \exp\left(\int_0^{\tau} m(\xi)\, d\xi\right) d\tau\right], \quad t \geq 0. \tag{41}$$

Proof. Let $z(t) = a(t) + (b(t)/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} g(s)\omega(u(s))ds$. Then we have

$$u(t) \leq z(t) + \int_0^t m(s)\, u(s)\, ds. \tag{42}$$

Since $z(t)$ is nondecreasing, then furthermore we have

$$u(t) \leq z(t) \exp\left(\int_0^t m(s)\, ds\right). \tag{43}$$

So

$$z(t) \leq a(t) + \frac{b(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, \omega(z(s)) \times \exp\left(\int_0^s m(\xi)\, d\xi\right) ds. \tag{44}$$

Since ω is submultiplicative, then furthermore we get that

$$z(t) \le a(t) + \frac{b(t)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} g(s)\,\omega(z(s)) \times \omega\left(\exp\left(\int_0^s m(\xi)\,d\xi\right)\right) ds. \tag{45}$$

Noticing that the structure of (45) is similar to that of the inequality (29), a suitable application of Lemma 9 to (45) yields that

$$z(t) \le G^{-1}\left[G(a(t)) + b(t)\frac{1}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} g(\tau)\,\omega\left(\exp\left(\int_0^\tau m(\xi)\,d\xi\right)\right) d\tau\right]. \tag{46}$$

Combining (43) and (46) we can obtain the desired result. □

3. Applications

In this section, we apply the inequalities established above to research of boundedness, uniqueness, continuous dependence on the initial value for solutions to certain fractional differential and integral equations. Let us first consider the following IVP of fractional differential equation:

$$D_t^{0.5} u^3(t) = L\left(t, u(t), \int_0^t M(\xi, u(\xi))\,d\xi\right), \quad t \ge 0, \qquad u(0) = C, \tag{47}$$

where $u \in C([0,\infty), R)$, $M \in C(R\times R, R)$, and $L \in C([0,\infty)\times R^2, R)$.

Theorem 11. *Suppose that $u(t)$ is a solution of the IVP (47). If $|L(t,u,v)| \le g(t)|u|^3 + |v|$, and $|M(t,u)| \le h(t)|u|^3$, where g, h are nonnegative continuous functions on $[0,\infty)$, then we have the following estimate for $u(t)$:*

$$|u(t)| \le \sqrt[3]{|C| \exp\left[\int_0^{\sqrt{t}/\Gamma(1.5)} H_2\left((s\Gamma(1.5))^2\right) ds\right]}, \quad t \ge 0, \tag{48}$$

where $H_2(t) = g(t) + \int_0^t h(\xi)d\xi$.

Proof. Similar to [28, Equation (5.1)], we can obtain the equivalent integral form of the IVP (47) as follows:

$$u^3(t) = C + \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5} \times L\left(s, u(s), \int_0^s M(\xi, u(\xi))\,d\xi\right) ds. \tag{49}$$

So

$$\begin{aligned}
|u(t)|^3 &\le |C| + \frac{1}{\Gamma(0.5)} \times \int_0^s (t-s)^{-0.5} \times \left|L\left(s, u(s), \int_0^s M(\xi, u(\xi))\,d\xi\right)\right| ds \\
&\le |C| + \frac{1}{\Gamma(0.5)} \times \int_0^s (t-s)^{-0.5} \times \left[g(s)|u(s)|^3 + \left|\int_0^s M(\xi, u(\xi))\,d\xi\right|\right] ds \\
&\le |C| + \frac{1}{\Gamma(0.5)} \times \int_0^s (t-s)^{-0.5} \left[g(s)|u(s)|^3 + \int_0^s |M(\xi, u(\xi))|\,d\xi\right] ds \\
&\le |C| + \frac{1}{\Gamma(0.5)} \times \int_0^s (t-s)^{-0.5} \left[g(s)|u(s)|^3 + \int_0^s h(\xi)|u(\xi)|^3 d\xi\right] ds.
\end{aligned} \tag{50}$$

Then a suitable application of Lemma 5 to (50) (with $\alpha = 0.5$, $p = q = r = 3$) yields the desired result. □

Remark 12. At the end of the proof of Theorem 11, if we apply the result of Remark 6 instead of Lemma 5 to (50), then we obtain the following estimate:

$$|u(t)| \le \left(|C| \left\{1 + \frac{1}{\Gamma(\alpha)} \times \int_0^t (t-\tau)^{\alpha-1} H_2(\tau)\right.\right.$$

$$\times \exp\left[-\int_{\tau^{\alpha}/\Gamma(1+\alpha)}^{t^{\alpha}/\Gamma(1+\alpha)} H_2 \times \left(\left(s\Gamma(1+\alpha)\right)^{1/\alpha}\right) ds\right] d\tau\Bigg\}\Bigg)^{1/3}, \quad t \geq 0, \tag{51}$$

where $H_2(t)$ is defined as in Theorem 11.

Remark 13. In Theorem 11, if we change the conditions by $|L(t,u,v)| \leq g(t)|u| + |v|$, and $|M(t,u)| \leq h(t)|u|$, where $g,\ h$ are nonnegative continuous functions on $[0,\infty)$, then we can obtain the following estimate for $u(t)$:

$$|u(t)| \leq \Bigg\{|C| \exp\left[\int_0^{\sqrt{t}/\Gamma(1.5)} H_2\left(\left(s\Gamma(1.5)\right)^2\right) ds\right] + \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{-0.5} H_1(\tau) \times \exp\left[-\int_{\sqrt{\tau}/\Gamma(1.5)}^{\sqrt{t}/\Gamma(1.5)} H_2 \times \left(\left(s\Gamma(1.5)\right)^2\right) ds\right] d\tau\Bigg\}^{1/3}, \quad t \geq 0, \tag{52}$$

where $H_1(t) = (2/3)K^{1/3}[g(t) + \int_0^t h(\xi)d\xi]$, $H_2(t) = (1/3)K^{-2/3}[g(t) + \int_0^t h(\xi)d\xi]$, and $K > 0$ is an arbitrary constant.

Theorem 14. *If $|L(t,u_1,v_1) - L(t,u_2,v_2)| \leq g(t)|u_1^3 - u_2^3| + |v_1 - v_2|$, $|M(t,u_1) - M(t,u_2)| \leq h(t)|u_1^3 - u_2^3|$, where $g,\ h$ are nonnegative continuous functions defined on $[0,\infty)$, then the IVP (47) has at most one solution.*

Proof. Suppose that the IVP (47) has two solutions $u_1(t), u_2(t)$. Then similar to Theorem 11, we can obtain that

$$u_1^3(t) = C + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times L\left(s, u_1(s), \int_0^s M(\xi, u_2(\xi))\, d\xi\right) ds,$$

$$u_2^3(t) = C + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times L\left(s, u_2(s), \int_0^s M(\xi, u_2(\xi))\, d\xi\right) ds. \tag{53}$$

Furthermore,

$$u_1^3(t) - u_2^3(t) = \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \times \left[L\left(s, u_1(s), \int_0^s M(\xi, u_1(\xi))\, d\xi\right) - L\left(s, u_2(s), \int_0^s M(\xi, u_2(\xi))\, d\xi\right)\right] ds, \tag{54}$$

which implies

$$\begin{aligned} \left|u_1^3(t) - u_2^3(t)\right| &\leq \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \times \left|L\left(s, u_1(s), \int_0^s M(\xi, u_1(\xi))\, d\xi\right) - L\left(s, u_2(s), \int_0^s M(\xi, u_2(\xi))\, d\xi\right)\right| ds \\ &\leq \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \times \left[g(s)\left|u_1^3(s) - u_2^3(s)\right| + \left|\int_0^s M(\xi, u_1(\xi))\, d\xi - \int_0^s M(\xi, u_2(\xi))\, d\xi\right|\right] ds \end{aligned}$$

$$\leq \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \times \left[g(s) \left| u_1^3(s) - u_2^3(s) \right| + \int_0^s h(\xi) \left| u_1^3(\xi) - u_2^3(\xi) \right| d\xi \right] ds. \quad (55)$$

Treating $|u_1^3(t) - u_2^3(t)|$ as one whole function, a suitable application of Lemma 5 to (55) (with $\alpha = 0.5$) yields $|u_1^3(t) - u_2^3(t)| \leq 0$, which implies $u_1(t) \equiv u_2(t)$. So the proof is complete. □

Now we research the continuous dependence on the initial value for the IVP (47).

Theorem 15. *Suppose that $u(t)$ is the solution of the IVP (47), and $\tilde{u}(t)$ is the solution of following IVP of fractional differential equation:*

$$D_t^{0.5} \tilde{u}^3(t) = L\left(t, \tilde{u}(t), \int_0^t M(\xi, \tilde{u}(\xi))\, d\xi\right), \quad t \geq 0, \qquad \tilde{u}(0) = \widetilde{C}, \quad (56)$$

where $\tilde{u} \in C([0,\infty), R)$, $M \in C(R\times R, R)$, and $L \in C([0,\infty)\times R^2, R)$. If $|L(t,u_1,v_1) - L(t,u_2,v_2)| \leq g(t)|u_1^3 - u_2^3| + |v_1 - v_2|$, $|M(t,u_1) - M(t,u_2)| \leq h(t)|u_1^3 - u_2^3|$, where g, h are nonnegative continuous functions on $[0,\infty)$ and $|C - \widetilde{C}| < \varepsilon$, we have the following estimate:

$$|u(t)| \leq \varepsilon \left\{ \exp\left[\int_0^{\sqrt{t}/\Gamma(1.5)} H_2\left((s\Gamma(1.5))^2\right) ds \right] \right\}, \quad t \geq 0, \quad (57)$$

where $H_2(t) = g(t) + \int_0^t h(\xi) d\xi$.

Proof. For the IVP (56), we have the following integral form:

$$\tilde{u}^3(t) = \widetilde{C} + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times L\left(s, \tilde{u}(s), \int_0^s M(\xi, \tilde{u}(\xi))\, d\xi\right) ds. \quad (58)$$

So by (49) and (58), we deduce that

$$u^3(t) - \tilde{u}^3(t) = C - \widetilde{C} + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times \left[L\left(s, u(s), \int_0^s M(\xi, u(\xi))\, d\xi\right) - L\left(s, \tilde{u}(s), \int_0^s M(\xi, \tilde{u}(\xi))\, d\xi\right) \right] ds. \quad (59)$$

Furthermore,

$$\begin{aligned} \left| u^3(t) - \tilde{u}^3(t) \right| &\leq \left| C - \widetilde{C} \right| + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times \left| L\left(s, u(s), \int_0^s M(\xi, u(\xi))\, d\xi\right) - L\left(s, \tilde{u}(s), \int_0^s M(\xi, \tilde{u}(\xi))\, d\xi\right) \right| ds \\ &\leq \varepsilon + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times \left[g(s) \left| u^3(s) - \tilde{u}^3(s) \right| + \left| \int_0^s M(\xi, u(\xi))\, d\xi - \int_0^s M(\xi, \tilde{u}(\xi))\, d\xi \right| \right] ds \\ &\leq \varepsilon + \frac{1}{\Gamma(0.5)} \times \int_0^t (t-s)^{-0.5} \times \left[g(s) \left| u^3(s) - \tilde{u}^3(s) \right| + \int_0^s h(\xi) \left| u^3(\xi) - \tilde{u}^3(\xi) \right| d\xi \right] ds. \end{aligned} \quad (60)$$

A suitable application of Lemma 5 to (60) yields the desired result. □

Remark 16. Theorem 15 indicates that the solution of the IVP (47) depends continuously on the initial value $u(0) = C$.

Example 17. Consider the following fractional integral equation:

$$u^3(t) = C + \int_0^t L_1(s, u(s))\, ds + I^\alpha L_2\left(t, u(t), \int_0^t M(\xi, u(\xi))\, d\xi\right), \quad 0 < \alpha < 1. \tag{61}$$

Theorem 18. *Suppose that $u(t)$ is a solution of the fractional integral equation (61). If $|L_1(t,u)| \le m(t)|u|^3$, $|L_2(t,u,v)| \le g(t)|u| + |v|$, and $|M(t,u)| \le h(t)|u|$, where m, g, and h are nonnegative continuous functions on $[0,\infty)$, then we have the following estimate for $u(t)$:*

$$\begin{aligned} |u(t)| &\le \exp\left(\frac{1}{3}\int_0^t m(s)\, ds\right) \\ &\quad \times \Bigg\{ |C| \exp\left[\int_0^{t^\alpha/\Gamma(1+\alpha)} \widetilde{H}_2\left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \widetilde{H}_1(\tau) \\ &\quad \times \exp\left[-\int_{\tau^\alpha/\Gamma(1+\alpha)}^{t^\alpha/\Gamma(1+\alpha)} \widetilde{H}_2 \right. \\ &\quad \left. \times \left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right] d\tau \Bigg\}^{1/3}, \\ &\qquad t \ge 0, \end{aligned} \tag{62}$$

where

$$\begin{aligned} \widetilde{H}_1(t) &= \frac{2K^{1/3}}{3}\left[\widetilde{g}(t) + \int_0^t \widetilde{h}(\xi)\, d\xi\right], \\ \widetilde{H}_2(t) &= \frac{K^{-2/3}}{3}\left[\widetilde{g}(t) + \int_0^t \widetilde{h}(\xi)\, d\xi\right], \\ \widetilde{g}(t) &= g(t) \exp\left(\frac{1}{3}\int_0^s m(\tau)\, d\tau\right), \\ \widetilde{h}(t) &= h(t) \exp\left(\frac{1}{3}\int_0^t m(\tau)\, d\tau\right). \end{aligned} \tag{63}$$

Proof. From (61), we have

$$\begin{aligned} \left|u^3(t)\right| &\le |C| + \int_0^t \left|L_1(s, u(s))\right| ds + \left|I^\alpha L_2\left(t, u(t), \int_0^t M(\xi, u(\xi))\, d\xi\right)\right| \\ &\le |C| + \int_0^t m(s) u^3(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left|L_2\left(s, u(s), \int_0^s M(\xi, u(\xi))\, d\xi\right)\right| ds \\ &\le |C| + \int_0^t m(s) u^3(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left[g(s)|u(s)| + \int_0^s h(\xi)\left|u(\xi)\right| d\xi\right] ds. \end{aligned} \tag{64}$$

Then a suitable application of Theorem 8 (with $p = 3$, $q = r = 1$) to (64) yields the desired estimate (62). □

Theorem 19. *For the fractional integral equation (62), if $|L_1(t,u_1) - L_1(t,u_2)| \le m(t)|u_1^3 - u_2^3||L_2(t,u_1,v_1) - L_2(t,u_2,v_2)| \le g(t)|u_1^3 - u_2^3| + |v_1 - v_2|$, $|M(t,u_1) - M(t,u_2)| \le h(t)|u_1^3 - u_2^3|$, where m, g, and h are nonnegative continuous functions on $[0,\infty)$, then the solution of (61) depends continuously on the initial value C.*

Proof. Let $\widetilde{u}(t)$ be the solution of the following fractional integral equation

$$\widetilde{u}^3(t) = \widetilde{C} + \int_0^t L_1(s, \widetilde{u}(s))\, ds + I^\alpha L_2\left(t, \widetilde{u}(t), \int_0^t M(\xi, \widetilde{u}(\xi))\, d\xi\right), \quad 0 < \alpha < 1. \tag{65}$$

A combination of (61) and (65) yields

$$u^3(t) - \widetilde{u}^3(t) = C - \widetilde{C} + \int_0^t \left[L_1(s, u(s)) - L_1(s, \widetilde{u}(s))\right] ds$$

$$
\begin{aligned}
&+ I^{\alpha}\left[L_2\left(t, u(t), \int_0^t M(\xi, u(\xi))\, d\xi\right) - L_2\left(t, \widetilde{u}(t), \int_0^t M(\xi, \widetilde{u}(\xi))\, d\xi\right)\right] \\
&= C - \widetilde{C} + \int_0^t \left[L_1(s, u(s)) - L_1(s, \widetilde{u}(s))\right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left[L_2\left(s, u(s), \times \int_0^s M(\xi, u(\xi))\, d\xi\right) - L_2\left(s, \widetilde{u}(s), \int_0^s M(\xi, \widetilde{u}(\xi))\, d\xi\right)\right] ds.
\end{aligned} \tag{66}
$$

Furthermore, we have

$$
\begin{aligned}
&\left|u^3(t) - \widetilde{u}^3(t)\right| \\
&\leq \left|C - \widetilde{C}\right| + \int_0^t \left|L_1(s, u(s)) - L_1(s, \widetilde{u}(s))\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left|L_2\left(s, u(s), \int_0^s M(\xi, u(\xi))\, d\xi\right) - L_2\left(s, \widetilde{u}(s), \int_0^s M(\xi, \widetilde{u}(\xi))\, d\xi\right)\right| \\
&\leq \left|C - \widetilde{C}\right| + \int_0^t m(s) \left|u^3(s) - \widetilde{u}^3(s)\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left\{g(s) \left|u^3(s) - \widetilde{u}^3(s)\right| + \left|\int_0^s \left[M(\xi, u(\xi)) - M(\xi, \widetilde{u}(\xi))\right] d\xi\right|\right\} \\
&\leq \left|C - \widetilde{C}\right| + \int_0^t m(s) \left|u^3(s) - \widetilde{u}^3(s)\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left\{g(s) \left|u^3(s) - \widetilde{u}^3(s)\right| + \int_0^s h(\xi) \left|u^3(\xi) - \widetilde{u}^3(\xi)\right| d\xi\right\}.
\end{aligned} \tag{67}
$$

Then treating $|u^3(t) - \widetilde{u}^3(t)|$ as one whole function, applying Theorem 8 to (67), we get that

$$
\begin{aligned}
&\left|u^3(t) - \widetilde{u}^3(t)\right| \\
&\leq \left|C - \widetilde{C}\right| \exp\left(\int_0^t m(s)\, ds\right) \times \left\{\exp\left[\int_0^{t^{\alpha}/\Gamma(1+\alpha)} \widetilde{H}_2 \times \left((s\Gamma(1+\alpha))^{1/\alpha}\right) ds\right]\right\}, \quad t \geq 0,
\end{aligned} \tag{68}
$$

where

$$
\begin{aligned}
\widetilde{H}_2(t) &= \widetilde{g}(t) + \int_0^t \widetilde{h}(\xi)\, d\xi, \\
\widetilde{g}(t) &= g(t) \exp\left(\int_0^s m(\tau)\, d\tau\right), \\
\widetilde{h}(t) &= h(t) \exp\left(\int_0^t m(\tau)\, d\tau\right).
\end{aligned} \tag{69}
$$

So the continuous dependence on the initial value C for the solution of (61) can be obtained from (68). □

Remark 20. From the two examples presented above, one can see that the main results established in Section 2 are mainly used in the qualitative analysis as well as quantitative analysis of the solutions to some certain fractional differential or integral equations, such as the bound estimate, the number of the solutions, and the continuous dependence on the initial value for unknown solutions. On the other hand, by the variational iteration method and the homotopy perturbation method, approximate solutions for some fractional differential equations can be obtained (see the examples in [33–36], e.g.), while in few cases, the closed form of these approximate solutions can be obtained. So to this extent, we note that the starting point of establishing the main results in this paper is different from the variational iteration method and the homotopyperturbation method.

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