

GRAPHS AND PROJECTIVE PLAINES IN 3-MANIFOLDS

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(Received March 17, 1986 and in revised form April 18, 1986)

ABSTRACT. Proper homotopy equivalent compact P^2 -irreducible and sufficiently large 3-manifolds are homeomorphic. The result is not known for irreducible 3-manifolds that contain 2-sided projective planes, even if one assumes the Poincaré conjecture. In this paper to such a 3-manifold M is associated a graph $G(M)$ that specifies how a maximal system of mutually disjoint non-isotopic projective planes is embedded in M , and it is shown that $G(M)$ is an invariant of the homotopy type of M . On the other hand it is shown that any given graph can be realized as $G(M)$ for infinitely many irreducible and boundary irreducible M .

As an application it is shown that any closed irreducible 3-manifold M that contains 2-sided projective planes can be obtained from a P^2 -irreducible 3-manifold and $P^2 \times S^1$ by removing a solid Klein bottle from each and gluing together the resulting boundaries; furthermore M contains an orientation preserving simple closed curve α such that any nontrivial Dehn surgery along α yields a P^2 -irreducible 3-manifold.

KEY WORDS AND PHRASES. P^2 -irreducible 3-manifolds, incompressible surfaces.
1986 AMS SUBJECT CLASSIFICATION CODE. 57N10.

1. INTRODUCTION

The classification theorem for compact P^2 -irreducible, sufficiently large 3-manifolds M and N asserts that if $(M, \partial M)$ is homotopy equivalent to $(N, \partial N)$, then M is homeomorphic to N [11], [3]. The assumption of being " P^2 -irreducible and sufficiently large" makes M and N aspherical which allows for modifications

of homotopy equivalences. Also, the assumption of being "irreducible" is needed to avoid the Poincaré conjecture.

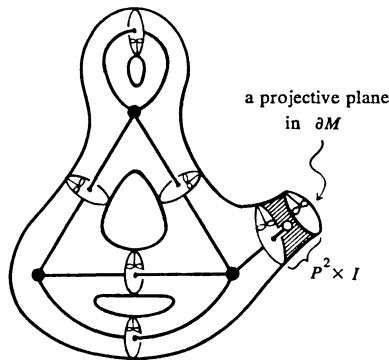
In this paper, we consider P^2 -containing 3-manifolds, that is, 3-manifolds that contain a 2-sided projective plane. Epstein [1] showed that any irreducible, compact, non-orientable 3-manifold with finite fundamental group is homotopy equivalent to $P^2 \times I$, and the Poincaré conjecture would imply that it is in fact homeomorphic to $P^2 \times I$.

As a first step in trying to prove a classification theorem modulo the Poincaré conjecture for irreducible P^2 -containing 3-manifolds M , we associate to M a P^2 -graph that specifies how a maximal system of mutually disjoint non-isotopic projective planes is embedded in M , and show that it is an invariant of homotopy type. On the other hand, it is shown that any given graph can be realized as a P^2 -graph of infinitely many irreducible and ∂ -irreducible 3-manifolds. As an application, it is shown that any closed irreducible P^2 -containing 3-manifold M can be obtained from a P^2 -irreducible 3-manifold and $P^2 \times S^1$ by removing a solid Klein bottle from each and gluing together the resulting boundaries.

2. P^2 -GRAPHS

Two disjoint 2-sided projective planes P_0 and P_1 in a 3-manifold M are parallel (pseudo-parallel resp.) if there is a submanifold Q of M homeomorphic (homotopy equivalent) to $P^2 \times I$ such that $\partial Q = P_0 \cup P_1$. A maximal set of pairwise disjoint non-parallel (non pseudo-parallel) 2-sided projective planes in $\text{Int } M$ is called a complete (pseudo-complete) system of projective planes in M . Since 2-sided projective planes are incompressible, it follows from Haken's Theorem [2] that a (pseudo-)complete system in a compact 3-manifold M is finite. Moreover if M is also irreducible, a complete system is uniquely determined up to ambient isotopy [9].

We define the (coloured) P^2 -graph $G(M)$ of an irreducible 3-manifold M as follows: Choose a vertex v_i in the interior of each component C_i of M cut open along a complete system \mathcal{P} . Let v_i be coloured white if $C_i \cong P^2 \times I$ and $\partial C_i \cap \partial M$ contains a projective plane; otherwise let v_i be coloured black. Join v_i and v_j by an edge if C_i and C_j meet along a common $P^2 \in \mathcal{P}$. The resulting graph $G(M)$ may be embedded in M so that each edge intersects its corresponding P^2 transversely in a single point. (See Figure 1.) Note that a white vertex must have degree 1.



In a similar way, we define the *pseudo P^2 -graph* $G'(M)$ by starting with a pseudo-complete system P' . More explicitly, let P be a complete system,

$$P = \{P_{1,1}, \dots, P_{1,\alpha}, P_{2,1}, \dots, P_{2,\alpha}, \dots, P_{m,1}, \dots, P_{m,\alpha_m}\}$$

where the $P_{i,j}$'s have been numbered so that $P_{i,k}$ is pseudo-parallel to $P_{i,1}$ for $1 \leq k, l \leq \alpha_i$, but $P_{i,k}$ is not pseudo-parallel to $P_{j,1}$ for $i \neq j$. We can also assume that the $P_{i,j}$'s are numbered in such a way that for each $i = 1, \dots, m$ the projective planes $P_{i,1}$ and P_{i,α_i} bound a submanifold \tilde{C}_i homotopy equivalent to $P^2 \times I$ in M that contains the other $P_{i,j}$ in its interior. Then $P' = \{P_{1,1}, P_{2,1}, \dots, P_{m,1}\}$ is a pseudo-complete system for M . Let M' be the quotient obtained from M by collapsing the components \tilde{C}_i onto $P_{i,1}$ ($i = 1, \dots, m$). The P' is a complete system in M' and clearly $G(M')$ is isomorphic to $G'(M)$ as graphs.

Note that $G(M)$ and $G'(M)$ are homeomorphic as topological spaces. However if there are 3-manifolds homotopy equivalent to but not homeomorphic to $P^2 \times I$ - (any such example would provide a counter example to the Poincaré conjecture) - then $G(M)$ is obtained from $G'(M)$ by subdividing edges, and the two graphs may not be isomorphic as graphs.

Observe also that the homomorphisms on fundamental groups induced by the inclusion $i: G(M) \rightarrow M$ are injective and that the natural projection $q: M \rightarrow M'$ is a homotopy equivalence.

We now show that the pseudo-graph of M is an invariant of the relative homotopy type of M :

THEOREM 1. Let M and N be compact, irreducible 3-manifolds and let $f: (N, \partial N) \rightarrow (M, \partial M)$ be a map such that $f_*: \pi_1(N) \rightarrow \pi_1(M)$ is an isomorphism. Then $G'(N)$ and $G'(M)$ are isomorphic as coloured graphs.

COROLLARY 2. If $f: (N, \partial N) \rightarrow (M, \partial M)$ is a homotopy equivalence then $G'(N) \cong G'(M)$.

For the proof, we need a generalization of the lemma on homotopy surgery of maps [11], [5, Lemma 6.5]:

LEMMA 3. Let M be a compact 3-manifold, X a p.l. k -manifold containing a properly embedded 2-sided p.l. $(k-1)$ -submanifold Y with $\ker(\pi_1(Y) \rightarrow \pi_1(X)) = 1$. Let $f: M \rightarrow X$ be a map. Then there are disjoint homotopy 3-balls B_1, \dots, B_n with B_i in $\text{Int } M$ or $B_i \cap \partial M = \partial B_i \cap \partial M$ a disk, and there is a map $g: M_0 = M - \cup_i B_i \rightarrow X$ such that

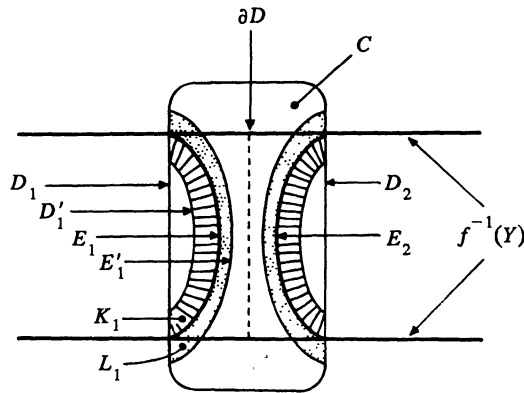
- (i) $g_* = (f|_{M_0})_*: \pi_1(M_0) \rightarrow \pi_1(X)$,
- (ii) each component of $g^{-1}(Y)$ is a properly embedded, 2-sided incompressible surface in M_0 , and
- (iii) g maps each fiber $p \times [-1,1]$ of a product neighbourhood $g^{-1}(Y) \times [-1,1]$ of $g^{-1}(Y)$ homeomorphically to the fiber $g(p) \times [-1,1]$ of a product neighborhood $Y \times [-1,1]$ of Y .

To see this, we follow the proof of Lemma 6.5 in [5] and indicate the modifications. By a homotopy, we may assume that $f^{-1}(X)$ is a system of 2-sided surfaces.

(1) If $f^{-1}(Y)$ contains a compressible 2-sphere F , then F bounds a homotopy 3-cell C in M and we let $M'_0 = \overline{M - U}$ where U is a smaller regular neighbourhood of C , and $g = f|M'_0$.

(2) If $f^{-1}(Y)$ contains a compressible 2-cell F then F bounds together with a disk on ∂M a 3-cell C in M and we proceed as in (1).

(3) If there is a compressible disk D in $\text{Int } M$ with $D \cap f^{-1}(Y) = \partial D$ and ∂D not contractible in $f^{-1}(Y)$, choose a regular neighbourhood C of D with $C \cap f^{-1}(Y)$ an annulus A properly embedded in C . Let D_1, D_2 be the disks in ∂C with $\partial A = \partial D_1 \cup \partial D_2$ and let E_1, E_2 be two disjoint disks properly embedded in C with $\partial E_i = \partial D_i$. Let K_i and L_i be relative collars on E_i in C such that $K_i \cup L_i$ is a relative bicollar of E_i in C and let $D'_i \cup E'_i$ be the relative boundary of $K_i \cup L_i$ in C (see Figure 2). The spheres $(D_i - D_i \cap K_i) \cup D'_i$ ($i = 1, 2$) and $(\partial C - D_1 \cup D_2 \cup L_1 \cup L_2) \cup E'_1 \cup E'_2$ bound balls B_i and B respectively in C . Let $M'_0 = \overline{M - B_1 \cup B_2 \cup B}$ and let $g|M - \overline{C} = f|M - \overline{C}$. Since $\ker(\pi_1(Y) \rightarrow \pi_1(X)) = 1$, we may extend $g|_{\partial E_i}$ to map E_i into Y and extending g to the bicollar $K_i \cup L_i$ of E_i into the product neighbourhood of Y we obtain $g: M'_0 \rightarrow X$ as required. We have $g_* = (f|M'_0)_*$ since $f(\partial E_i) = 0$ in X .



PROOF OF THEOREM 1. If $\pi_1(M) = \mathbb{Z}_2$ then both M and N are homotopy equivalent to $P^2 \times I$ and the assertion is trivial. Thus assume $\pi_1(M) \neq \mathbb{Z}_2$. Consider the map $f' = q \circ f$, where $q: M \rightarrow M'$ is the natural projection that collapses fake homotopy $P^2 \times I$'s in M onto a projective plane. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a complete system in M' (i.e., \mathcal{P} is a pseudo-complete system in M). By Lemma 3, there are 3-balls B_1, \dots, B_k in $\text{Int } N$ such that for $N_0 = \overline{N - \cup_i B_i}$ there is a map $g: (N_0, \partial N) \rightarrow (M', \partial M')$ with $g_*: \pi_1(N_0) \rightarrow \pi_1(M')$ an isomorphism and $g^{-1}(\mathcal{P})$ a system of incompressible surfaces. We may assume that $G'(N)$ is embedded in N_0 . For each component F of $g^{-1}(\mathcal{P})$ the map $(g|_F)_*: \pi_1(F) \rightarrow \pi_1(\mathcal{P})$ is injective (for suitably chosen base points). Since $g|_{\partial N} = f|_{\partial N}$ the component F is closed and therefore F is a projective plane. Suppose that $F_{i,1}, \dots, F_{i,\alpha_i}$ are the components of $g^{-1}(F_i)$. Then all the $g_*\pi_1(F_{i,j})$ are conjugate to $\pi_1(P_i)$ in $\pi_1(M')$, and hence $\pi_1(F_{i,j})$ is conjugate to $\pi_1(F_{i,1})$ in $\pi_1(N_0)$ ($1 \leq j, l \leq \alpha_i$). Thus the nontrivial loops carried by $F_{i,1}, \dots, F_{i,\alpha_i}$ are homotopic to each other in N_0 and it follows from Theorem 4.1 of [10] that there is

a submanifold Q_i in N_0 , homotopy equivalent to a punctured $P^2 \times I$, with $\partial Q_i = F_{i,1} \cup F_{i,\alpha_i}$ and containing the other $F_{i,j}$'s in its interior. Collapsing each Q_i in N_0 to $F_{i,1}$, we obtain a quotient map $p: N_0 \rightarrow N'_0$ that induces an isomorphism on fundamental groups (but note that this N'_0 may still contain fake homotopy $P^2 \times I$'s). Again we may assume that $G'(N)$ is embedded in N'_0 . We may assume that $g|_{F_{i,j}}$ is a homeomorphism [5, Theorem 13.1]. If we collapse Q_i so that $x \in F_{i,\alpha_i}$ is identified with $(g|_{F_{i,1}})^{-1}(g|_{F_{i,\alpha_i}})(x) \in F_{i,1}$, we obtain a map $g': (N'_0, \partial N) \rightarrow (M', \partial M)$ induced by g with $g'_*: \pi_1(N'_0) \rightarrow \pi_1(M')$ an isomorphism and such that $(g')^{-1}(P_i)$ is empty or consists of one projective plane F_i in N'_0 ($i = 1, \dots, m$). Suppose $(g')^{-1}(P_i)$ is empty. If F_i does not separate M' then g' factors as $N'_0 \rightarrow M' - U(P_i) \xrightarrow{\cong} M'$ (where $U(\dots)$ is a regular neighbourhood) and i_* would be an isomorphism, which cannot be. If P_i separates M' into M'_1 and M'_2 then g' maps N'_0 into M'_1 , say, and it follows that $\pi_1(M'_2) = \mathbb{Z}_2$, hence $\partial M'_2$ consists of P_i and another projective plane P'_i , homotopic to P_i . For the nontrivial loop β of P_i , $g_*^{-1}(\beta)$ is carried by a projective plane P in N'_0 . If P is not homotopic to a boundary component then P separates (by an argument similar to the above) N'_0 into N_1 and N_2 with $\pi_1(N_1) \neq \mathbb{Z}_2$ and the isomorphism g'_* from the free product of $\pi_1(N_1)$ and $\pi_1(N_2)$ with amalgamation over $\pi_1(P)$ would induce a splitting of M'_1 over a projective plane homotopic to P_i into two submanifolds each with fundamental group different from \mathbb{Z}_2 (by Swarup [10], Theorem 5.4), which cannot be. If P is homotopic to a boundary component, by a similar argument we would get the excluded case $\pi_1(M') = \mathbb{Z}_2$.

Therefore $(g')^{-1}(P_i) = F_i$, a projective plane in N'_0 . The nontrivial loops of F_i and F_j are not homotopic in N'_0 since their images are not homotopic in M' . Therefore (again by [10]) the system $\mathbb{F} = \{F_1, \dots, F_m\}$ is a pseudo-complete system in N'_0 . Also F_i is pseudo-parallel to a boundary component of N if and only if P_i is parallel to a component of ∂M .

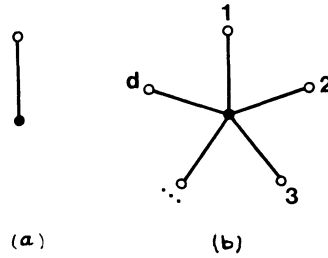
Now if M_j is a component of M' cut along P then $N_j = (g')^{-1}(M_j)$ is a component of N'_0 cut along \mathbb{F} . We may embed $G'(N)$ in N'_0 such that each edge intersects \mathbb{F} transversely in one point and such that g' maps each edge to an arc in M' that intersects P transversely in one point. Then for $G'(M)$ we may choose the graph $g'(C'(N))$ in M' and thus g' induces an isomorphism $G'(N) \rightarrow G'(M)$ of coloured graphs. This proves Theorem 1. \square

REMARK. We wish to thank J. Kalliongis and D. McCullough for pointing out a gap in the original proof of Theorem 1. Also it follows from their recent result (Theorem 5.1 of [12]) that Theorem 1 is in fact equivalent to Corollary 2.

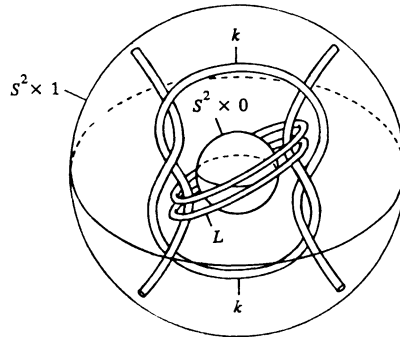
If M is closed then each component of M cut along a complete system has an even number of projective planes in its boundary, since the Euler characteristic of a compact 3-manifold is even. It follows that every vertex of $G(M)$ has even degree and hence that $G(M)$ is an Euler graph (i.e. a connected graph each of whose vertices is incident to an even number of edges). Conversely, given any Euler graph G , there are infinitely many distinct, closed irreducible 3-manifolds M with $G(M)$ isomorphic to G (see [9]). It is also shown in [9] that any given graph G can be

realized as $G(M)$ for some irreducible 3-manifold M with possible compressible boundary. We strengthen this in:

THEOREM 4. For any connected graph G with a given colouring of the vertices of degree 1, there are infinitely many distinct, irreducible and ∂ -irreducible 3-manifolds M such that $G(M)$ is isomorphic to G .



PROOF. We first construct such a 3-manifold corresponding to the graph of Figure 3(a). Let $\hat{M}_0 = S^2 \times I - \text{Int } U(k)$ and $\hat{M}_n = S^2 \times I - \text{Int } U(k \cup L)$, where U is a regular neighbourhood for the link $L = \bigcup_{i=1}^n l_i$ and the two properly embedded arcs k as shown in Figure 4. Here U is invariant under the antipodal map $\rho \times \text{id}: S^2 \times I \rightarrow S^2 \times I$. Let $g: \hat{M}_n \rightarrow M_n$ be the covering map of \hat{M}_n onto the quotient $M_n = \hat{M}_n / (\rho \times \text{id})$, then ∂M_n consists of one projective plane $g(S^2 \times 0)$, one nonorientable surface of genus 3 namely $F = g((S^2 \times 1 \cup U(k)) \cap \hat{M}_n)$, and n Klein bottles $K_i = g(\partial U(l_i) \cap \hat{M}_n)$. We claim that all boundary components of M_n are incompressible, that M_n is irreducible, and that $\{P_1^2\}$ is a complete system, where P_1^2 is a projective plane in $\text{Int } M_n$ parallel to $g(S^2 \times 0)$; it then follows that $G(M_n)$ is isomorphic to the graph of Figure 3(a).



Let \hat{M}_n be obtained from \hat{M}_n by capping of $S^2 \times 0$ by a 3-ball. Lambert shows in [6] that $\partial \hat{M}_0$ is incompressible in \hat{M}_0 . Therefore $\partial \hat{M}_0$ is incompressible in M_0 and $F = g(\partial \hat{M}_0)$ is incompressible in M_0 . Now \hat{M}_0 is the complementary space of the prime tangle k (see [7]) and is in particular irreducible. To see that \hat{M}_n is irreducible, close off the components of k by arcs in $S^2 \times 1$ to get a link \bar{k} and observe that the linking number of l_i and each component of \bar{k} is ± 1 . Any 2-sphere S in $\text{Int } \hat{M}_n$ separates \hat{M}_n into two components $C_j = B_j \cap \hat{M}_n$ ($j = 1, 2$),

where B_j is a 3-ball in S^3 . If l_j lies in B_1 , then so does \tilde{k} . Hence $L \cup \tilde{k} \subset B_1$ and $B_2 \cap (L \cup \tilde{k}) = \emptyset$; therefore $C_2 \cap \hat{M}_n = \emptyset$ and $C_2 = B_2$ in \hat{M}_n . It follows that any 2-sphere S in $\text{Int } \tilde{M}_n$ either bounds a 3-ball or is parallel to $S^2 \times 0$. This implies that M_n is irreducible, that $\{P_1^2\}$ is a complete system, and that each K_i is incompressible.

We now realize the graph of Figure 3(b) that consists of one vertex of degree $d \geq 2$ and d white vertices each of degree 1. Let B_i be a copy of M_1 for $i = 1, \dots, d-1$ and let B_d be a copy of M_{n+d} . Denote the Klein bottles of ∂B_i by K_i ($i = 1, \dots, n+d$). Construct the 3-manifold $B(d, n)$ from the B_j by identifying K_i with the Klein bottle of ∂B_j ($i = 1, \dots, d-1$). Since each K_i is incompressible, any 2-sided projective plane in $B(d, n)$ can be deformed off $U_i K_i$ (see e.g. [9]) and is therefore parallel to one of the projective planes of $\partial B(d, n)$. Thus $G(B(d, n))$ is isomorphic to the graph in Figure 3(b). Note that $\partial B(d, n)$ consists of n (incompressible) Klein bottles, d (incompressible) nonorientable surfaces of genus 3 and d projective planes. Note that $B(1, n) = M_n$.

Now suppose that v_1, \dots, v_k are the vertices of G of degree ≥ 2 and that among the neighbours of v_i exactly c_i are of degree 1 each. Let f_i be the number of self-loops of G based at v_i . Suppose that vertex v_i is joined to v_j by d_{ij} edges and let g_i be the sum of the d_{ij} over all neighbours v_j of v_i . Then $\deg v_i = d_i = c_i + g_i + 2f_i$. For each v_i let C_i be the 3-manifold obtained from $B(d_i, m_i)$ by identifying $2f_i$ projective planes of $\partial B(d_i, m_i)$ in pairs. Now construct a 3-manifold M' from C_1, \dots, C_k as follows: If vertex v_i is joined to v_j by d_{ij} edges in G , identify d_{ij} projective planes in ∂C_i with d_{ij} projective planes in ∂C_j . The resulting 3-manifold M' is irreducible and $\partial M'$ contains $n = m_1 + \dots + m_k$ (incompressible) Klein bottles. Now $G(M')$ is isomorphic to G , except that each vertex of degree 1 of $G(M')$ is colored white. To change the colour of such a vertex w , attach $B(1, 0)$ to M' by identifying the projective plane of M' corresponding to the vertex w with the projective plane of $\partial B(1, 0)$. In this way we construct the desired M such that $G(M) \cong G$.

Note that ∂M contains exactly n Klein bottles, where n is an arbitrary integer (independent of G). Therefore, by varying n , we have proved Theorem 4. \square

3. KLEIN BOTTLE SUM AND DEHN SURGERY

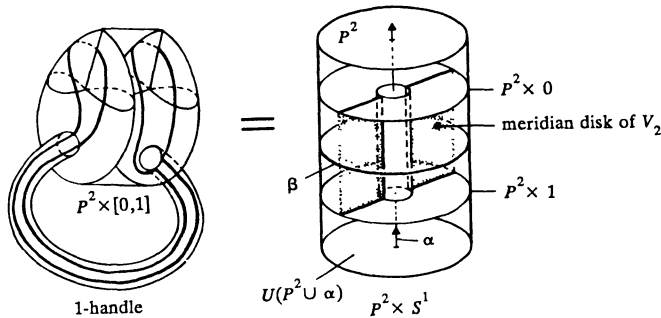
In this section, we shall show two methods to construct a P^2 -containing, irreducible, closed 3-manifold M from a P^2 -irreducible closed 3-manifold M_1 . One is a Klein bottle sum with $P^2 \times S^1$ and the other is a usual Dehn surgery. Both of them are modifications of a regular neighbourhood $U(\alpha)$ of a knot α in M_1 and are applied when $U(\alpha)$ is a solid Klein bottle or a solid torus, respectively. Lickorish's result [8] implies that any two nonorientable closed 3-manifolds can be transformed into each other by a finite sequence of Dehn surgeries along knots. However we shall show that M can be obtained from suitably chosen M_1 and α by a single Klein bottle sum with $P^2 \times S^1$ or a single Dehn surgery, as applications of the P^2 -graph $G(M)$.

Let M_1 and M_2 be two nonorientable compact 3-manifolds and let V_1 and V_2 be solid Klein bottles in $\text{Int } M_1$ and $\text{Int } M_2$, respectively. If a 3-manifold M is

obtained from $\overline{M_1 - V_1}$ and $\overline{M_2 - V_2}$ by sewing them up along the Klein bottles ∂V_1 and ∂V_2 , then M is said to be a *Klein bottle sum* of M_1 and M_2 and we write $M = M_1 \# M_2$. The homeomorphism type of M depends on a sewing map $\phi: \partial V_1 \rightarrow \partial V_2$, but is not so various. For there are only four isotopy classes of homeomorphisms of a Klein bottle. (See [8]).

Conversely, suppose that M splits into two submanifolds M'_1 and M'_2 along a 2-sided Klein bottle K^2 (i.e. $M = M'_1 \cup M'_2$, $M'_1 \cap M'_2 = K^2$). Let M_1 and M_2 denote the 3-manifolds obtained from M'_1 and M'_2 by capping off each K^2 on them with a solid Klein bottle V_i ($i = 1, 2$). Then we have a Klein bottle sum decomposition $M = M_1 \# M_2$. Since any homeomorphism of a Klein bottle extends to a homeomorphism of a solid Klein bottle [8], the homeomorphism type of M_1 and M_2 is uniquely determined, only depending on the choice of K^2 in M .

Now consider a canonical Klein bottle sum with $P^2 \times S^1$. Let V_2 be a regular neighbourhood of a nontrivial simple loop β on a fiber P^2 in $P^2 \times S^1$. Figure 5 shows that $M_2 = P^2 \times S^1 - V_2$ is a regular neighbourhood $U(P^2 \cup \alpha)$ for another fiber P^2 and a simple loop α in $P^2 \times S^1$ and is homeomorphic to $P^2 \times [0,1]$ with a 1-handle attached to both $P^2 \times 0$ and $P^2 \times 1$. Notice that the boundary of any meridian disk of V_2 runs twice through the 1-handle.



Now let M be a closed 3-manifold containing a projective plane P^2 . Since P^2 does not separate M , there is a simple loop α in M that intersects P^2 in a single point and a regular neighbourhood V of $P^2 \cup \alpha$ is homeomorphic to M_2 . Thus for $M'_1 = \overline{M - V}$ we have $M = M_1 \# P^2 \times S^1$.

By the Mayer-Vietoris exact sequence of (M_1, M_2) , we have

$$\dim H_1(M; \mathbb{Z}_2) = \dim H_1(M_1; \mathbb{Z}_2) + 1.$$

Roughly speaking, $[\alpha]$ is an extra generator for $H_1(M; \mathbb{Z}_2)$. This implies that there is a nonorientable closed 3-manifold M_1 such that M_1 contains no 2-sided projective plane and that

$$M = M_1 \# P^2 \times S^1 \# \dots \# P^2 \times S^1$$

for finitely many $P^2 \times S^1$'s.

The number of $P^2 \times S^1$'s does not exceed $\dim H_1(M_1; \mathbb{Z}_2) - 1$ since any nonorientable 3-manifold M_1 has nontrivial H_1 . Furthermore, we can decrease it to only one:

THEOREM 5. Every P^2 -containing, irreducible closed 3-manifold M can be obtained as a Klein bottle sum of a P^2 -irreducible closed 3-manifold M_1 and $P^2 \times S^1$.

$$M = M_1 \natural P^2 \times S^1.$$

PROOF. Suppose that $G(M)$ is embedded in M naturally. Since M is closed, $G(M)$ is an Euler graph. As is well-known, an Euler graph G has an Euler circuit, that is, a closed reduced edge path that contains each edge of G exactly once. Tracing an Euler circuit of $G(M)$, we can find a simple closed curve α in M which crosses each projective plane in a complete system P of M at a point. Furthermore, we make a local knot on α so that for a ball B^3 in M with $B^3 \cap \alpha$ a knotted arc, $B^3 - U(\alpha)$ contains no incompressible, ∂ -incompressible, planar surface. For example, the complements of most of the 2-bridge knots k , except torus knots, have this property [4]. So we can take the connected sum $(M, \alpha) \# (S^3, k)$ as a new α .

Let P^2 be any member of P , disjoint from B^3 . Since every 2-sided projective plane has to meet α with intersection number 1 mod 2, there is no 2-sided projective plane in M disjoint from α . Since a ball cannot contain a projective plane, if a 2-sphere S^2 in M does not meet $P^2 \cup \alpha$, then S^2 bounds a ball in M disjoint from $P^2 \cup \alpha$ by the irreducibility of M . Thus, the submanifold $M'_1 = M - U(P^2 \cup \alpha)$ is P^2 -irreducible.

Let $M_1 = M'_1 \cup V$ be the closed 3-manifold obtained from M'_1 by capping off its boundary with a solid Klein bottle V . If M_1 is not P^2 -irreducible, then there is either a 2-sided projective plane or an incompressible 2-sphere in M_1 which meets V along several meridian disks of V . Figure 5 shows that the boundary curve of a meridian disk of V must pass through B^3 twice along the local knot of α . This implies that $B^3 \cap M_1$ contains an incompressible, ∂ -incompressible, planar surface, contrary to the assumption of k . Therefore, M_1 is P^2 -irreducible and we have a decomposition $M = M_1 \natural P^2 \times S^1$. \square

We notice that M_1 in Theorem 5 cannot be taken to be universal, that is, there is no closed nonorientable 3-manifold M_1 such that every P^2 -containing closed 3-manifold M admits a Klein bottle sum decomposition $M_1 \natural P^2 \times S^1$. For if $M = M_1 \natural P^2 \times S^1$, then

$$\dim H_1(M_1; \mathbb{Z}_2) + 1 = \dim H_1(M; \mathbb{Z}_2) \geq \dim h_1(G(M); \mathbb{Z}_2).$$

The last inequality holds since there is a retraction of M onto $G(M)$. However, we can take any large value as $\dim H_1(G(M); \mathbb{Z}_2)$ by Theorem 4. Is there such a universal 3-manifold M_1 if we do not restrict the number of $P^2 \times S^1$'s?

Let α be a knot in M . A 3-manifold M_1 is called the result of a (nontrivial) *Dehn surgery* along α if M_1 can be obtained from $\overline{M - U(\alpha)}$ by sewing back $U(\alpha)$ along $\partial U(\alpha)$ in a different way. If $U(\alpha)$ is a solid Klein bottle, then any Dehn surgery does not change the homeomorphism type of M . So we shall treat only Dehn surgeries along a knot α with $U(\alpha)$ a solid torus.

By the same idea as in the proof of Theorem 5, we can show that:

THEOREM 6. Every P^2 -containing, irreducible, closed 3-manifold M contains a knot α with a regular neighbourhood $U(\alpha)$ homeomorphic to $D^2 \times S^1$ such that any nontrivial Dehn surgery along α yields a P^2 -irreducible closed 3-manifold.

PROOF. Take the same knot α as in the proof of Theorem 5. If $U(\alpha)$ is a solid Klein bottle, we add a nontrivial loop on some 2-sided projective plane to α . Then $\overline{M - U(\alpha)}$ is P^2 -irreducible. By similar arguments, we conclude that the result of a Dehn surgery along α is P^2 -irreducible whenever its surgery instruction passes through the part of the local knot of α , that is, whenever the surgery is not trivial. \square

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