Research Article

Stability Bound Analysis and Synthesis for Singularly Perturbed Systems with Time-Varying Delay

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This paper addresses the problems of stability bound analysis and synthesis for singularly perturbed systems with time-varying delay. First, by constructing an appropriate Lyapunov-Krasovskii functional, a sufficient condition is derived for the system to be stable when the singular perturbation parameter is lower than a predefined upper bound which is referred to as the stability bound of the singularly perturbed system. The proposed criterion needs less computational cost than the existing ones. Then, a state feedback controller design method is proposed to achieve a prescribed stability bound, which can be applied to both standard and nonstandard singularly perturbed systems with time-varying delay. Finally, the effectiveness and merits of the proposed approach are shown through numerical examples.

1. Introduction

Singularly perturbed systems (SPSs) have attracted much attention due to their extensive applications in chemical processes, robots, aerospace engineering, power systems, magnetic-ball suspension systems, and so forth. Consequently, a vast amount of effort has been devoted to deriving stability criteria for SPSs with constant or time-varying delays. Therefore, the relevant theory and application research have attracted much attention [1]. Stability problem for SPSs is known as stability bound problem, which is referred to as the problem of determining the stability bound $\bar{\varepsilon}$ such that the system is stable for $\forall \varepsilon \in (0, \bar{\varepsilon}]$. Many results for stability bound problem of SPSs have been reported [2, 3].

Delay is a widespread physical phenomenon in practical systems [4, 5]. In [6–10], by a reduction technique, some significant advances have been achieved in analysis and synthesis of SPSs where delays are proportional to perturbed parameter. For a more general case in which time delay and perturbed parameter are independent, there are two main classes

of approaches: frequency domain techniques and Lyapunov-Krasovskii-functional-based approaches. In [11–14], frequency domain techniques were applied to establish stability criteria for SPSs with time delay in slow state. Lyapunov-Krasovskii-functional-based approaches which are usually expressed in terms of linear matrix inequalities (LMIs) are popular in recent years because LMI conditions can be easily verified by using convex optimization algorithms [2, 15]. In [16], robust stability of uncertain SPSs with state delays has been studied by Lyapunov-Krasovskii stability theory and LMI technique. In [17], a delay-dependent LMI criterion for the stability of singularly perturbed differential-difference systems has been derived. In [18], a stability criterion was proposed for uncertain SPSs with time delays. Based on the state space approach, the exact stability bound for discrete-time SPSs with multiple time delays was studied in [19]. In [20], the problem of stabilization for SPSs with multiple time delays was considered. Instead of applying the Lyapunov theory, the well-known comparison theorem and matrix measure are employed to investigate the problem.

It should be emphasized that most of the reported results on SPSs with time delay require the time delay to be time invariant [6–20]. Little works deal with the stability problem of SPSs with time-varying delays. In [22], a stability criterion was proposed for SPSs with time-varying delay by transforming the system into an equivalent descriptor system. But the proposed results cannot produce the stability bound of SPSs. In [23], an approach to estimate the stability bound of SPSs with time-varying delays was proposed under model transformation. However, this constructive method is not easy to perform. Stability bound problem of SPSs with time-varying delay was considered in [21] and an LMI-based method was proposed, which is easy to be solved by LMI solvers. This paper will follow the line of [21] and will try to improve the existing results.

In this paper, we consider the problems of stability bound analysis and design of SPSs with time-varying delay. One contribution of this paper is that through the use of a Lyapunov-Krasovskii functional, the obtained results have turned out to be with less computational cost than the existing methods. Another contribution is that the proposed methods do not depend on the system decomposition and thus they can be applied to both standard and nonstandard SPSs with time-varying delay. The presented examples show that the obtained methods are of less computational cost than some existing ones, which results from newly developed Lyapunov-Krasovskii functionals.

2. Main Results

Consider the following SPSs with tine-varying delay:

$$E(\varepsilon)\dot{x}(t) = Ax(t) + Dx(t - d(t)) + Bu(t),$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0), \tag{1}$$

where $x(t) \in R^n$ is the system state, and $u(t) \in R^m$ is the control input. $\phi(t)$ is initial condition of the system. $E(\varepsilon) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix} \in R^{n \times n}$, $A \in R^{n \times n}$, $D \in R^{n \times n}$, and $B \in R^{n \times m}$ are known constant matrices. $d(t)$ is the time-varying delay and satisfies

$$0 \le d(t) \le \tau, \quad \dot{d}(t) \le \mu < 1, \tag{2}$$

where τ and μ are known constants.

Condition (2) is common in practice and is widely used in the investigations on analysis and design of systems subject to time-varying delays [24–26]. In this paper, we aim at providing stability bound analysis and design method for system (1). Some useful lemmas are provided for subsequent technical development of the paper.

Lemma 1 (see [1]). *Given $\bar{\varepsilon} > 0$, symmetric matrices S_1, S_2, and S_3, if*

$$S_1 \ge 0,$$
$$S_1 + \bar{\varepsilon}S_2 > 0, \tag{3}$$
$$S_1 + \bar{\varepsilon}S_2 + \bar{\varepsilon}^2 S_3 > 0,$$

hold, then

$$S_1 + \varepsilon S_2 + \varepsilon^2 S_3 > 0, \quad \forall \varepsilon \in (0, \bar{\varepsilon}]. \tag{4}$$

Lemma 2 (see [1]). *If there exist matrices Z_i ($i = 1, 2, \ldots, 5$) of appropriate dimensions with $Z_i = Z_i^T$ ($i = 1, 2, 3, 4$) satisfying the following LMIs:*

$$Z_1 > 0,$$
$$\begin{bmatrix} Z_1 + \bar{\varepsilon}Z_3 & \bar{\varepsilon}Z_5^T \\ \bar{\varepsilon}Z_5 & \bar{\varepsilon}Z_2 \end{bmatrix} > 0, \tag{5}$$
$$\begin{bmatrix} Z_1 + \bar{\varepsilon}Z_3 & \bar{\varepsilon}Z_5^T \\ \bar{\varepsilon}Z_5 & \bar{\varepsilon}Z_2 + \bar{\varepsilon}^2 Z_4 \end{bmatrix} > 0,$$

then

$$E(\varepsilon)Z(\varepsilon) = (E(\varepsilon)Z(\varepsilon))^T = Z^T(\varepsilon)E(\varepsilon) > 0, \quad \forall \varepsilon \in (0, \bar{\varepsilon}], \tag{6}$$

where

$$Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}, \qquad E(\varepsilon) = \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix}. \tag{7}$$

2.1. Stability Analysis. Consider the following SPS with time-varying delay:

$$E(\varepsilon)\dot{x}(t) = Ax(t) + Dx(t - d(t)),$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0). \tag{8}$$

Theorem 3. *Given $\bar{\varepsilon} > 0$, system (8) is asymptotically stable for $\forall \varepsilon \in (0, \bar{\varepsilon}]$, and any $d(t)$ satisfying (2), if there exist symmetric positive-definite matrices $Q > 0$, $M > 0$, and matrices Z_i ($i = 1, \ldots, 5$) with $Z_i = Z_i^T$ ($i = 1, 2, 3, 4$) satisfying the following LMIs:*

$$Z_1 > 0,$$
$$\begin{bmatrix} Z_1 + \bar{\varepsilon}Z_3 & \bar{\varepsilon}Z_5^T \\ \bar{\varepsilon}Z_5 & \bar{\varepsilon}Z_2 \end{bmatrix} > 0, \tag{9}$$
$$\begin{bmatrix} Z_1 + \bar{\varepsilon}Z_3 & \bar{\varepsilon}Z_5^T \\ \bar{\varepsilon}Z_5 & \bar{\varepsilon}Z_2 + \bar{\varepsilon}^2 Z_4 \end{bmatrix} > 0,$$

$$\begin{bmatrix} A^T Z(0) + Z^T(0)A + Q + \tau A^T MA & Z^T(0)D + \tau A^T MD \\ * & -(1-\mu)Q + \tau D^T MD \end{bmatrix}$$
$$< 0,$$

$$\begin{bmatrix} A^T Z(\bar{\varepsilon}) + Z^T(\bar{\varepsilon})A + Q + \tau A^T MA & Z^T(\bar{\varepsilon})D + \tau A^T MD \\ * & -(1-\mu)Q + \tau D^T MD \end{bmatrix}$$
$$< 0,$$

$$\tag{10}$$

where $Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}$.

Proof. Define a quadratic Lyapunov-Krasovskii functional as follows:

$$V(x_t) = x^T(t) E(\varepsilon) Z(\varepsilon) x(t)$$
$$+ \int_{t-d(t)}^{t} x^T(s) Q x(s) \, ds$$
$$+ \int_{-\tau}^{0} \int_{t-d(t)+\theta}^{t} (E(\varepsilon) \dot{x}(s))^T M E(\varepsilon) \dot{x}(s) \, ds \, d\theta,$$
$$(11)$$

where $Q > 0$, $M > 0$.

By Lemma 2, LMIs (9) imply that

$$E(\varepsilon) Z(\varepsilon) = Z^T(\varepsilon) E(\varepsilon) > 0, \quad \forall \varepsilon \in (0, \bar{\varepsilon}]. \quad (12)$$

Thus $V(x_t)$ is a positive definite Lyapunov-Krasovskii functional.

By Lemma 1, it follows from (10) that

$$\begin{bmatrix} A^T Z(\varepsilon) + Z^T(\varepsilon) A + Q + \tau A^T M A & Z^T(\varepsilon) D + \tau A^T M D \\ * & -(1-\mu)Q + \tau D^T M D \end{bmatrix}$$
$$< 0. \quad (13)$$

Taking the derivative of $V(x_t)$ along the trajectories of the system (8), we have

$$\dot{V}(x_t)\big|_{(8)}$$
$$= \frac{d}{dt}\left(x^T(t) E(\varepsilon) Z(\varepsilon) x(t)\right)$$
$$+ \frac{d}{dt}\left(\int_{t-d(t)}^{t} x^T(s) Q x(s) \, ds\right)$$
$$+ \frac{d}{dt}\left(\int_{-\tau}^{0} \int_{t-d(t)+\theta}^{t} (E(\varepsilon) \dot{x}(s))^T M E(\varepsilon) \dot{x}(s) \, ds \, d\theta\right). \quad (14)$$

It is easy to show that

$$\frac{d}{dt}\left(x^T(t) E(\varepsilon) Z(\varepsilon) x(t)\right)$$
$$= \frac{d}{dt}\left(x^T(t) E(\varepsilon)\right) Z(\varepsilon) x(t)$$
$$+ x^T(t) E(\varepsilon) \frac{d}{dt}(Z(\varepsilon) x(t))$$
$$= \dot{x}^T(t) E(\varepsilon) Z(\varepsilon) x(t)$$
$$+ x^T(t) E(\varepsilon) Z(\varepsilon) \dot{x}(t)$$
$$= (E(\varepsilon) \dot{x}(t))^T Z(\varepsilon) x(t)$$
$$+ x^T(t) Z^T(\varepsilon) (E(\varepsilon) \dot{x}(t))$$
$$= (Ax(t) + Dx(t-d(t)))^T Z(\varepsilon) x(t)$$
$$+ x^T(t) Z^T(\varepsilon) (Ax(t) + Dx(t-d(t)))$$

$$= x^T(t)\left(A^T Z(\varepsilon) + Z^T(\varepsilon) A\right) x(t)$$
$$+ (Dx(t-d(t)))^T Z(\varepsilon) x(t)$$
$$+ (Z(\varepsilon) x(t))^T Dx(t-d(t))$$
$$= x^T(t)\left(A^T Z(\varepsilon) + Z^T(\varepsilon) A\right) x(t)$$
$$+ 2x^T(t) Z^T(\varepsilon) Dx(t-d(t)),$$

$$\frac{d}{dt}\left(\int_{t-d(t)}^{t} x^T(s) Q x(s) \, ds\right)$$
$$= x^T(t) Q x(t) - \left(1 - \dot{d}(t)\right) x^T$$
$$\times (t-d(t)) Q x(t-d(t))$$
$$\leq x^T(t) Q x(t) - (1-\mu) x^T$$
$$\times (t-d(t)) Q x(t-d(t)),$$

$$\frac{d}{dt}\left(\int_{-\tau}^{0} \int_{t-d(t)+\theta}^{t} (E(\varepsilon) \dot{x}(s))^T M E(\varepsilon) \dot{x}(s) \, ds \, d\theta\right)$$
$$= \tau(E(\varepsilon) \dot{x}(t))^T M E(\varepsilon) \dot{x}(t) - \left(1 - \dot{d}(t)\right)$$
$$\times \int_{-\tau}^{0} (E(\varepsilon) \dot{x}(t-d(t)+\theta))^T$$
$$\times M E(\varepsilon) \dot{x}(t-d(t)+\theta) \, d\theta$$
$$\leq \tau(E(\varepsilon) \dot{x}(t))^T M E(\varepsilon) \dot{x}(t) - (1-\mu)$$
$$\times \int_{-\tau}^{0} (E(\varepsilon) \dot{x}(t-d(t)+\theta))^T$$
$$\times M E(\varepsilon) \dot{x}(t-d(t)+\theta) \, d\theta. \quad (15)$$

Since $M > 0$, we have

$$\int_{-\tau}^{0} (E(\varepsilon) \dot{x}(t-d(t)+\theta))^T M E(\varepsilon) \dot{x}(t-d(t)+\theta) \, d\theta > 0,$$

$$\frac{d}{dt}\left(\int_{-\tau}^{0} \int_{t-d(t)+\theta}^{t} (E(\varepsilon) \dot{x}(s))^T M E(\varepsilon) \dot{x}(s) \, ds \, d\theta\right)$$
$$\leq \tau(E(\varepsilon) \dot{x}(t))^T M E(\varepsilon) \dot{x}(t). \quad (16)$$

Therefore,

$$\dot{V}(x_t)\big|_{(8)} \leq x^T(t)\left(A^T Z(\varepsilon) + Z^T(\varepsilon) A\right) x(t)$$
$$+ 2x^T(t) Z^T(\varepsilon) Dx(t-d(t))$$
$$+ x^T(t) Q x(t) - (1-\mu) x^T$$
$$\times (t-d(t)) Q x(t-d(t))$$
$$+ \tau(E(\varepsilon) \dot{x}(t))^T M E(\varepsilon) \dot{x}(t)$$
$$= \xi^T(t) H(\varepsilon) \xi(t), \quad (17)$$

where $\xi(t) = (x^T(t) \quad x^T(t-d))^T$,

$H(\varepsilon)$

$$= \begin{bmatrix} A^T Z(\varepsilon) + Z^T(\varepsilon) A + Q + \tau A^T M A & Z^T(\varepsilon) D + \tau A^T M D \\ * & -(1-\mu)Q + \tau D^T M D \end{bmatrix}. \tag{18}$$

From (13), we get $H(\varepsilon) < 0$, which yields

$$\xi^T(t) H(\varepsilon) \xi(t) < 0. \tag{19}$$

Therefore, $\dot{V}(x_t)|_{(8)} < 0$.

Hence, system (8) is asymptotically stable for all $\varepsilon \in (0, \bar{\varepsilon}]$. $\square$

Remark 4. By Theorem 3, an estimate of the stability bound can be obtained. However, the results in [16–18, 22] are only sufficient conditions for the system to be stable for small enough singular perturbation parameter.

By Theorem 3, it is easy to see that the following corollary holds.

Corollary 5. *Given $\bar{\varepsilon} > 0$, system (8) is asymptotically stable for $\forall \varepsilon \in (0, \bar{\varepsilon}]$ and any $d(t)$ satisfying $d(t) \geq 0$ and $\dot{d}(t) \leq \mu < 1$, if there exist symmetric positive-definite matrices $Q > 0$, and matrices Z_i $(i = 1, \ldots, 5)$ with $Z_i = Z_i^T$ $(i = 1, 2, 3, 4)$ satisfying the following LMIs:*

$$Z_1 > 0,$$

$$\begin{bmatrix} Z_1 + \bar{\varepsilon} Z_3 & \bar{\varepsilon} Z_5^T \\ \bar{\varepsilon} Z_5 & \bar{\varepsilon} Z_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} Z_1 + \bar{\varepsilon} Z_3 & \bar{\varepsilon} Z_5^T \\ \bar{\varepsilon} Z_5 & \bar{\varepsilon} Z_2 + \bar{\varepsilon}^2 Z_4 \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} A^T Z(0) + Z^T(0) A + Q & Z^T(0) D \\ * & -(1-\mu)Q \end{bmatrix} < 0,$$

$$\begin{bmatrix} A^T Z(\bar{\varepsilon}) + Z^T(\bar{\varepsilon}) A + Q & Z^T(\bar{\varepsilon}) D \\ * & -(1-\mu)Q \end{bmatrix} < 0,$$

where $Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}$.

Remark 6. In [21], stability bound problem was considered for SPSs with time delay, and LMI-based results were proposed. A new Lyapunov-Krasovskii functional is used to derive the analysis method. It is known that computational cost of LMI conditions increases in direct proportion to the number of decision variables and lines of the LMI conditions. In this sense, when the results of [21] are specialized to study stability of system (1), they need more computational cost than Theorem 3 and Corollary 5 of the present paper as will be illustrated by an example in the next section. In addition, the proposed Lyapunov-Krasovskii functional is expected to be used to study the problems of robust stability and H_∞ control for singularly perturbed systems with time-varying delay.

Remark 7. When $\varepsilon = 0$, system (8) becomes a singular system [27, 28]. Stability of singular systems with time-varying delay has been considered by many researchers (see, e.g., [4]). Theorem 3 and Corollary 5 present sufficient conditions for system (8) to be stable when $\varepsilon \in (0, \bar{\varepsilon}]$, which excludes the case $\varepsilon = 0$. In the future, we will study the stability problem of system (8) for $\varepsilon \in [0, \bar{\varepsilon}]$, which is also an interesting problem.

2.2. State Feedback Controller Design. In this subsection, we will design a state feedback controller for system (1) to achieve a given stability bound. A state feedback controller under consideration is in the form of

$$u(t) = Kx(t), \tag{21}$$

where $K \in R^{m \times n}$ is the controller gain to be designed. Then the closed-loop system is as follows:

$$E(\varepsilon) \dot{x}(t) = (A + BK) x(t) + Dx(t - d(t)). \tag{22}$$

Theorem 8. *Given $\bar{\varepsilon} > 0$, if there exist matrices $Q > 0$, $\widetilde{K}$ and Z_i $(i = 1, 2, \ldots, 5)$ with $Z_i = Z_i^T$ $(i = 1, 2, 3, 4)$, satisfying the following LMIs:*

$$Z_1 > 0, \tag{23}$$

$$\begin{bmatrix} Z_1 + \bar{\varepsilon} Z_3 & \bar{\varepsilon} Z_5^T \\ \bar{\varepsilon} Z_5 & \bar{\varepsilon} Z_2 \end{bmatrix} > 0, \tag{24}$$

$$\begin{bmatrix} Z_1 + \bar{\varepsilon} Z_3 & \bar{\varepsilon} Z_5^T \\ \bar{\varepsilon} Z_5 & \bar{\varepsilon} Z_2 + \bar{\varepsilon}^2 Z_4 \end{bmatrix} > 0, \tag{25}$$

$$\begin{bmatrix} AZ(0) + B\widetilde{K} + Z^T(0) A^T + \widetilde{K} B^T + Q & DZ(0) & Z^T(0) A^T + \widetilde{K} B^T \\ * & -(1-\mu)Q & Z^T(0) D^T \\ * & * & -\tau^{-1}\overline{M} \end{bmatrix} < 0,$$

$$\begin{bmatrix} AZ(\bar{\varepsilon}) + B\widetilde{K} + Z^T(\bar{\varepsilon}) A^T + \widetilde{K} B^T + Q & DZ(\bar{\varepsilon}) & Z^T(\bar{\varepsilon}) A^T + \widetilde{K} B^T \\ * & -(1-\mu)Q & Z^T(\bar{\varepsilon}) D^T \\ * & * & -\tau^{-1}\overline{M} \end{bmatrix} < 0, \tag{26}$$

where $Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}$, then the closed-loop system (22) is asymptotically stable for $\forall \varepsilon \in (0, \bar{\varepsilon}]$. Under such conditions, the gain matrix of controller (21) is $K = \widetilde{K} Z^{-1}(\varepsilon)$.

Proof. Choose the following Lyapunov functional candidate:

$$V(x_t) = x^T(t) Z^{-T}(\varepsilon) E(\varepsilon) x(t)$$

$$+ \int_{t-d(t)}^{t} x^T(s) Z^{-T}(\varepsilon) QZ^{-1}(\varepsilon) x(s) ds$$

$$+ \int_{-\tau}^{0} \int_{t-d(t)+\theta}^{t} (E(\varepsilon) \dot{x}(s))^T M E(\varepsilon) \dot{x}(s) ds d\theta, \tag{27}$$

where $Q > 0$, $M > 0$.

Now, considering the derivative of the Lyapunov functional candidate along the solution of SPSs with respect to t, we obtain

$$\dot{V}\left(x_t\right)\big|_{(22)}$$
$$= \frac{d}{dt}\left(x^T(t)\,Z^{-T}\left(\varepsilon\right)E\left(\varepsilon\right)x(t)\right)$$
$$+ \frac{d}{dt}\left(\int_{t-d(t)}^{t} x^T(s)\,Z^{-T}\left(\varepsilon\right)QZ^{-1}\left(\varepsilon\right)x(s)\,ds\right)$$
$$+ \frac{d}{dt}\left(\int_{-\tau}^{0}\int_{t-d(t)+\theta}^{t}\left(E\left(\varepsilon\right)\dot{x}(s)\right)^T ME\left(\varepsilon\right)\dot{x}(s)\,ds\,d\theta\right). \tag{28}$$

By Lemma 2, LMIs (23), (24), and (25) imply that

$$E\left(\varepsilon\right)Z\left(\varepsilon\right) = \left(E\left(\varepsilon\right)Z\left(\varepsilon\right)\right)^T = Z^T\left(\varepsilon\right)E\left(\varepsilon\right), \tag{29}$$

which shows

$$Z^{-T}\left(\varepsilon\right)E\left(\varepsilon\right)Z\left(\varepsilon\right) = Z^{-T}\left(\varepsilon\right)Z^T\left(\varepsilon\right)E\left(\varepsilon\right) = E\left(\varepsilon\right). \tag{30}$$

Then

$$Z^{-T}\left(\varepsilon\right)E\left(\varepsilon\right) = E\left(\varepsilon\right)Z^{-1}\left(\varepsilon\right). \tag{31}$$

Consequently,

$$\frac{d}{dt}\left(x^T(t)\,Z^{-T}\left(\varepsilon\right)E\left(\varepsilon\right)x(t)\right)$$
$$= \dot{x}^T(t)\,Z^{-1}\left(\varepsilon\right)E\left(\varepsilon\right)x(t) + x^T(t)\,Z^{-T}\left(\varepsilon\right)E\left(\varepsilon\right)\dot{x}(t)$$
$$= \dot{x}^T(t)\,E\left(\varepsilon\right)Z^{-1}\left(\varepsilon\right)x(t) + \left(Z^{-1}\left(\varepsilon\right)x(t)\right)^T E\left(\varepsilon\right)\dot{x}(t)$$
$$= 2\left(Z^{-1}\left(\varepsilon\right)x(t)\right)^T E\left(\varepsilon\right)\dot{x}(t)$$
$$= 2\left(Z^{-1}\left(\varepsilon\right)x(t)\right)^T\left[(A+BK)\,x(t)+Dx(t-d(t))\right]$$
$$= 2x^T(t)\,Z^{-T}\left(\varepsilon\right)(A+BK)\,x(t)$$
$$\quad + 2x^T(t)\,Z^{-T}\left(\varepsilon\right)Dx(t-d(t)),$$

$$\frac{d}{dt}\left(\int_{t-d(t)}^{t} x^T(s)\,Z^{-T}\left(\varepsilon\right)QZ^{-1}\left(\varepsilon\right)x(s)\,ds\right)$$
$$= x^T(t)\,Z^{-T}(t)\,QZ^{-1}(t)\,x(t)$$
$$\quad - \left(1-\dot{d}(t)\right)x^T(t-d(t))\,Z^{-T}(t-d(t))$$
$$\quad \times QZ^{-1}(t-d(t))\,x(t-d(t))$$
$$\leq x^T(t)\,Z^{-T}(t)\,QZ^{-1}(t)\,x(t)$$
$$\quad - \left(1-\mu\right)x^T(t-d(t))\,Z^{-T}(t-d(t))$$
$$\quad \times QZ^{-1}(t-d(t))\,x(t-d(t)),$$

$$\frac{d}{dt}\left(\int_{-\tau}^{0}\int_{t-d(t)+\theta}^{t}\left(E\left(\varepsilon\right)\dot{x}(s)\right)^T ME\left(\varepsilon\right)\dot{x}(s)\,ds\,d\theta\right)$$
$$= \tau(E\left(\varepsilon\right)\dot{x}(t))^T ME\left(\varepsilon\right)\dot{x}(t)$$
$$\quad - \left(1-\dot{d}(t)\right)\int_{-\tau}^{0}\left(E\left(\varepsilon\right)\dot{x}(t-d(t)+\theta)\right)^T ME\left(\varepsilon\right)$$
$$\qquad \times \dot{x}(t-d(t)+\theta)\,d\theta$$
$$\leq \tau(E\left(\varepsilon\right)\dot{x}(t))^T ME\left(\varepsilon\right)\dot{x}(t)$$
$$\quad - \left(1-\mu\right)\int_{-\tau}^{0}\left(E\left(\varepsilon\right)\dot{x}(t-d(t)+\theta)\right)^T ME\left(\varepsilon\right)$$
$$\qquad \times \dot{x}(t-d(t)+\theta)\,d\theta$$
$$\leq \tau(E\left(\varepsilon\right)\dot{x}(t))^T ME\left(\varepsilon\right)\dot{x}(t)$$
$$= \tau\left[(A+BK)\,x(t)+Dx(t-d(t))\right]^T$$
$$\quad \times M\left[(A+BK)\,x(t)+Dx(t-d(t))\right]$$
$$= \tau x^T(t)\,(A+BK)^T M(A+BK)\,x(t)$$
$$\quad + \tau 2x^T(t)\,(A+BK)^T MDx^T(t-d(t))$$
$$\quad + \tau x^T(t-d(t))\,D^T MDx(t-d(t)). \tag{32}$$

Therefore, taking the derivative of $V(x_t)$ along the trajectories of the system (22), we have

$$\dot{V}\left(x_t\right)\big|_{(22)} \leq 2x^T(t)\,Z^{-T}\left(\varepsilon\right)(A+BK)\,x(t)$$
$$\quad + 2x^T(t)\,Z^{-T}\left(\varepsilon\right)Dx(t-d(t))$$
$$\quad - \left(1-\mu\right)x^T(t-d(t))\,Z^{-T}(t-d(t))$$
$$\quad \times QZ^{-1}(t-d(t))\,x(t-d(t))$$
$$\quad + x^T(t)\,Z^{-T}(t)\,QZ^{-1}(t)\,x(t)$$
$$\quad + \tau x^T(t)\,(A+BK)^T M(A+BK)\,x(t)$$
$$\quad + \tau 2x^T(t)\,(A+BK)^T MDx^T(t-d(t))$$
$$\quad + \tau x^T(t-d(t))\,D^T MDx(t-d(t))$$
$$= \xi^T(t)\,G\left(\varepsilon\right)\xi(t), \tag{33}$$

where $\xi(t) = \left(x^T(t) \quad x^T(t-d)\right)^T$,

$$G\left(\varepsilon\right) = \begin{bmatrix} \Xi & Z^{-T}\left(\varepsilon\right)D+\tau(A+BK)^T MD \\ * & \Pi \end{bmatrix}, \tag{34}$$

with

$$\Xi = Z^{-T}\left(\varepsilon\right)(A+BK)+(A+BK)^T Z^{-1}\left(\varepsilon\right)$$
$$\quad + \tau(A+BK)^T M(A+BK)+Z^{-T}\left(\varepsilon\right)QZ^{-1}\left(\varepsilon\right),$$
$$\Pi = -\left(1-\mu\right)Z^{-T}\left(\varepsilon\right)QZ^{-1}\left(\varepsilon\right)+\tau D^T MD. \tag{35}$$

By defining

$$Y = \begin{bmatrix} Z^{-T}(\varepsilon)(A+BK) + (A+BK)^T Z^{-1}(\varepsilon) + Z^{-T}(\varepsilon)QZ^{-1}(\varepsilon) & Z^{-T}(\varepsilon)D \\ * & -(1-\mu)Z^{-T}(\varepsilon)QZ^{-1}(\varepsilon) \end{bmatrix}, \tag{36}$$

algebraic manipulation gives

$$
\begin{aligned}
G(\varepsilon) &= Y + \begin{bmatrix} \tau(A+BK)^T M(A+BK) & \tau(A+BK)^T MD \\ * & \tau D^T MD \end{bmatrix} \\
&= Y + \begin{bmatrix} \tau(A+BK)^T M(A+BK) & \tau(A+BK)^T MD \\ * & \tau D^T MD \end{bmatrix} \\
&= Y + \begin{bmatrix} (A+BK)^T \\ D^T \end{bmatrix} \tau M \begin{bmatrix} A+BK & D \end{bmatrix} \\
&= Y - \begin{bmatrix} (A+BK)^T \\ D^T \end{bmatrix} \left(-\tau^{-1}M^{-1}\right)^{-1} \begin{bmatrix} A+BK & D \end{bmatrix}.
\end{aligned}
\tag{37}
$$

Now comes the validation that $G(\varepsilon) < 0$, by the Schur complement, which is equivalent to

$$\begin{bmatrix} \Theta & Z^{-T}(\varepsilon)D & (A+BK)^T \\ * & -(1-\mu)Z^{-T}(\varepsilon)QZ^{-1}(\varepsilon) & D^T \\ * & * & -\tau^{-1}M^{-1} \end{bmatrix} < 0, \tag{38}$$

where $\Theta = Z^{-T}(\varepsilon)(A+BK) + (A+BK)^T Z^{-1}(\varepsilon) + Z^{-T}(\varepsilon)QZ^{-1}(\varepsilon)$.

Pre- and postmultiplying (38) with $\mathrm{diag}\{Z^T(\varepsilon), Z^T(\varepsilon), I\}$ and its transpose, respectively, we obtain

$$
\begin{aligned}
&\begin{bmatrix} (A+BK)Z(\varepsilon) + [(A+BK)Z(\varepsilon)]^T + Q & DZ(\varepsilon) & [(A+BK)Z(\varepsilon)]^T \\ * & -(1-\mu)Q & Z^T(\varepsilon)D^T \\ * & * & -\tau^{-1}M^{-1} \end{bmatrix} \\
&= \begin{bmatrix} AZ(\varepsilon) + BKZ(\varepsilon) + Z^T(\varepsilon)A^T + [BKZ(\varepsilon)]^T + Q & DZ(\varepsilon) & Z^T(\varepsilon)A^T + [BKZ(\varepsilon)]^T \\ * & -(1-\mu)Q & Z^T(\varepsilon)D^T \\ * & * & -\tau^{-1}M^{-1} \end{bmatrix} < 0.
\end{aligned}
\tag{39}
$$

Defining

$$\widetilde{K} = KZ(\varepsilon), \qquad \widetilde{M} = M^{-1}, \tag{40}$$

we can obtain

$$\widetilde{G}(\varepsilon) = \begin{bmatrix} AZ(\varepsilon) + B\widetilde{K} + Z^T(\varepsilon)A^T + \widetilde{K}B^T + Q & DZ(\varepsilon) & Z^T(\varepsilon)A^T + \widetilde{K}B^T \\ * & -(1-\mu)Q & Z^T(\varepsilon)D^T \\ * & * & -\tau^{-1}\widetilde{M} \end{bmatrix} < 0. \tag{41}$$

It follows from (26) that $\widetilde{G}(0) < 0$ and $\widetilde{G}(\bar{\varepsilon}) < 0$. Then, by Lemma 1, we have that inequality (41) holds, that is, $\widetilde{G}(\varepsilon) < 0$, which is equivalent to $G(\varepsilon) < 0$. Then

$$\dot{V}(x_t)\big|_{(22)} < 0. \tag{42}$$

This implies that the closed-loop system (22) is asymptotically stable for $\forall \varepsilon \in (0, \bar{\varepsilon}]$ with the gain matrix $K = \widetilde{K}Z^{-1}(\varepsilon)$. This completes the proof. □

Remark 9. It follows from LMIs (23) and (24) that Z_1 and Z_2 are nonsingular matrices, which guarantees that $Z(0)$ is nonsingular. Then $K = \widetilde{K}Z^{-1}(\varepsilon)$ is well defined for $\forall \varepsilon \in (0, \bar{\varepsilon}]$.

Remark 10. In [20], the synthesis of SPSs with multiple time delays was considered. The proposed method depends on the separation of the original system into fast and slow subsystems and therefore cannot be applied to nonstandard SPSs. In [23], an LMI-based controller design method was proposed. But it did not consider the stability bound of the closed-loop system. Using Theorem 8, a stabilization bound can be

TABLE 1: Stability bounds of Example 11.

μ	0	0.2	0.4	0.6	0.65
$\bar{\varepsilon}$	0.4999	0.3924	0.2636	0.1027	0.0557

TABLE 2: Comparison of the number of decision variables and lines of the LMI conditions.

	Theorem 3	Theorem 3 of [21]	Corollary 5	Theorem 6 in [21]
Variables	11	61	8	9
Lines	17	78	15	18

achieved by the proposed state feedback controller. Moreover, it is easy to see that no system separation is required in our results and the conditions obtained in Theorem 8 have turned out to deal with both standard and nonstandard SPSs.

3. Examples

In this section, we provide two examples to demonstrate the validity and the advantage of the proposed results in this paper.

Example 11. Consider the following system with time-varying delay:

$$\dot{x}_1(t) = x_2(t) + x_1(t - d(t)),$$
$$\varepsilon \dot{x}_2(t) = -x_2(t) + 0.5 x_2(t - d(t)) - 2 x_1(t). \tag{43}$$

This system can be transformed into system (8) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{44}$$

Solving the LMI conditions of Theorem 3 with $\bar{\varepsilon} = 0.4482$, $\tau = 0.55$, and $\mu = 0.1$, we obtain

$$Z_1 = 3.5345, \qquad Z_2 = 1.8589, \qquad Z_3 = 4.0958,$$
$$Z_4 = 1.9564, \qquad Z_5 = 2.9219,$$
$$Q = \begin{bmatrix} 5.5894 & 1.3679 \\ 1.3679 & 1.4026 \end{bmatrix}, \qquad M = \begin{bmatrix} 0.0120 & -0.0040 \\ -0.0040 & 0.0130 \end{bmatrix}. \tag{45}$$

Then by Theorem 3, the system is stable for any delay satisfying $0 \le d(t) \le 5.5$, $\dot{d}(t) \le 0.1$, and any $\varepsilon \in (0, 0.4482]$. We found that the largest stability bound obtained by Theorem 3 of [21] is also 0.4482 when the involved time delay satisfies $0 \le d(t) \le 5.5$, $\dot{d}(t) \le 0.1$.

When the time delay involved in this example satisfies $d(t) \ge 0$ and $\dot{d}(t) \le \mu < 1$, by Corollary 5, the stability bounds for different μ are shown in Table 1. It can be seen that the results are the same to those given by Theorem 6 in [21].

From Table 2, Theorem 3 and Corollary 5 in the present paper need less computational cost than Theorems 3 and 6 of [21], respectively. Therefore, it can be seen that the newly developed methods can produce the same stability bound with reduced computational cost than the existing method in [21].

Example 12. We now apply the proposed approach in Theorem 8 to find a state feedback controller to stabilize system (1) with

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{46}$$

Letting $\tau = 5$, $\mu = 0.2$, and $\bar{\varepsilon} = 0.4$ and using Theorem 8, we obtain

$$Z_1 = 0.0439, \qquad Z_2 = 0.1801, \qquad Z_3 = 0.0689,$$
$$Z_4 = 0.0548, \qquad Z_5 = -0.1787,$$
$$\widetilde{K} = [0.1713 \quad -0.4163], \qquad M = \begin{bmatrix} 1.3566 & -0.3795 \\ -0.3795 & 2.1505 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.1119 & -0.0772 \\ -0.0772 & 0.2276 \end{bmatrix}. \tag{47}$$

Then the controller gain is as follows:

$$K = [0.1713 \quad -0.4163]$$
$$\times \begin{bmatrix} 0.0439 + 0.0689\varepsilon & -0.1787\varepsilon \\ -0.1787 & 0.1801 + 0.0548\varepsilon \end{bmatrix}^{-1}. \tag{48}$$

By Theorem 8, the closed-loop system is stable for any $0 \le d(t) \le 5$, $\dot{d}(t) \le 0.2$, and any $\varepsilon \in (0, 0.4)$.

This example shows that Theorem 8 can be applied to nonstandard SPSs with time-varying delay.

4. Conclusion

In this paper, we have investigated the problems of stability bound analysis and design for singularly perturbed systems with time-varying delay. The results extend and improve the existing works. A new form of Lyapunov-Krasovskii functional has been constructed to improve delay-dependent stability analysis and design methods for SPSs with time-varying delay. Using the analysis methods, the stability bound of the

SPSs can be computed. By the design methods, a given stability bound can be achieved. The proposed methods do not depend on the system decomposition and therefore can be applied to both standard and nonstandard SPSs with time-varying delays. The numerically examples have illustrated the advantages and effectiveness of the proposed methods.

Acknowledgments

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