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Research Article

On Higher Dimensional Kaluza-Klein Theories

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We present a new method for the study of general higher dimensional Kaluza-Klein theories. Our new approach is based on the Riemannian adapted connection and on a theory of adapted tensor fields in the ambient space. We obtain, in a covariant form, the fully general 4D equations of motion in a $(4 + n)$D general gauge Kaluza-Klein space. This enables us to classify the geodesics of the $(4 + n)$D space and to show that the induced motions in the 4D space bring more information than motions from both the 4D general relativity and the 4D Lorentz force equations. Finally, we note that all the previous studies on higher dimensional Kaluza-Klein theories are particular cases of the general case considered in the present paper.

1. Introduction

As it is well known, by the Kaluza-Klein theory, the unification of Einstein's theory of general relativity with Maxwell's theory of electromagnetism was achieved. In a modern terminology, this theory is developed on a trivial principal bundle over the usual 4D spacetime, with $U(1)$ as fibre type. Thus, a natural generalization of Kaluza-Klein theory consists in replacing $U(1)$ by a nonabelian gauge group G (cf. [1–5]). There have been also some other generalizations wherein the internal space has been considered a homogeneous space of type G/H (cf. [6, 7]).

Two conditions have been imposed in the classical Kaluza-Klein theory and in most of the above generalizations: the "cylinder condition" and the "compactification condition." The former condition assumes that all the local components of the pseudo-Riemannian metric on the ambient space do not depend on the extra dimensions, while the latter requires that the fibre must be a compact manifold.

In 1938, Einstein and Bergmann [8] presented the first generalization in this direction. According to it, the local components of the 4D Lorentz metric in a 5D space are supposed to be periodic functions of the fifth coordinate. Later on, two other important generalizations have been intensively studied. One is called brane-world theory and assumes that the observable universe is a 4-surface (the "brane") embedded in a $(4 + n)$-dimensional spacetime (the "bulk") with particles and fields trapped on the brane, while gravity is free to access the bulk (cf. [9]). The other one is called space-time-matter theory and assumes that matter in the 4D spacetime is a manifestation of the fifth dimension (cf. [10, 11]).

Recently, we presented a new point of view on a general Kaluza-Klein theory in a 5D space (cf. [12]). We removed both the above conditions and gave a new method of study based on the Riemannian horizontal connection. This enabled us to give a new definition of the fifth force in 4D physics (cf. [13]) and to obtain a classification of the warped 5D spaces satisfying Einstein equations with cosmological constant (cf. [14]).

The present paper is the first in a series of papers devoted to the study of general Kaluza-Klein theory with arbitrary gauge group. More precisely, our approach is developed on a principal bundle $\overline{M}$ over the 4D spacetime M, with an n-dimensional Lie group G as fibre type. Moreover, both the cylinder condition and the compactification condition are removed. In other words, the theory we develop here contains as particular cases all the other generalizations of Kaluza-Klein theory that have been presented above.

The whole study is based on the Riemannian adapted connection that we construct in this paper and on a 4D tensor calculus that we introduce via a natural splitting of the tangent bundle of the ambient space. We obtain, in a covariant form and in their full generality, the 4D equations of motion as part of equations of motion in a $(4+n)$D space. We analyze these equations and deduce that the induced motions on the base manifold bring more information than both the motions from general relativity and the motions from Lorentz force equations. Moreover, these equations show the existence of an extra force, which, in a particular case, is perpendicular to the 4D velocity. The general study of the extra force will be presented in a forthcoming paper.

Now, we outline the content of the paper. In Section 2 we present the general gauge Kaluza-Klein space $(\overline{M}, \overline{g}, H\overline{M})$, where $\overline{M}$ is the total space of a principal bundle over a 4D space time with a Lie group G as fibre type. The pseudo-Riemannian metric $\overline{g}$ determines the orthogonal splitting (5) and enables us to construct the adapted frame field $\{\delta/\delta x^{\alpha}, \partial/\partial y^{i}\}$ (see (12)). Our study is based on a 4D tensor calculus developed in Section 3. The electromagnetic tensor field $F = (F^{k}{}_{\alpha\beta})$ given by (41) and the adapted tensor fields H and V given by (44) and (45), respectively, play an important role in our approach. In Section 4 we construct the Riemannian adapted connection, that is, a metric connection with respect to which both distributions $H\overline{M}$ and $V\overline{M}$ are parallel, and its torsion is given by (58a), (58b), and (58c). Section 5 is the main section of the paper and presents the 4D equations of motions in $(\overline{M}, \overline{g}, H\overline{M})$ (cf. (85a) and (85b)). Also, in a particular case, we show that the extra force is orthogonal to the 4D velocity and therefore does not contradict the 4D physics. Finally, in Section 6 we show that the set of geodesics in $(\overline{M}, \overline{g}, H\overline{M})$ splits into three categories: horizontal, vertical, and oblique geodesics. Both, the horizontal and oblique geodesics induce some new motions on the 4D spacetime. We close the paper with conclusions.

2. General Gauge Kaluza-Klein Space

Let M be a 4-dimensional manifold and G an n-dimensional Lie group. The Kaluza-Klein theory we present in the paper is developed on a principal bundle $\overline{M}$ with base manifold M and structure group G. Any coordinate system (x^{α}) on M will define a coordinate system (x^{α}, y^{i}) on $\overline{M}$, where (y^{i}) are the fibre coordinates. Two such coordinate systems (x^{α}, y^{i}) and $(\tilde{x}^{\mu}, \tilde{y}^{j})$ are related by the following general transformations:

$$\tilde{x}^{\alpha} = \tilde{x}^{\alpha}\left(x^{\mu}\right), \tag{1a}$$

$$\tilde{y}^{i} = \tilde{y}^{i}\left(x^{\alpha}, y^{j}\right). \tag{1b}$$

Then, the transformations of the natural frame and coframe fields on $\overline{M}$ have the forms

$$\frac{\partial}{\partial x^{\alpha}} = \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\gamma}} + \frac{\partial \tilde{y}^{i}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{y}^{i}}, \tag{2a}$$

$$\frac{\partial}{\partial y^{j}} = \frac{\partial \tilde{y}^{i}}{\partial y^{j}}\frac{\partial}{\partial \tilde{y}^{i}}, \tag{2b}$$

$$d\tilde{x}^{\gamma} = \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}dx^{\alpha}, \tag{3a}$$

$$d\tilde{y}^{i} = \frac{\partial \tilde{y}^{i}}{\partial x^{\alpha}}dx^{\alpha} + \frac{\partial \tilde{y}^{i}}{\partial y^{j}}dy^{j}, \tag{3b}$$

respectively.

Throughout the paper we use the ranges of indices: $\alpha, \beta, \gamma, \dots \in \{0,1,2,3\}$, $i, j, k, \dots \in \{4, \dots, n+3\}$, $\dots A, B, C, \dots, \in \{0, \dots, 3+n\}$. By $T(x, y)$ we denote a function T that is locally defined on $\overline{M}$. Also, for any vector bundle E over $\overline{M}$, we denote by $\Gamma(E)$ the $\mathcal{F}(\overline{M})$-module of smooth sections of E, where $\mathcal{F}(\overline{M})$ is the algebra of smooth functions on $\overline{M}$.

Next, from (2b) we see that there exists a vector bundle $V\overline{M}$ over $\overline{M}$ of rank n which is locally spanned by $\{\partial/\partial y^{i}\}$. We call $V\overline{M}$ the *vertical distribution* on $\overline{M}$. Then, we suppose that there exists on $\overline{M}$ a pseudo-Riemannian metric $\overline{g}$ whose restriction to $V\overline{M}$ is a Riemannian metric $g^{\star}$. Denote by $H\overline{M}$ the complementary orthogonal distribution to $V\overline{M}$ in $T\overline{M}$, and call it the *horizontal distribution* on $\overline{M}$. Suppose that $H\overline{M}$ is invariant with respect to the action of G on $\overline{M}$ on the right; that is, we have

$$\left(R_{a}\right)_{\star}\left(H\overline{M}\right) = H\overline{M}, \quad \forall a \in G, \tag{4}$$

where $R_{a^{\star}}$ is the differential of the right translation R_{a} of G. Thus $H\overline{M}$ defines an Ehresmann connection on $\overline{M}$ (cf. [15, p. 359]). Also, suppose that the restriction of $\overline{g}$ to $H\overline{M}$ is a Lorentz metric g; that is, g is nondegenerate of signature $(+,+,+,-)$. Thus $T\overline{M}$ is endowed with a Lorentz distribution $(H\overline{M}, g)$ and a Riemannian distribution $(V\overline{M}, g^{\star})$ and admits the orthogonal direct decomposition

$$T\overline{M} = H\overline{M} \oplus V\overline{M}. \tag{5}$$

As we apply the above objects to physics, we need a coordinate presentation for them. First, we recall (cf. [15, p. 359], [16, p. 64]) that the Ehresmann connection defined by $H\overline{M}$ is completely determined by a 1-form ω on $\overline{M}$ with values in the Lie algebra $L(G)$ of G, satisfying the conditions

$$\omega\left(A^{\star}\right) = A, \quad \forall A \in L\left(G\right), \tag{6a}$$

$$\omega\left(\left(R_{a^{\star}}\right)X\right) = ad\left(a^{-1}\right)\omega\left(X\right), \quad \forall a \in G,\ \forall X \in \Gamma\left(T\overline{M}\right), \tag{6b}$$

where $A^{\star}$ is the fundamental vector field corresponding to A and ad denotes the adjoint representation of G in $L(G)$. Now, suppose that $\{K_{i}\}$ is a basis of left invariant vector fields in $L(G)$ and put

$$K_{i}^{\star} = K_{i}^{j}\left(y\right)\frac{\partial}{\partial y^{j}}, \tag{7}$$

where $[K_i^j(y)]$ is a nonsingular matrix whose inverse we denote by $[\overline{K}_i^j(y)]$. Then we put $\omega = \omega^i K_i$, and by using (6a), (6b), and (7) we deduce that

$$\omega^i\left(\frac{\partial}{\partial y^j}\right) = \overline{K}_j^i, \tag{8a}$$

$$\omega^i\left(\left(R_{a^*}\right)\frac{\partial}{\partial y^j}\right) = ad\left(a^{-1}\right)\overline{K}_j^i. \tag{8b}$$

As it is well known, $H\overline{M}$ is the kernel of the connection form ω. In order to present two other local characterizations of $H\overline{M}$, we consider a local basis $\{E_\alpha\}$ in $\Gamma(H\overline{M})$ and put

$$\frac{\partial}{\partial x^\alpha} = L_\alpha^\gamma(x,y)\,E_\gamma + L_\alpha^i(x,y)\frac{\partial}{\partial y^i}. \tag{9}$$

As the transition matrix from $\{E_\alpha,\ \delta/\delta y^i\}$ to the natural frame field $\{\partial/\partial x^\gamma,\ \partial/\partial y^i\}$ has the form

$$\begin{bmatrix} L_\alpha^\gamma & 0 \\ L_\alpha^i & \delta_j^i \end{bmatrix}, \tag{10}$$

we infer that the 4×4 matrix $[L_\alpha^\gamma]$ is nonsingular. Hence the vector fields

$$\frac{\delta}{\delta x^\alpha} = L_\alpha^\gamma E_\gamma, \quad \alpha \in \{0,1,2,3\}, \tag{11}$$

form a local basis in $\Gamma(H\overline{M})$, too. Moreover, from (9) we obtain

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - L_\alpha^i\frac{\partial}{\partial y^i}. \tag{12}$$

Note that $\delta/\delta x^\alpha$ is just the projection of $\partial/\partial x^\alpha$ on $H\overline{M}$. Also, we define the local 1-forms

$$\delta y^i = dy^i + L_\alpha^i dx^\alpha, \tag{13}$$

and by using (8a) and (13), we deduce that

$$\omega^i = \overline{K}_j^i \delta y^j. \tag{14}$$

Hence, $H\overline{M}$ is locally represented by the kernel of the 1-forms $\{\delta y^i\}$. Now, by using the fundamental vector fields $\{K_i^\star\}$ we put

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A_\alpha^i(x,y)\,K_i^\star, \tag{15}$$

and comparing (12) with (15) we obtain

$$L_\alpha^i = A_\alpha^j K_j^i, \tag{16}$$

via (7). The frame fields $\{\delta/\delta x^\alpha,\ \partial/\partial y^i\}$ and $\{\delta/\delta x^\alpha,\ K_i^\star\}$ are called *adapted frame fields* with respect to the decomposition (5). The commutation formulas for these vector fields will have a great role in the study. First, by direct calculations using (12) we obtain

$$\left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right] = L_i{}^k{}_\alpha \frac{\partial}{\partial y^k}, \tag{17a}$$

$$\left[\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right] = F^k{}_{\alpha\beta}\frac{\partial}{\partial y^k}, \tag{17b}$$

$$\left[\frac{\partial}{\partial x^\alpha}, K_i^\star\right] = 0, \tag{17c}$$

where we put

$$L_i{}^k{}_\alpha = \frac{\partial L_\alpha^k}{\partial y^i}, \tag{18a}$$

$$F^k{}_{\alpha\beta} = \frac{\delta L_\beta^k}{\delta x^\alpha} - \frac{\delta L_\alpha^k}{\delta x^\beta}. \tag{18b}$$

Next, we show that

$$\left[\frac{\delta}{\delta x^\alpha}, K_i^\star\right] = 0. \tag{19}$$

First, according to a general result stated in page 78 in the book of Kobayashi and Nomizu [16], we deduce that the vector fields in the left hand side of (19) must be horizontal. On the other hand, by using (7) and (17a), we obtain

$$\left[\frac{\delta}{\delta x^\alpha}, K_i^\star\right] = \left(\frac{\delta K_i^h}{\delta x^\alpha} + K_i^j L_j{}^h{}_\alpha\right)\frac{\partial}{\partial y^h}. \tag{20}$$

That is, these vectors fields are vertical, too. This proves (19) via (5). As $L(G)$ is isomorphic to the Lie algebra of vertical vector fields, we have

$$\left[K_i^\star, K_j^\star\right] = C_i{}^k{}_j K_k^\star, \tag{21}$$

where $C_i{}^k{}_j$ are the structure constants of the Lie group G. Then, by using (17a)–(17c), (15), (19), (21), and (7), we deduce that

$$F^k{}_{\alpha\beta} = F^{\star h}{}_{\alpha\beta} K_h^k, \tag{22}$$

where we put

$$F^{\star h}{}_{\alpha\beta} = \frac{\partial A_\beta^h}{\partial x^\alpha} - \frac{\partial A_\alpha^h}{\partial x^\beta} + A_\alpha^i A_\beta^j C_i{}^h{}_j. \tag{23}$$

Now, taking into account (17c), from (19), we obtain

$$K_i^\star\left(A_\alpha^h\right) = -C_i{}^h{}_j A_\alpha^j, \tag{24}$$

which together with (23) implies

$$K_i^\star\left(F^{\star h}{}_{\alpha\beta}\right) = -C_i{}^h{}_j F^{\star j}{}_{\alpha\beta}. \tag{25}$$

By using (24) and (25) we are entitled to call $F^{\star h}{}_{\alpha\beta}$ the *Yang-Mills fields* corresponding to *gauge potentials* A_α^h. Also by (18b) we may call $F^k{}_{\alpha\beta}$ the *electromagnetic tensor field* corresponding to the *electromagnetic potentials* L_α^k. It is important to note that these objects come from different physical theories, and they are related by (22) and (16).

Remark 1. By a different method, the above Yang-Mills fields have been first introduced by Cho [4]. On the other hand, we should stress that we find it more convenient to use ${F^k}_{\alpha\beta}$ and L^k_α instead of ${F^{\star h}}_{\alpha\beta}$ and A^h_α.

Next, we express the pseudo-Riemannian metric $\overline{g}$ on $\overline{M}$ with respect to the adapted frame field $\{\delta/\delta x^\alpha,\ \partial/\partial y^i\}$; that is, we have

$$g_{\alpha\beta}(x,y) = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) = \overline{g}\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right), \quad (26a)$$

$$g_{ij}(x,y) = g^\star\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \overline{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right), \quad (26b)$$

$$\overline{g}\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = 0. \quad (26c)$$

Thus the local line element representing $\overline{g}$ has the form

$$\begin{aligned} d\overline{s}^2 &= g_{\alpha\beta}(x,y)\,dx^\alpha dx^\beta + g_{ij}(x,y)\,\delta y^i \delta y^j \\ &= g_{\alpha\beta}(x,y)\,dx^\alpha dx^\beta + g_{ij}(x,y)\left(dy^i + L^i_\alpha dx^\alpha\right) \\ &\quad \times \left(dy^j + L^j_\beta dx^\beta\right) \\ &= g_{\alpha\beta}(x,y)\,dx^\alpha dx^\beta + g_{ij}(x,y)\left(dy^i + A^h_\alpha K^i_h dx^\alpha\right) \\ &\quad \times \left(dy^j + A^h_\beta K^j_h dx^\beta\right). \end{aligned} \quad (27)$$

Hence $\overline{g}$ is locally given by the matrices

$$\begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{ij} \end{bmatrix}, \quad (28)$$

and

$$\begin{bmatrix} g_{\alpha\beta} + g_{ij}L^i_\alpha L^j_\beta & g_{ij}L^i_\alpha \\ g_{ij}L^j_\beta & g_{ij} \end{bmatrix} = \begin{bmatrix} g_{\alpha\beta} + g_{ij}K^i_h K^j_k A^h_\alpha A^k_\beta & g_{ij}K^i_h A^h_\alpha \\ g_{ij}K^j_k A^k_\beta & g_{ij} \end{bmatrix}, \quad (29)$$

with respect to the frame fields $\{\delta/\delta x^\alpha,\ \partial/\partial y^i\}$ and $\{\partial/\partial x^\alpha,\ \partial/\partial y^i\}$, respectively. Formally, (29) is identical to(13.31) from [17], but in the latter the local components are supposed to be functions of (x^α) alone. So $\overline{g}$ given by (27) is the most general Kaluza-Klein metric considered in any Kaluza-Klein theory. The principal bundle $\overline{M}$, together with the metric $\overline{g}$ and the Ehresmann connection defined by the horizontal distribution $H\overline{M}$, is denoted by $(\overline{M}, \overline{g}, H\overline{M})$, and it is called a *general gauge Kaluza-Klein space*.

Finally, we consider two coordinate systems (x^α, y^j) and $(\widetilde{x}^\gamma, \widetilde{y}^i)$ and by using (12), (13), (2a), (2b), (3a), and (3b), we obtain

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha}\frac{\delta}{\delta \widetilde{x}^\gamma}, \quad (30a)$$

$$\delta \widetilde{y}^i = \frac{\partial \widetilde{y}^i}{\partial y^j}\delta y^j, \quad (30b)$$

$$L^j_\alpha \frac{\partial \widetilde{y}^i}{\partial y^j} = \widetilde{L}^i_\gamma \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha} + \frac{\partial \widetilde{y}^i}{\partial x^\alpha}. \quad (30c)$$

Now, we put

$$K^\star_i = \widetilde{K}^j_i \frac{\partial}{\partial \widetilde{y}^j}, \quad (31)$$

and by using (16) into (30c) we deduce that

$$\left(A^i_\alpha - \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha}\widetilde{A}^i_\gamma\right)\widetilde{K}^j_i = \frac{\partial \widetilde{y}^j}{\partial x^\alpha}. \quad (32)$$

The transformations (30c) and (32) have a gauge character. Apart from them we will meet transformations with tensorial character. Here we observe that by using (26a), (26b), (30a), and (2b) we obtain the first such transformations

$$g_{\alpha\beta} = \widetilde{g}_{\mu\nu}\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha}\frac{\partial \widetilde{x}^\nu}{\partial x^\beta}, \quad (33a)$$

$$g_{ij} = \widetilde{g}_{hk}\frac{\partial \widetilde{y}^h}{\partial y^i}\frac{\partial \widetilde{y}^k}{\partial y^j}. \quad (33b)$$

3. Adapted Tensor Fields on $(\overline{M},\ \overline{g},\ H\overline{M})$

In the present section we develop a tensor calculus on $\overline{M}$ that is adapted to the decomposition (5). For example, we construct some adapted tensor fields which have an important role in the general Kaluza-Klein theory which we develop in a series of papers. In particular, we show that the electromagnetic tensor field is indeed an adapted tensor field.

First, we consider the dual vector bundles $H\overline{M}^\star$ and $V\overline{M}^\star$ of $H\overline{M}$ and $V\overline{M}$, respectively. Then, an $\mathscr{F}(\overline{M}) - (p+q)$-linear mapping

$$T : \Gamma\left(H\overline{M}^\star\right)^p \times \Gamma\left(H\overline{M}\right)^q \longrightarrow \mathscr{F}\left(\overline{M}\right) \quad (34)$$

is called a *horizontal tensor field* of type (p, q). Similarly, an $\mathscr{F}(\overline{M}) - (r+s)$-linear mapping

$$T : \Gamma\left(V\overline{M}^\star\right)^r \times \Gamma\left(V\overline{M}\right)^s \longrightarrow \mathscr{F}\left(\overline{M}\right) \quad (35)$$

is called a *vertical tensor field* of type (r, s). For example, g (resp., $g^\star$) is a horizontal (resp., vertical) tensor field of type $(0, 2)$. Also, dx^α (resp., δy^i) are horizontal (resp., vertical) covector fields, while $\delta/\delta x^\alpha$ (resp., $\partial/\partial y^i$) are horizontal (resp., vertical) vector fields, locally defined on $\overline{M}$. More generally, an $\mathscr{F}(\overline{M}) - (p+q+r+s)$-linear mapping

$$\begin{aligned} T : \Gamma\left(H\overline{M}^\star\right)^p \times \Gamma\left(H\overline{M}\right)^q \times \Gamma\left(V\overline{M}^\star\right)^r \\ \times \Gamma\left(V\overline{M}\right)^s \longrightarrow \mathscr{F}\left(\overline{M}\right) \end{aligned} \quad (36)$$

is an *adapted tensor field* of type $(p, q; r, s)$ on $\overline{M}$. Locally, T is given by the functions

$$T^{\alpha_1\cdots\alpha_p i_1\cdots i_r}_{\beta_1\cdots\beta_q j_1\cdots j_s}(x, y) = T\left(dx^{\alpha_1}, \dots, dx^{\alpha_p}, \frac{\delta}{\delta x^{\beta_1}}, \dots, \frac{\delta}{\delta x^{\beta_q}}, \delta y^{i_1}, \dots, \delta y^{i_r}, \frac{\partial}{\partial y^{j_1}}, \dots, \frac{\partial}{\partial y^{j_s}}\right). \quad (37)$$

Then by using (2b), (3a), (30a), and (30b) we deduce that there exists an adapted tensor field of type $(p, q; r, s)$ on $\overline{M}$, if and only if, on the domain of each coordinate system, there exist $4^{p+q} \cdot n^{r+s}$ functions $T^{\alpha_1\cdots\alpha_p i_1\cdots i_r}_{\beta_1\cdots\beta_q j_1\cdots j_s}$ satisfying

$$T^{\alpha_1\cdots\alpha_p i_1\cdots i_r}_{\beta_1\cdots\beta_q j_1\cdots j_s} \frac{\partial \widetilde{x}^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \widetilde{x}^{\gamma_p}}{\partial x^{\alpha_p}} \frac{\partial \widetilde{y}^{k_1}}{\partial y^{i_1}} \cdots \frac{\partial \widetilde{y}^{k_r}}{\partial y^{i_r}} = \widetilde{T}^{\gamma_1\cdots\gamma_p k_1\cdots k_r}_{\mu_1\cdots\mu_q h_1\cdots h_s} \frac{\partial \widetilde{x}^{\mu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \widetilde{x}^{\mu_q}}{\partial x^{\beta_q}} \frac{\partial \widetilde{y}^{h_1}}{\partial y^{j_1}} \cdots \frac{\partial \widetilde{y}^{h_s}}{\partial y^{j_s}}, \quad (38)$$

with respect to the transformations (1a) and (1b). Also, we note that any $\mathcal{F}(\overline{M}) - (q + r + s)$-linear mapping

$$T : \Gamma(H\overline{M})^q \times \Gamma(V\overline{M}^\star)^r \times \Gamma(V\overline{M})^s \longrightarrow \Gamma(H\overline{M}) \quad (39)$$

defines an adapted tensor field of type $(1, q; r, s)$. Similarly, any $\mathcal{F}(\overline{M}) - (p + q + s)$-linear mapping

$$T : \Gamma(H\overline{M}^\star)^p \times \Gamma(H\overline{M})^q \times \Gamma(V\overline{M})^s \longrightarrow \Gamma(V\overline{M}) \quad (40)$$

defines an adapted tensor field of type $(p, q; 1, s)$. More about adapted tensor fields can be found in the book of Bejancu and Farran [18].

Next, we will construct some adapted tensor fields which are deeply involved in our study. First, we denote by h and v the projection morphisms of $T\overline{M}$ on $H\overline{M}$ and $V\overline{M}$, respectively. Then, we consider the mapping

$$F : \Gamma(H\overline{M})^2 \longrightarrow \Gamma(V\overline{M}),$$
$$F(hX, hY) = -v[hX, hY], \quad \forall X, Y \in \Gamma(T\overline{M}). \quad (41)$$

It is easy to check that F is $\mathcal{F}(\overline{M})$-bilinear mapping. Thus F is an adapted tensor field of type $(0, 2; 1, 0)$. By using (17b) and (41) we obtain

$$F\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) = F^k{}_{\alpha\beta} \frac{\partial}{\partial y^k}, \quad (42)$$

where $F^k{}_{\alpha\beta}$ are given by (18b). Hence the electromagnetic tensor field is indeed an adapted tensor field. Next, we define the mappings:

$$H : \Gamma(H\overline{M})^2 \times \Gamma(V\overline{M}) \longrightarrow \mathcal{F}(\overline{M}),$$
$$V : \Gamma(V\overline{M})^2 \times \Gamma(H\overline{M}) \longrightarrow \mathcal{F}(\overline{M}), \quad (43)$$

given by

$$H(hX, hY, vZ) = \frac{1}{2}\{vZ(g(hX, hY)) - g(h[vZ, hX], hY) - g(h[vZ, hY], hX)\}, \quad (44)$$

$$V(vX, vY, hZ) = \frac{1}{2}\{hZ(g^\star(vX, vY)) - g^\star(v[hZ, vX], vY) - g^\star(v[hZ, vY], vX)\}, \quad (45)$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. It is easy to verify that both H and V are $\mathcal{F}(\overline{M})$-3-linear mappings and therefore define the adapted tensor fields of types $(0, 2; 0, 1)$ and $(0, 1; 0, 2)$, respectively. By using H and V and the metrics on $H\overline{M}$ and $V\overline{M}$, we define two adapted tensor fields denoted by the same symbols and given by

$$H : \Gamma(H\overline{M}) \times \Gamma(V\overline{M}) \longrightarrow \Gamma(H\overline{M}),$$
$$V : \Gamma(V\overline{M}) \times \Gamma(H\overline{M}) \longrightarrow \Gamma(V\overline{M}), \quad (46)$$

$$g(hX, H(hY, vZ)) = H(hX, hY, vZ), \quad (47a)$$

$$g^\star(vX, V(vY, hZ)) = V(vX, vY, hZ), \quad (47b)$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

We close this section with a local presentation of the adapted tensor fields H and V. First, from (33a) and (33b) we deduce that entries $g^{\alpha\beta}$ (resp., g^{ij}) of the inverse of the matrix $[g_{\alpha\beta}]$ (resp., $[g_{ij}]$) define a horizontal (resp., vertical) tensor field of type $(2, 0)$. Then, we put

$$H\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = H_{i\alpha\beta}, \quad (48a)$$

$$H\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = H_{i\alpha}{}^\gamma \frac{\delta}{\delta x^\gamma}, \quad (48b)$$

$$V\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^\alpha}\right) = V_{\alpha ij}, \quad (48c)$$

$$V\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^\alpha}\right) = V_{\alpha i}{}^k \frac{\partial}{\partial y^k}, \quad (48d)$$

and by using (44), (45), (47a), (47b), (48a)–(48d), (26a), (26b), and (17a)–(17c), we obtain

$$H_{i\alpha\beta} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial y^i}, \quad (49a)$$

$$H_{i\alpha}{}^\gamma = g^{\gamma\beta} H_{i\alpha\beta}, \quad (49b)$$

$$V_{\alpha ij} = \frac{1}{2}\left\{\frac{\delta g_{ij}}{\delta x^\alpha} - g_{kj} L_i{}^k{}_\alpha, - g_{ik} L_j{}^k{}_\alpha\right\}, \quad (50a)$$

$$V_{\alpha i}{}^k = g^{kj} V_{\alpha ij}. \quad (50b)$$

Remark 2. If in particular $G = U(1)$, then $H_{1\alpha\beta}$ represent the local components of the extrinsic curvature used in brane-world theory (cf. [9]) and in space-time-matter theory (cf. [12, 19]). For this reason we call H given by (44) the *extrinsic curvature* of the horizontal distribution.

Remark 3. In all the papers published so far on Kaluza-Klein theories with nonabelian gauge group, the local components $g_{\alpha\beta}$ of the Lorentz metric g are supposed to be independent of y^i (cf. [2, 4–7]). From (49a) and (49b) we see that this particular case occurs if and only if the extrinsic curvature of $H\overline{M}$ vanishes identically on $\overline{M}$.

4. A Remarkable Linear Connection on $(\overline{M}, \overline{g}, H\overline{M})$

In a previous paper (cf. [12]), we constructed the Riemannian horizontal connection on the horizontal distribution of a 5D general Kaluza-Klein theory and obtain both the 4D equations of motion and 4D Einstein equations. As in that case the vertical bundle was of rank 1, it was not necessary to consider a linear connection on it. On the contrary, the geometric configuration of $(\overline{M}, \overline{g}, H\overline{M})$ from the present paper requires such connections on both $H\overline{M}$ and $V\overline{M}$. The construction of these connections is the purpose of this section.

First, we denote by $\overline{\nabla}$ the Levi-Civita connection on $(\overline{M}, \overline{g}, H\overline{M})$ given by (cf. [20, p. 61])

$$\begin{aligned} 2\overline{g}\left(\overline{\nabla}_X Y, Z\right) = {} & X\left(\overline{g}(Y,Z)\right) + Y\left(\overline{g}(Z,X)\right) - Z\left(\overline{g}(X,Y)\right) \\ & + \overline{g}([X,Y],Z) - \overline{g}([Y,Z],X) \\ & + \overline{g}([Z,X],Y), \end{aligned} \tag{51}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. Recall that $\overline{\nabla}$ is the unique linear connection on $\overline{M}$ which is metric and torsion free.

Next, we say that ∇ is an *adapted linear connection* on $(\overline{M}, \overline{g}, H\overline{M})$ if both distributions $H\overline{M}$ and $V\overline{M}$ are parallel with respect to ∇; that is, we have

$$\nabla_X hY \in \Gamma\left(H\overline{M}\right), \tag{52a}$$

$$\nabla_X vY \in \Gamma\left(V\overline{M}\right), \tag{52b}$$

for all $X, Y \in \Gamma(T\overline{M})$. Then there exist two linear connections $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ on $H\overline{M}$ and $V\overline{M}$, respectively, given by

$$\overset{h}{\nabla}_X hY = \nabla_X hY, \tag{53a}$$

$$\overset{v}{\nabla}_X vY = \nabla_X vY. \tag{53b}$$

Conversely, given two linear connections $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ on $H\overline{M}$ and $V\overline{M}$, respectively, there exists an adapted linear connection ∇ on $\overline{M}$ given by

$$\nabla_X Y = \overset{h}{\nabla}_X hY + \overset{v}{\nabla}_X vY. \tag{54}$$

Also, it is easy to show that an adapted connection $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ is metric; that is,

$$(\nabla_X \overline{g})(Y,Z) = 0, \quad \forall X, Y, Z \in \Gamma\left(T\overline{M}\right), \tag{55}$$

if and only if both $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ are metric connections; that is,

$$\left(\overset{h}{\nabla}_X g\right)(hY, hZ) = 0, \tag{56a}$$

$$\left(\overset{v}{\nabla}_X g^{\star}\right)(vY, vZ) = 0, \tag{56b}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. The torsion tensor field of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{57}$$

Now, we can prove the following important result.

Theorem 4. *Let $(\overline{M}, \overline{g}, H\overline{M})$ be a general gauge Kaluza-Klein space. Then there exists a unique metric adapted linear connection $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ whose torsion tensor field T is given by*

$$T(hX, hY) = F(hX, hY), \tag{58a}$$

$$T(vX, vY) = 0, \tag{58b}$$

$$T(hX, vY) = V(vY, hX) - H(hX, vY), \tag{58c}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Proof. First, define $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ as follows:

$$\overset{h}{\nabla}_{hX} hY = h\overline{\nabla}_{hX} hY, \tag{59a}$$

$$\overset{h}{\nabla}_{vX} hY = h[vX, hY] + H(hY, vX), \tag{59b}$$

$$\overset{v}{\nabla}_{vX} vY = v\overline{\nabla}_{vX} vY, \tag{59c}$$

$$\overset{v}{\nabla}_{hX} vY = v[hX, vY] + V(vY, hX), \tag{59d}$$

for all $X, Y \in \Gamma(T\overline{M})$. Then, it is easy to check that $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ given by (59a)–(59d) is a metric adapted linear connection whose torsion tensor field satisfies (58a)–(58c). Next, suppose that $\nabla' = (\overset{h}{\nabla}', \overset{v}{\nabla}')$ is an another metric adapted linear connection satisfying (58a)–(58c). Then, from (58c) we deduce that

$$\begin{aligned} & \overset{v}{\nabla}_{hX} vY - v[hX, vY] - V(vY, hX) \\ & = \overset{h}{\nabla}_{vY} hX - h[vY, hX] - H(hX, vY), \end{aligned} \tag{60}$$

which implies both (59b) and (59d) for ∇', via (5). Now, we note that (58a) is equivalent to

$$\overset{h}{\nabla}_{hX}hY - \overset{h}{\nabla}_{hY}hX - h[hX, hY] = 0, \quad \forall X, Y \in \Gamma(T\overline{M}). \tag{61}$$

Then by using (56a) and (61) for $\overset{h}{\nabla}'$ and taking into account (51), we obtain

$$\begin{aligned} 0 &= \left(\overset{h}{\nabla}'_{hX}g\right)(hY, hZ) + \left(\overset{h}{\nabla}'_{hY}g\right)(hZ, hX) \\ &\quad - \left(\overset{h}{\nabla}'_{hZ}g\right)(hX, hY) \\ &= hX(\overline{g}(hY, hZ)) + hY(\overline{g}(hZ, hX)) \\ &\quad - hZ(\overline{g}(hX, hY)) \\ &\quad + \overline{g}([hX, hY], hZ) - \overline{g}([hY, hZ], hX) \\ &\quad + \overline{g}([hZ, hX], hY) \\ &\quad - 2\overline{g}\left(\overset{h}{\nabla}'_{hX}hY, hZ\right) \\ &= 2\overline{g}\left(h\overline{\nabla}_{hX}hY - \overset{h}{\nabla}'_{hX}hY, hZ\right), \end{aligned} \tag{62}$$

which proves (59a) for $\overset{h}{\nabla}'$. In a similar way (59c) is proved for $\overset{v}{\nabla}'$. Thus $\nabla' = \nabla$, and the proof is complete. □

As $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ satisfy (56a) and (56b), we call them the *Riemannian horizontal connection* and the *Riemannian vertical connection*, respectively. Also, $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ given by (59a), (59b), (59c), and (59d) is called *Riemannian adapted connection* on $(\overline{M}, \overline{g}, H\overline{M})$.

Remark 5. It is important to note that both $\nabla_{hX}T$ and $\nabla_{vX}T$ are adapted tensor fields, where T is an adapted tensor field and ∇ is given by (59a)–(59d).

Remark 6. Throughout the paper, all local components for linear connections and adapted tensor fields are defined with respect to the adapted frame field $\{\delta/\delta x^{\alpha}, \partial/\partial y^{i}\}$ and the adapted coframe field $\{dx^{\alpha}, \delta y^{i}\}$.

Next, we consider $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ given by (59a)–(59d) and put

$$\overset{h}{\nabla}_{\delta/\delta x^{\beta}} \frac{\delta}{\delta x^{\alpha}} = \Gamma_{\alpha\ \beta}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}}, \tag{63a}$$

$$\overset{h}{\nabla}_{\partial/\partial y^{i}} \frac{\delta}{\delta x^{\alpha}} = \Gamma_{\alpha\ i}^{\ \gamma} \frac{\delta}{\delta x^{\gamma}}, \tag{63b}$$

$$\overset{v}{\nabla}_{\partial/\partial y^{j}} \frac{\partial}{\partial y^{i}} = \Gamma_{i\ j}^{\ k} \frac{\partial}{\partial y^{k}}, \tag{63c}$$

$$\overset{v}{\nabla}_{\delta/\delta x^{\alpha}} \frac{\partial}{\partial y^{i}} = \Gamma_{i\ \alpha}^{\ k} \frac{\partial}{\partial y^{k}}. \tag{63d}$$

Then, we take $X = \delta/\delta x^{\beta}, Y = \delta/\delta x^{\alpha}$, and $Z = \delta/\delta x^{\mu}$ into (51), and using (59a), (63a) (26a), and (17b), we obtain

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \frac{1}{2} g^{\gamma\mu} \left\{ \frac{\delta g_{\mu\alpha}}{\delta x^{\beta}} + \frac{\delta g_{\mu\beta}}{\delta x^{\alpha}} - \frac{\delta g_{\alpha\beta}}{\delta x^{\mu}} \right\}. \tag{64}$$

Similarly, we take $X = \partial/\partial y^{j}, Y = \partial/\partial y^{i}$, and $Z = \partial/\partial y^{k}$ in (51), and by using (59c), (63c), and (26b), we infer that

$$\Gamma_{i\ j}^{\ k} = \frac{1}{2} g^{kh} \left\{ \frac{\partial g_{hi}}{\partial y^{j}} + \frac{\partial g_{hj}}{\partial y^{i}} - \frac{\partial g_{ij}}{\partial y^{h}} \right\}. \tag{65}$$

Also, by direct calculations using (59b), (59d), (63b), (63d), (17a), (48b), and (48d), we deduce that

$$\Gamma_{\alpha\ i}^{\ \gamma} = H_{i\alpha}^{\ \ \gamma}, \tag{66a}$$

$$\Gamma_{i\ \alpha}^{\ k} = L_{i\ \alpha}^{\ k} + V_{\alpha i}^{\ \ k}. \tag{66b}$$

According to the splitting in (5), the Riemannian adapted connection $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ defines two types of covariant derivatives. More precisely, if $T_{\beta j}^{\gamma k}$ are the local components of an adapted tensor field of type $(1, 1; 1, 1)$, then we have

$$T_{\beta j|_{\alpha}}^{\gamma k} = \frac{\delta T_{\beta j}^{\gamma k}}{\delta x^{\alpha}} + T_{\beta j}^{\mu k} \Gamma_{\mu\ \alpha}^{\ \gamma} + T_{\beta j}^{\gamma h} \Gamma_{h\ \alpha}^{\ k} - T_{\mu j}^{\gamma k} \Gamma_{\beta\ \alpha}^{\ \mu} - T_{\beta h}^{\gamma k} \Gamma_{j\ \alpha}^{\ h}, \tag{67a}$$

$$T_{\beta j|_{i}}^{\gamma k} = \frac{\partial T_{\beta j}^{\gamma k}}{\partial y^{i}} + T_{\beta j}^{\mu k} \Gamma_{\mu\ i}^{\ \gamma} + T_{\beta j}^{\gamma h} \Gamma_{h\ i}^{\ k} - T_{\mu j}^{\gamma h} \Gamma_{\beta\ i}^{\ \mu} - T_{\beta h}^{\gamma k} \Gamma_{j\ i}^{\ h}. \tag{67b}$$

In particular, from (56a) and (56b) we deduce that

$$g_{\alpha\beta|_{\gamma}} = 0, \tag{68a}$$

$$g^{\alpha\beta}{}_{|_{\gamma}} = 0, \tag{68b}$$

$$g_{ij|_{\alpha}} = 0, \tag{68c}$$

$$g^{ij}{}_{|_{\alpha}} = 0, \tag{68d}$$

$$g_{\alpha\beta|_{i}} = 0, \tag{69a}$$

$$g^{\alpha\beta}{}_{|_{i}} = 0, \tag{69b}$$

$$g_{ij|_{k}} = 0, \tag{69c}$$

$$g^{ij}{}_{|_{k}} = 0. \tag{69d}$$

Throughout the paper we use $g_{\alpha\beta}$, $g^{\alpha\beta}$, g_{ij}, and g^{ij} for raising and lowering indices of adapted tensor fields as follows:

$$H^{k}{}_{\alpha\beta} = g^{ki} H_{i\alpha\beta}, \tag{70a}$$

$$V^{\gamma}{}_{ij} = g^{\gamma\alpha} V_{\alpha ij}, \tag{70b}$$

$$F_{i\alpha}{}^{\gamma} = g_{ik} g^{\gamma\beta} F^{k}{}_{\alpha\beta}. \tag{70c}$$

Now, we state the following.

Theorem 7. *The Levi-Civita connection $\overline{\nabla}$ on the general gauge Kaluza-Klein space $(\overline{M}, \overline{g}, H\overline{M})$ is expressed as follows:*

$$\overline{\nabla}_{\delta/\delta x^{\beta}} \frac{\delta}{\delta x^{\alpha}} = \Gamma_{\alpha}{}^{\gamma}{}_{\beta} \frac{\delta}{\delta x^{\gamma}} + \left(\frac{1}{2} F^{k}{}_{\alpha\beta} - H^{k}{}_{\alpha\beta}\right) \frac{\partial}{\partial y^{k}}, \tag{71a}$$

$$\overline{\nabla}_{\partial/\partial y^{i}} \frac{\delta}{\delta x^{\alpha}} = \left(H_{i\alpha}{}^{\gamma} + \frac{1}{2} F_{i\alpha}{}^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} + V_{\alpha i}{}^{k} \frac{\partial}{\partial y^{k}}, \tag{71b}$$

$$\overline{\nabla}_{\delta/\delta x^{\alpha}} \frac{\partial}{\partial y^{i}} = \left(H_{i\alpha}{}^{\gamma} + \frac{1}{2} F_{i\alpha}{}^{\gamma}\right) \frac{\delta}{\delta x^{\gamma}} + \Gamma_{i}{}^{k}{}_{\alpha} \frac{\partial}{\partial y^{k}}, \tag{71c}$$

$$\overline{\nabla}_{\partial/\partial y^{j}} \frac{\partial}{\partial y^{i}} = -V^{\gamma}{}_{ij} \frac{\delta}{\delta x^{\gamma}} + \Gamma_{i}{}^{k}{}_{j} \frac{\partial}{\partial y^{k}}. \tag{71d}$$

Proof. According to decomposition (5) we put

$$\overline{\nabla}_{\delta/\delta x^{\beta}} \frac{\delta}{\delta x^{\alpha}} = \overline{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta} \frac{\delta}{\delta x^{\gamma}} + \overline{\Gamma}_{\alpha}{}^{k}{}_{\beta} \frac{\partial}{\partial y^{k}}. \tag{72}$$

Then by using (59a) and (63a), we deduce that

$$\overline{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta} = \Gamma_{\alpha}{}^{\gamma}{}_{\beta}. \tag{73}$$

Next, take $X = \delta/\delta x^{\beta}, Y = \delta/\delta x^{\alpha}$, and $Z = \partial/\partial y^{i}$ in (51) and by using (26a), (26c), (17a), (17b), (48a), and (70a), we obtain

$$\overline{\Gamma}_{\alpha}{}^{k}{}_{\beta} = \frac{1}{2} F^{k}{}_{\alpha\beta} - H^{k}{}_{\alpha\beta}. \tag{74}$$

Thus (71a) is obtained from (72). Similarly, we put

$$\overline{\nabla}_{\partial/\partial y^{i}} \frac{\delta}{\delta x^{\alpha}} = \overline{\Gamma}_{\alpha}{}^{\gamma}{}_{i} \frac{\delta}{\delta x^{\gamma}} + \overline{\Gamma}_{\alpha}{}^{k}{}_{i} \frac{\partial}{\partial y^{k}}. \tag{75}$$

Then, take $X = \partial/\partial y^{i}, Y = \delta/\delta x^{\alpha}$, and $Z = \delta/\delta x^{\mu}$ in (51) and by using (26a), (26c), (17a), (17b), (49a), and (70c), we infer that

$$\overline{\Gamma}_{\alpha}{}^{\gamma}{}_{i} = H_{i\alpha}{}^{\gamma} + \frac{1}{2} F_{i\alpha}{}^{\gamma}. \tag{76}$$

Also, take $X = \partial/\partial y^{i}, Y = \delta/\delta x^{\alpha}$, and $Z = \partial/\partial y^{i}$ in (51) and by using (26b), (26c), (17a), and (50b), we deduce that

$$\overline{\Gamma}_{\alpha}{}^{k}{}_{i} = V_{\alpha i}{}^{k}. \tag{77}$$

Thus (71b) is obtained from (75). Now, taking into account that $\overline{\nabla}$ is a torsion-free connection and using (17a), (71b), and (66b) we obtain (71c). Finally, (71d) is deduced in a similar way as (71a). □

5. 4D Equations of Motion in $(\overline{M}, \overline{g}, H\overline{M})$

In this section we present the first achievement of the new method which we develop on general $(4 + n)$D Kaluza-Klein theories. We obtain, in a covariant form, the 4D equations of motion induced by the equations of motion in $(\overline{M}, \overline{g}, H\overline{M})$. This enables us to study the geodesics of the ambient space according to their positions with respect to horizontal distribution. It is noteworthy that the geodesics which are tangent to $H\overline{M}$ must be autoparallel curve for the Riemannian horizontal connection $\overset{h}{\nabla}$. The motions on the base manifold are defined as projections of the motions in $(\overline{M}, \overline{g}, H\overline{M})$.

Let $\overline{C}$ be a smooth curve in $\overline{M}$ given by parametric equations

$$x^{\alpha} = x^{\alpha}(t), \tag{78a}$$

$$y^{i} = y^{i}(t), \quad t \in [a, b], \ \alpha \in \{0, 1, 2, 3\}, \ i \in \{4, \ldots, 3 + n\}. \tag{78b}$$

Then, we express the tangent vector field d/dt to $\overline{C}$ with respect to the natural frame field as follows:

$$\frac{d}{dt} = \frac{dx^{\alpha}}{dt} \frac{\partial}{\partial x^{\alpha}} + \frac{dy^{i}}{dt} \frac{\partial}{\partial y^{i}}. \tag{79}$$

Taking into account decomposition (5) and using (12) into (79), we obtain

$$\frac{d}{dt} = \frac{dx^{\alpha}}{dt} \frac{\delta}{\delta x^{\alpha}} + \frac{\delta y^{i}}{\delta t} \frac{\partial}{\partial y^{i}}, \tag{80}$$

where we put

$$\frac{\delta y^{i}}{\delta t} = \frac{dy^{i}}{dt} + L^{i}_{\alpha} \frac{dx^{\alpha}}{dt}. \tag{81}$$

Next, by direct calculations using (71a)–(71d) and (80), we deduce that

$$\overline{\nabla}_{d/dt} \frac{\delta}{\delta x^{\alpha}} = \left\{\Gamma_{\alpha}{}^{\gamma}{}_{\beta} \frac{dx^{\beta}}{dt} + \left(H_{i\alpha}{}^{\gamma} + \frac{1}{2} F_{i\alpha}{}^{\gamma}\right) \frac{\delta y^{i}}{\delta t}\right\} \frac{\delta}{\delta x^{\gamma}} + \left\{\left(\frac{1}{2} F^{k}{}_{\alpha\beta} - H^{k}{}_{\alpha\beta}\right) \frac{dx^{\beta}}{dt} + V_{\alpha i}{}^{k} \frac{\delta y^{i}}{\delta t}\right\} \frac{\partial}{\partial y^{k}}, \tag{82a}$$

$$\overline{\nabla}_{d/dt} \frac{\partial}{\partial y^{i}} = \left\{\left(H_{i\alpha}{}^{\gamma} + \frac{1}{2} F_{i\alpha}{}^{\gamma}\right) \frac{dx^{\alpha}}{dt} - V^{\gamma}{}_{ij} \frac{\delta y^{j}}{\delta t}\right\} \frac{\delta}{\delta x^{\gamma}} + \left\{\Gamma_{i}{}^{k}{}_{\alpha} \frac{dx^{\alpha}}{dt} + \Gamma_{i}{}^{k}{}_{j} \frac{\delta y^{j}}{\delta t}\right\} \frac{\partial}{\partial y^{k}}, \tag{82b}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $(\overline{M}, \overline{g}, H\overline{M})$. Then, by using (80), (82a), and (82b) and taking into account that

$F^{k}{}_{\alpha\beta}$ are skew symmetric with respect to Greek indices, we obtain

$$\begin{aligned}
&\overline{\nabla}_{d/dt}\frac{d}{dt} \\
&= \overline{\nabla}_{d/dt}\left\{\frac{dx^\alpha}{dt}\frac{\delta}{\delta x^\alpha}+\frac{\delta y^i}{\delta t}\frac{\partial}{\partial y^i}\right\} \\
&= \frac{d^2x^\gamma}{dt^2}\frac{\delta}{\delta x^\gamma}+\frac{d}{dt}\left(\frac{\delta y^k}{\delta t}\right)\frac{\partial}{\partial y^k}+\frac{dx^\alpha}{dt}\overline{\nabla}_{d/dt}\frac{\delta}{\delta x^\alpha} \\
&\quad+\frac{\delta y^i}{\delta t}\overline{\nabla}_{d/dt}\frac{\partial}{\partial y^i} \\
&= \left\{\frac{d^2x^\gamma}{dt^2}+\Gamma_\alpha{}^\gamma{}_\beta\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right. \\
&\quad\left.+\left(2H_{i\alpha}{}^\gamma+F_{i\alpha}{}^\gamma\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{dt}-V^\gamma{}_{ij}\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t}\right\}\frac{\delta}{\delta x^\gamma} \\
&\quad+\left\{\frac{d}{dt}\left(\frac{\delta y^k}{\delta t}\right)-H^k{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right. \\
&\quad\left.+\left(\Gamma_i{}^k{}_\alpha+V_{\alpha i}{}^k\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}+\Gamma_i{}^k{}_j\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t}\right\}\frac{\partial}{\partial y^k}.
\end{aligned} \tag{83}$$

Now, we recall that $\overline{C}$ is a geodesic of $(\overline{M},\overline{g},H\overline{M})$ if and only if it is a curve of acceleration zero; that is, we have (cf. [20, p. 67])

$$\overline{\nabla}_{d/dt}\frac{d}{dt}=0. \tag{84}$$

Thus, using (84), (83), and decomposition (5), we can state the main result of this section.

Theorem 8. *The equations of motion in a general gauge Kaluza-Klein space $(\overline{M},\overline{g},H\overline{M})$ are expressed as follows:*

$$\begin{aligned}
&\frac{d^2x^\gamma}{dt^2}+\Gamma_\alpha{}^\gamma{}_\beta\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \\
&\quad+\left(2H_{i\alpha}{}^\gamma+F_{i\alpha}{}^\gamma\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{dt}-V^\gamma{}_{ij}\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t}=0,
\end{aligned} \tag{85a}$$

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\delta y^k}{\delta t}\right)-H^k{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \\
&\quad+\left(\Gamma_i{}^k{}_\alpha+V_{\alpha i}{}^k\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}+\Gamma_i{}^k{}_j\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t}=0.
\end{aligned} \tag{85b}$$

We call (85a) the 4D *equations of motion in* $(\overline{M},\overline{g},H\overline{M})$. We justify this name as follows. Suppose that the following conditions are satisfied:

$$H_{i\alpha}{}^\gamma=0, \tag{86a}$$

$$F_{i\alpha}{}^\gamma=0, \tag{86b}$$

$$V^\gamma{}_{ij}=0, \tag{86c}$$

for all $\alpha,\gamma\in\{0,1,2,3\}$ and $i,j\in\{4,\ldots,3+n\}$. Note that all these conditions have geometrical (physical) meaning, because they are invariant with respect to the transformations (1a) and (1b). Taking into account (86a), (49a), (49b), and (26a), we deduce that the Lorentz metric g on $H\overline{M}$ can be considered as a Lorentz metric on the base manifold M. Thus, in this particular case, $\Gamma_\alpha{}^\gamma{}_\beta$ given by (64) are functions of (x^α) alone and they are given by

$$\Gamma_\alpha{}^\gamma{}_\beta\left(x^\mu\right)=\frac{1}{2}g^{\gamma\mu}\left\{\frac{\partial g_{\mu\alpha}}{\partial x^\beta}+\frac{\partial g_{\mu\beta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right\}. \tag{87}$$

Moreover, (85a) becomes

$$\frac{d^2x^\gamma}{dt^2}+\Gamma_\alpha{}^\gamma{}_\beta\left(x^\mu\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}=0. \tag{88}$$

That is, we obtain the equations of motion in the 4D spacetime $(M,g=g_{\alpha\beta}(x^\mu))$. Hence, *the projections of geodesics of $(\overline{M},\overline{g},H\overline{M})$ on M coincide with the geodesics of the spacetime (M,g)*. This justifies the name 4D equations of motion for (85a).

Next, we suppose that only (86a) and (86c) are satisfied. Then (85a) becomes

$$\frac{d^2x^\gamma}{dt^2}+\Gamma_\alpha{}^\gamma{}_\beta\left(x^\mu\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}+F_{i\alpha}{}^\gamma\left(x,y\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}=0. \tag{89}$$

In this case, we show that there exists an extra force which does not contradict the 4D physics. First, we define the 4D *velocity* along a geodesic $\overline{C}$ as the horizontal vector field $U(t)$ given by

$$U\left(t\right)=\frac{dx^\alpha}{dt}\frac{\delta}{\delta x^\alpha}. \tag{90}$$

Then, define the *extra force* induced by extra dimensions as the horizontal vector field F given by

$$F\left(t\right)=\overset{h}{\nabla}_{d/dt}U\left(t\right), \tag{91}$$

where d/dt is given by (80) and $\overset{h}{\nabla}$ is the Riemannian horizontal connection. Now, we put

$$F\left(t\right)=F^\gamma\left(t\right)\frac{\delta}{\delta x^\gamma}, \tag{92}$$

and by using (92), (90), (80), and (91), we deduce that

$$F^\gamma\left(t\right)=\frac{d^2x^\gamma}{dt^2}+\Gamma_\alpha{}^\gamma{}_\beta\left(x^\mu\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}. \tag{93}$$

Thus, from (89) we obtain

$$F^\gamma\left(t\right)=-F_{i\alpha}{}^\gamma\left(x,y\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}. \tag{94}$$

Then, by using (90), (92), and (94) and taking into account that $F_{i\alpha\beta}$ are skew symmetric with respect to Greek indices, we infer that

$$g(F(t), U(t)) = -g_{\gamma\beta} F_{i\alpha}{}^{\gamma}(x, y) \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{\delta y^i}{\delta t} = -F_{i\alpha\beta}(x, y)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{\delta y^i}{\delta t} = 0. \quad (95)$$

Thus the extra force is perpendicular to the 4D velocity, which is a well-known property of the extra force in classical Kaluza-Klein theory. The above result on the extra force enables us to call (89) *the Lorentz force equations induced in the space time* (M, g). Finally, in this particular case, we see that our equations (89) coincide with (44) obtained by Kerner [2].

6. Motions in $(\overline{M}, \overline{g}, H\overline{M})$ and Induced Motions on the Base Manifold M

In this section we show that the set of geodesics in $(\overline{M}, \overline{g}, H\overline{M})$ splits into three categories and state characterizations of each category. Also, we define and study the induced motions on the base manifold.

The study of geodesics of $(\overline{M}, \overline{g}, H\overline{M})$ is based on their positions with respect to the distributions $H\overline{M}$ and $V\overline{M}$. First, we see from (80) that, apart from the 4D velocity $U(t)$ given by (90), there exists an nD *velocity* $W(t)$ given by

$$W(t) = \frac{\delta y^i}{\delta t}\frac{\partial}{\partial y^i}. \quad (96)$$

The whole study is developed in a coordinate neighbourhood $\overline{\mathcal{U}}$ around a point $P_0 \in \overline{M}$. We say that a curve $\overline{C}$ passing through P_0 is *horizontal* (resp.,*vertical*) if its nD velocity (resp., 4D velocity) vanishes on $\overline{\mathcal{U}}$. By (80) and (81) we see that $\overline{C}$ is a horizontal curve if and only if one of the following conditions is satisfied:

$$\frac{d}{dt} = \frac{dx^\alpha}{dt}\frac{\delta}{\delta x^\alpha} \quad (97a)$$

or

$$\frac{\delta y^i}{\delta t} = \frac{dy^i}{dt} + L^i_\alpha \frac{dx^\alpha}{dt} = 0. \quad (97b)$$

Similarly, $\overline{C}$ is a vertical curve if and only if we have

$$\frac{d}{dt} = \frac{\delta y^i}{\delta t}\frac{\partial}{\partial y^i}, \quad (98a)$$

or

$$\frac{dx^\alpha}{dt} = 0. \quad (98b)$$

Then, by using (85a), (85b), (97b), and (98b) we can state the following.

Theorem 9. *(i) A curve $\overline{C}$ is a horizontal geodesic in $(\overline{M}, \overline{g}, H\overline{M})$ if and only if* (97b) *and the following equations are satisfied:*

$$\frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x, y)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad \forall \gamma \in \{0, 1, 2, 3\}, \quad (99a)$$

$$H^k{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \quad \forall k \in \{4, \dots, 3+n\}. \quad (99b)$$

(ii) A curve $\overline{C}$ is a vertical geodesic in $(\overline{M}, \overline{g}, H\overline{M})$ if and only if (98b) *and the following equations are satisfied:*

$$\frac{d}{dt}\left(\frac{\delta y^k}{\delta t}\right) + \Gamma_i{}^k{}_j(x, y)\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t} = 0, \quad \forall k \in \{4, \dots, 3+n\}, \quad (100a)$$

$$V^\gamma{}_{ij}\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t} = 0, \quad \forall \gamma \in \{0, 1, 2, 3\}. \quad (100b)$$

It is noteworthy that the equations in (99a) and (99b) are related to the geometry of the horizontal distribution. To emphasize this, we give some definitions. First, we say that a curve $\overline{C}$ in $\overline{M}$ is an *autoparallel curve* with respect to the Riemannian horizontal connection $\overset{h}{\nabla}$ if it is a horizontal curve satisfying

$$\overset{h}{\nabla}_{d/dt}\frac{d}{dt} = 0, \quad (101)$$

where d/dt is given by (97a). Then, by direct calculations using (97a) and (63a), we deduce that (101) is equivalent to (99a). Now, according to (71a) we may say that

$$K^k{}_{\alpha\beta} = \frac{1}{2}F^k{}_{\alpha\beta} - H^k{}_{\alpha\beta} \quad (102)$$

are local components of the *second fundamental form* of the distribution $H\overline{M}$. Note that $K^k{}_{\alpha\beta}$ are symmetric with respect to Greek indices if and only if $\overline{M}$ is an integrable distribution. If this is the case and $n = 1$, then $-K^1{}_{\alpha\beta}$ is just the extrinsic curvature which has been intensively used in both the brane-world theory (cf. [9]) and space-time-matter theory (cf. [19]).

Coming back to the general case, we say that a curve $\overline{C}$ in $\overline{M}$ is an *asymptotic line* for $H\overline{M}$ if it is a horizontal curve satisfying

$$K^k{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \quad \forall k \in \{4, \dots, 3+n\}. \quad (103)$$

Then taking into account the skew symmetry of $F^k{}_{\alpha\beta}$, we deduce that (103) is equivalent to (99b). Summing up this discussion and using assertion (i) in Theorem 9, we can state the following characterization of horizontal geodesics.

Corollary 10. *A curve $\overline{C}$ is a horizontal geodesic of $(\overline{M}, \overline{g}, H\overline{M})$ if and only if the following conditions are satisfied:*

(a) $\overline{C}$ is an autoparallel curve with respect to the Riemannian horizontal connection $\overset{h}{\nabla}$ on $H\overline{M}$;

(b) $\overline{C}$ is an asymptotic line for $H\overline{M}$.

Remark 11. A similar characterization can be given for vertical geodesics in $(\overline{M}, \overline{g}, H\overline{M})$. However, we omit it here because as we will see in the last part of the paper the vertical geodesics do not induce any motion on the base manifold.

Next, we consider the case of the integrable horizontal distribution; that is, (86b) is satisfied. Then, any leaf of $H\overline{M}$ is locally given by the equations

$$y^i = c^i, \quad i \in \{4, \ldots, 3+n\}, \tag{104}$$

and it is denoted by $M(c)$. In this case, any horizontal geodesic must lie in only one leaf of $H\overline{M}$, and by Theorem 9 it is given by the following system of equations:

$$\frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}(x,c)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \tag{105a}$$

$$H^k_{\ \alpha\beta}(x,c)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \tag{105b}$$

$$L^k_\alpha(x,c)\frac{dx^\alpha}{dt} = 0, \tag{105c}$$

for all $\gamma \in \{0,1,2,3\}$ and $k \in \{4, \ldots, 3+n\}$. By (105a) we see that horizontal geodesics in $(\overline{M}, \overline{g}, H\overline{M})$ are in fact some particular geodesics of the 4D Lorentz manifolds $(M(c), g_{\alpha\beta}(x,c))$.

Now, we say that $\overline{C}$ is an *oblique geodesic* through a point P_0 if both the 4D velocity and nD velocity are nonzero at P_0. By continuity, we deduce that $\overline{C}$ is an oblique geodesic if and only if both $U(t)$ and $W(t)$ are nonzero for any $t \in [a,b]$. It is important to note that both velocities $U(t)$ and $W(t)$ are involved in the equations of motion in $(\overline{M}, \overline{g}, H\overline{M})$. First, by using (90), (96), and the Riemannian adapted connection $\nabla = (\overset{h}{\nabla}, \overset{v}{\nabla})$ given by (63a), (63b), (63c), and (63d), we obtain

$$\overset{h}{\nabla}_{d/dt}U(t) = \left\{\frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + H_{i\alpha}^{\ \ \gamma}\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}\right\}\frac{\delta}{\delta x^\gamma}, \tag{106a}$$

$$\overset{v}{\nabla}_{d/dt}W(t) = \left\{\frac{d}{dt}\left(\frac{\delta y^k}{\delta t}\right) + \Gamma_{i\ \alpha}^{\ k}\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t} + \Gamma_{i\ j}^{\ k}\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t}\right\}\frac{\partial}{\partial y^k}. \tag{106b}$$

Then, taking into account (106a) and (106b) in (85a) and (85b) we can state the following.

Corollary 12. *An oblique geodesic of $(\overline{M}, \overline{g}, H\overline{M})$ is given by the system of equations*

$$\left(\overset{h}{\nabla}_{d/dt}U(t)\right)^\gamma = V^\gamma_{\ ij}\frac{\delta y^i}{\delta t}\frac{\delta y^j}{\delta t} - \left(H_{i\alpha}^{\ \ \gamma} + F_{i\alpha}^{\ \ \gamma}\right)\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}, \tag{107a}$$

$$\left(\overset{v}{\nabla}_{d/dt}W(t)\right)^k = H^k_{\ \alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{\delta} - V_{\alpha i}^{\ \ k}\frac{dx^\alpha}{dt}\frac{\delta y^i}{\delta t}. \tag{107b}$$

Next, we say that $\overline{C}$ passing through $P_0 \in \overline{U}$ is a *projectable curve* around P_0, if its 4D velocity is nonzero around P_0. Taking into account (90), we deduce that through the projection point Q_0 of P_0 on M is passing a smooth curve C in M given by the equations (see (78a) and (78b))

$$x^\alpha = x^\alpha(t), \quad t \in [a,b], \ \alpha \in \{0,1,2,3\}. \tag{108}$$

In case $\overline{C}$ is a geodesic in $(\overline{M}, \overline{g}, H\overline{M})$, we call C the *induced motion* on M by the motion $\overline{C}$ in $\overline{M}$. Taking into account the definitions of the above three categories of geodesics in $(\overline{M}, \overline{g}, H\overline{M})$ we conclude that horizontal geodesics and oblique geodesics are projectable curves, and therefore they will induce some motions in the base manifold M. Hence, the vertical geodesics have no influence on the 4D dynamics in M. According to the two particular cases considered at the end of Section 5 (see (88) and (89)) we conclude that, in general, the induced motions on M bring more information than both the motions from general relativity and the solutions of the Lorentz force equations. This is due to the existence of extra dimensions and to the action of the Lie group G on $\overline{M}$. Something interesting can be observed from the particular case, where $H\overline{M}$ is integrable (see (105a), (105b), and (105c)). Let $\overline{C}_1$ and $\overline{C}_2$ be two horizontal geodesics in $M(c_1)$ and $M(c_2)$, with initial conditions $\{(x_0^\alpha, c_1^i), (u^\alpha, v_1^i)\}$ and $\{(x_0^\alpha, c_2^i), (u^\alpha, v_2^i)\}$, respectively. Then the induced motions C_1 and C_2 on M have the same initial conditions (x_0^α, u^α), but they come from different systems of equations, and therefore they do not necessarily coincide. This might be used to detect extra dimensions experimentally.

7. Conclusions

In the present paper we obtain, for the first time in the literature, the fully general equations of motion in a general gauge Kaluza-Klein space (cf. (85a) and (85b)). We pay attention to the 4D equations of motion, which of course modify the well-known motions in 4D Einstein gravity. Comparing (85a) with usual 4D equations of motion (88), we note two important differences. First, the local coefficients of the Riemannian horizontal connection $(\Gamma_{\alpha\ \beta}^{\ \gamma}, H_{i\alpha}^{\ \ \gamma})$ do depend on the extra dimensions. Then, there are some extra terms given by the 4D tensor fields $(F_{i\alpha}^{\ \ \gamma}, V^\gamma_{\ ij})$, which, in principle, can be used to test the theory. Such terms in astrophysics might appear for usual velocities of galaxies or clusters of galaxies.

The method developed in the present paper opens new perspectives in the study of some other important concepts from higher dimensional physical theories. Here we have in mind an approach of the dynamics in such spaces under the effect of an extra force whose existence is guaranteed by the extra dimensions. In a particular case (see Section 5) we have seen that such force does not contradict the 4D physics. It is an open question whether this result is still valid in case of a general gauge Kaluza-Klein space. Also, we should stress that the Riemannian adapted connection constructed in Section 4 plays in this general theory the same role as the Levi-Civita connection on the 4D spacetime in the classical Kaluza-Klein theory. This connection together with theory of adapted tensor fields (see Section 3) enables us to think of some 4D Einstein equations induced by the $(4 + n)$D Einstein equations on the ambient space.

All these problems deserve further studies which might show how far the concepts induced by the extra dimensions can be related to the real matter.

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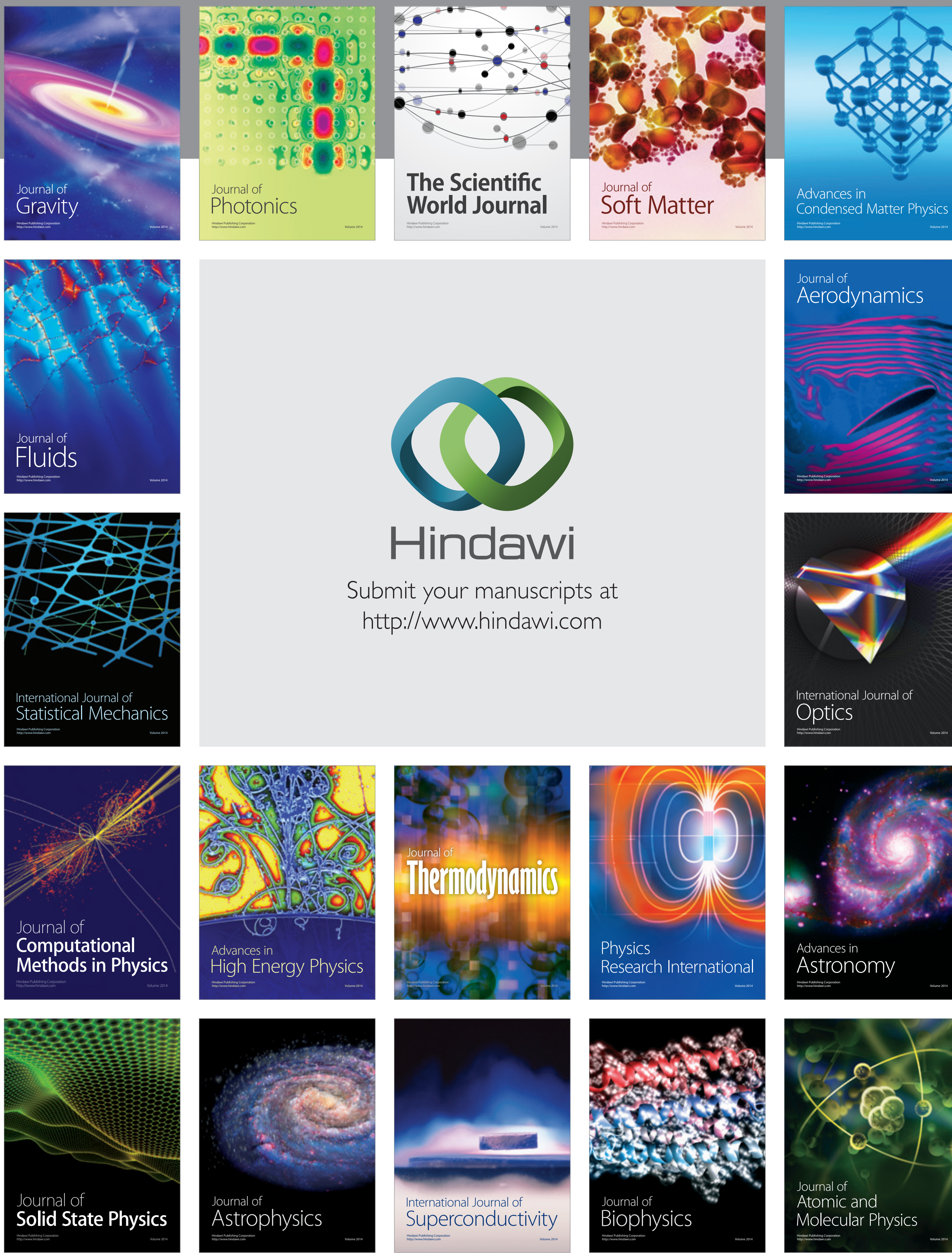

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