ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
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The heteroclinic connection problem for general double-well potentials

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THE HETEROCLINIC CONNECTION PROBLEM FOR GENERAL DOUBLE-WELL POTENTIALS

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ABSTRACT. By variational methods, we provide a simple proof of existence of a heteroclinic orbit to the Hamiltonian system $u'' = \nabla W(u)$ that connects the two global minima of a double-well potential W. Moreover, we consider several inhomogeneous extensions.

1. INTRODUCTION

1.1. **The problem.** In this paper, we will prove existence of solutions $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the following problem:

$$u_{xx} = \nabla W(u), \quad x \in \mathbb{R}, \quad \lim_{x \to \pm\infty} u(x) = a_\pm, \tag{1.1}$$

where

$$W \in C^1(\mathbb{R}^n),\ n \geq 1,\ \text{satisfy } W(a_-) = W(a_+) = 0,\ W(u) > 0 \text{ if } u \neq a_\pm, \tag{1.2}$$

for some $a_- \neq a_+$, and

$$\liminf_{|u| \to \infty} W(u) > 0. \tag{1.3}$$

Since $a_- \neq a_+$, such a solution is called a *heteroclinic connection*, as opposed to a homoclinic. Motivated from mechanics, in relation with Newton's second law of motion (where x plays the role of time), we will often refer to W as a double-well potential (see also [6] and the references therein).

We note that the quantity

$$\frac{1}{2}|u_x|^2 - W(u)$$

is constant along solutions of the equation, which easily implies that $W(a_-) = W(a_+)$ is a necessary condition for a heteroclinic connection to exist between a_- and a_+.

We will also study the inhomogeneous problem

$$u_{xx} = h(x)\nabla W(u), \quad \lim_{x \to \pm\infty} u(x) = a_\pm, \tag{1.4}$$

under various assumptions on h.

1.2. **Motivation.** The theory of phase transitions has led to the extensive study of singularly perturbed, non-convex energies of the form

$$J_\varepsilon(u) = \int_\Omega \left\{ \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right\} dx,$$

where W is a nonnegative potential with multiple global minima. In the scalar case, this problem was studied by Modica [33] using De Giorgi's notion of Γ-convergence (see also [3] and the references therein). In the vectorial case of two global minima, that is when (1.2)-(1.3) hold, the Γ-limit of this energy was studied in [13], [24]. The case where W has more than two wells was considered in [12] (see also [40]). In this context, the heteroclinic connections determine the interfacial energy.

In parallel, the interest in the heteroclinic connection problem stems also from the study of the vectorial Allen-Cahn equation that models multi-phase transitions (see [1], [3], [5], [6], [15], [19], and the references therein). Loosely speaking, the heteroclinic connections are expected to describe the way in which the solutions to the multi-dimensional parabolic system

$$u_t = \varepsilon^2 \Delta u - \nabla W(u),$$

for small $\varepsilon > 0$, transition from one state to the other (see [18]).

The heteroclinic connection problem also comes up when studying phase coexistence in consolidating porous medium (see [22] and the references therein), crystalline grain boundaries (see [17]), planar transition front solutions to the Cahn-Hilliard system [29], and domain walls in coupled Gross-Pitaevskii equations (see [2] and the references therein).

We emphasize that some of these applications require a triple-well potential. Nevertheless, under a reflection symmetry assumption on W (which is frequently inherited from the physical model), the problem can easily be reduced to the double-well case (see [39]).

For an application which requires one to consider potentials with degenerate minima, we refer to [11].

Our motivation for the inhomogeneous problems is twofold:

In [34], among other things, by employing singular perturbation techniques, the author constructed heteroclinic connections to the scalar spatially inhomogeneous Allen-Cahn equation

$$u_{xx} = h(\varepsilon x) W'(u) \ \text{ such that } \ \lim_{x \to \pm\infty} u(x) = a_{\pm}, \tag{1.5}$$

provided that $\varepsilon > 0$ is sufficiently small, where W has the same features as in the present paper but assuming non-degeneracy of the global minima; h is strictly positive, bounded, and having at least one non-degenerate local minimum. The result relies on the fact that the $\varepsilon = 0$ limit problem has a unique, asymptotically stable heteroclinic solution. Our results provide existence for all $\varepsilon > 0$ and hold for systems with more general W. Moreover, we believe that, with some more effort, they can provide information about the $\varepsilon \to 0$ asymptotic behavior of the solutions.

Recently, there has been an interest in constructing heteroclinic solutions to systems of semilinear equations (see [9]). In that case, in order to exclude the possibility of constructing the one dimensional heteroclinic, one has to impose some spatial inhomogeneity to the problem. For related results concerning solutions of the system

$$\Delta u = \nabla W(u), \quad u : \mathbb{R}^m \to \mathbb{R}^n, \ m, n \geq 1,$$

connecting global minima of the potential W along certain directions, we refer the interested reader to [1], [5], [7], [8], [19], [26], [27], [28], [36], and the references therein. We stress that, even though some of the results of the current paper were previously proven by different methods, our approach, a considerable refinement of that of [6], has the advantage of being flexible enough to potentially treat the case of these semilinear elliptic systems.

1.3. **Known results.** The problem (1.1) is completely understood if $n = 1$, see for instance [3], [16]; in fact, assumption (1.3) is not needed in that case.

If $n \geq 2$, under assumptions (1.2)-(1.3), the existence of a heteroclinic orbit was proven by Rabinowitz in [35] via a variational approach (see also [16, Thm. 2.3]).

Under various additional nondegeneracy or geometric conditions near the global minima of W, this problem has been dealt, mostly as a tangential issue, in several references. Under the assumption that

$$W(a_\pm + \rho\nu) \text{ is increasing in } \rho \in [0,\delta],\ \forall\, \nu \in \mathbb{S}^{n-1}, \tag{1.6}$$

(for some small $\delta > 0$), where $\mathbb{S}^{n-1}$ stands for the unit sphere, the existence of the heteroclinic connection was proven recently by Alikakos and Fusco in [6]. Their novelty was to employ constraints which are subsequently removed. If $W(a_\pm + \rho\nu) \geq c\rho^\gamma$, $\rho \in [0,\delta]$, for some $c, \gamma, \delta > 0$, and assuming that the level sets of W near $a_\pm$ are strictly convex, the existence of the heteroclinic connection was proven very recently by Katzourakis in [32] in the spirit of the concentrated compactness method. If the global minima of W are non-degenerate, that is the Hessian $\partial^2 W(a_\pm)$ is positive definite, the existence of the heteroclinic connection was proven by Sternberg in [40] by using techniques from Γ-convergence theory (an additional growth condition as $|u| \to \infty$ was also assumed). Other variational proofs which require non-degeneracy of the global minima can be found in [1], [2], [5], [19], and [37]. In fact, as is pointed out, the proof of [2] carries over to the case where W vanishes to finite order at $a_\pm$.

1.4. **The main result.** Our primary goal is to give a new simple proof of the following theorem.

Theorem 1.1. [35] Under assumptions (1.2) and (1.3), there exists a solution $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the problem (1.1).

Then, we adapt this proof to treat in a unified way a class of spatially inhomogeneous problems.

1.5. **Method of proof and outline of the paper.** Our proof is motivated from the constraint variational set up of [6] but, instead of using energy decreasing local replacement arguments as a substitute of the maximum principle, we will use energy controlling local replacements together with a clearing-out argument. In particular, we do not need to employ the polar representation that was used in [6] (see also the introduction in [15]), that is to write a function $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ as

$$u(x) = a_\pm + \rho_\pm(x)\Theta_\pm(x) \text{ whenever } \rho_\pm(x) = |u(x) - a_\pm| \neq 0;\quad u(x) = a_\pm \text{ otherwise},$$

which turns out to be a rather cumbersome issue (especially in the case of the corresponding elliptic problems, see [9]).

To the best of our knowledge, besides of rendering the most general result, our proof is the simplest available.

The outline of the paper is the following: In Section 2 we present the proof of Theorem 1.1, and in Section 3 we consider extensions to the inhomogeneous case.

2. Proof of the main result

Proof of Theorem 1.1. The main part of the proof will be devoted in showing that there exists a solution $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the equation

$$u_{xx} = \nabla W(u), \tag{2.1}$$

and an $L > 0$, such that

$$|u(x) - a_-| < \delta,\quad x \leq -L;\quad |u(x) - a_+| < \delta,\quad x \geq L, \tag{2.2}$$

for some small $\delta < |a_+ - a_-|$. To this end, as in [6], for $L > 2$, let

$$X_L^- = \left\{ u \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) \ : \ |u(x) - a_-| \leq \delta, \ \ x \leq -L \right\}, \tag{2.3}$$

$$X_L^+ = \left\{ u \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) \ : \ |u(x) - a_+| \leq \delta, \ \ x \geq +L \right\}. \tag{2.4}$$

It is standard to show that there exists a $u_L \in X_L^- \cap X_L^+$ such that

$$J(u_L) = \inf_{u \in X_L^- \cap X_L^+} J(u) < \infty,$$

where $J : W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) \to [0, \infty]$ is the associated energy functional

$$J(u) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |u_x|^2 + W(u) \right\} dx,$$

(see [6]). Our goal is to show that there exists $L \gg 1$ such that u_L (or a translate of it) satisfies (2.2), since this will imply that u_L is a classical solution to (2.1). We note that, a-priori, the minimizer u_L is C^2 and satisfies the Euler-Lagrange equation (2.1) *only in* $(-L, L)$ *and wherever it is away from the cylindrical boundary of the constraints.*

By constructing a piecewise linear competitor that is identically equal to a_- for $x \leq -1$ and equal to a_+ for $x \geq 1$, it is easy to show that

$$J(u_L) \leq C_1, \tag{2.5}$$

where the constant $C_1 > 0$ *is independent of* $L > 2$ (an analogous argument also appears in [20]).

We claim that, given any $d \in (0, \delta)$, there exists $\varepsilon \in (0, \frac{d}{2})$, *independent of* $L > 2$, such that

$$\text{if} \ \ x_2 - x_1 \geq 3 \ \ \text{and} \ \ |u_L(x_i) - a_\pm| \leq \varepsilon, \ \ i = 1, 2, \tag{2.6}$$

then

$$|u_L(x) - a_\pm| < d, \quad x \in [x_1, x_2]. \tag{2.7}$$

It is clear that we only have to verify this claim for the $+$ case. To this end, assume $(2.6)_+$. The minimality property of u_L implies that there exists a constant $C_2 > 0$, *independent of* ε, x_1, x_2, L, such that

$$\int_{x_1}^{x_2} \left\{ \frac{1}{2} \left| (u_L)_x \right|^2 + W(u_L) \right\} dx \leq C_2 \varepsilon, \tag{2.8}$$

(if W was C^2, we would have ε^2). (Indeed, one can easily cook up a suitable competitor which agrees with u_L outside of (x_1, x_2) and is equal to a_+ over $[x_1 + 1, x_2 - 1]$, see also [10, Rem. 2.3]). The desired claim now follows by applying the clearing-out lemma in [15] (see Lemma 1 therein). For the sake of completeness, let us present a different argument. Suppose to the contrary that there exists $x_* \in (x_1, x_2)$ such that

$$|u_L(x) - a_+| < d, \quad x \in [x_1, x_*), \quad \text{and} \quad |u_L(x_*) - a_+| = d. \tag{2.9}$$

Note that there exists a $V \in C[0, \delta]$, $V > 0$ on $(0, \delta]$, such that

$$W(a_\pm + \rho\nu) \geq V(\rho) \ \ \forall \ \rho \in [0, \delta], \ \ \nu \in \mathbb{S}^{n-1}. \tag{2.10}$$

Indeed, plainly set $V(\rho) = \min\{V_-(\rho), V_+(\rho)\}$, where

$$V_\pm(\rho) = \min_{\nu \in \mathbb{S}^{n-1}} W(a_\pm + \rho\nu), \quad \rho \in [0, \delta].$$

Moreover, observe that $u_L(x) \neq a_+$, $x \in [x_1, x_*]$ (if not and $u_L(\bar{x}) = a_+$ for some $\bar{x}$, the function which is identically u_L for $x < \bar{x}$ and a_+ for $x \geq \bar{x}$ would have less energy than the minimizer u_L). Armed with this information, we have

$$\begin{array}{rcl} \int_{x_1}^{x_*}\left\{\frac{1}{2}\left|(u_L)_x\right|^2 + W(u_L)\right\}dx & \overset{(2.10)}{\geq} & \int_{x_1}^{x_*}\left\{\frac{1}{2}\left|(u_L-a_+)_x\right|^2 + V\left(|u_L-a_+|\right)\right\}dx \\ \text{via the diamagnetic inequality [25, Prop. 2.1.2]:} & \geq & \int_{x_1}^{x_*}\left\{\frac{1}{2}\left|u_L-a_+\right|_x^2 + V\left(|u_L-a_+|\right)\right\}dx \\ \text{as in [33]:} & \geq & \sqrt{2}\int_{x_1}^{x_*}|u_L-a_+|_x V^{\frac{1}{2}}\left(|u_L-a_+|\right)dx \\ & \geq & \sqrt{2}\int_{\frac{d}{2}}^{d} V^{\frac{1}{2}}(\rho)d\rho. \end{array}$$

Therefore, on account of (2.8), we can exclude the possibility (2.9) by choosing

$$\varepsilon \in \left(0, \frac{d}{2}\right) \quad \text{such that} \quad \varepsilon < \frac{1}{C_2}\int_{\frac{d}{2}}^{d} V^{\frac{1}{2}}(\rho)d\rho. \tag{2.11}$$

In fact, if (1.6) holds, as in [6], it follows from (2.6) that $|u_L(x) - a_\pm| \leq \varepsilon, \quad x \in [x_1, x_2]$.

Next, we claim that, for any $\zeta > 0$ sufficiently small, there exists $M > 3$, *independent of* L, and a sequence of positive numbers $x_1^+ < x_2^+ < \cdots$, with

$$M < x_{i+1}^+ - x_i^+ < 3M, \quad i \geq 1, \tag{2.12}$$

such that

$$W\left(u_L(x_i^+)\right) \leq \zeta, \quad i \geq 1. \tag{2.13}$$

To see this, plainly take

$$M = C_1\zeta^{-1}, \tag{2.14}$$

where C_1 is as in (2.5) (we may assume that $M > 3$), and apply the mean value theorem in the intervals $[0, M], [2M, 3M], \cdots$. Analogously, given $\zeta > 0$ sufficiently small, we can find negative numbers $\cdots < x_2^- < x_1^-$, with $M < x_i^- - x_{i+1}^- < 3M$ (increasing the value of M if needed), such that $W\left(u_L(x_i^-)\right) \leq \zeta$, $i \geq 1$.

Let $\varepsilon > 0$ be as in (2.11) with $d = \delta$, and $\zeta > 0$ be such that

$$W(u) \leq \zeta \quad \text{implies that} \quad |u - a_-| \leq \varepsilon \text{ or } |u - a_+| \leq \varepsilon. \tag{2.15}$$

We then choose

$$L = 1000M,$$

where $M > 3$ is as in (2.14). From (2.6), (2.7), (2.12), (2.13), and (2.15), we certainly have that

$$|u_L(x) - a_-| < \delta \quad \text{if} \quad x \leq -1010M; \quad |u_L(x) - a_+| < \delta \quad \text{if} \quad x \geq 1010M. \tag{2.16}$$

In view of (2.13) and (2.15), only two possibilities can occur:

(1) $|u_L(x_1^+) - a_+| \leq \varepsilon$. Then, by the property (2.6)–(2.7) and the second part of (2.16), we infer that $|u_L(x) - a_+| < \delta$ for $x \geq x_1^+ \in (0, M)$. Hence, by abusing notation and replacing u_L with its translate $u_L(\cdot - 20M) \in X_L^- \cap X_L^+$, if necessary (they have the same energy), via the first part of (2.16), we deduce that (2.2) holds, as desired.

(2) $|u_L(x_1^+) - a_-| \leq \varepsilon$. Then, we have that $|u_L(x) - a_-| < \delta$ for $x \leq x_1^+ \in (0, M)$ (from (2.6)-(2.7) and the first part of (2.16)). In that case, as before, replacing u_L by the translated minimizer $u_L(\cdot + 20M)$, if necessary, we find that (2.2) holds, as desired.

We have thus shown that the minimizer u_L satisfies (2.2). In particular, by standard arguments (see [6]), it induces a classical solution to (2.1). To complete the proof of the theorem, we will show that $\lim_{x\to\pm\infty} u_L(x) = a_\pm$. Indeed, for any arbitrarily small $d > 0$, by (2.13)–(2.15), there exists a sequence $x_i \to \infty$ such that $|u_L(x_i) - a_+| < \varepsilon$, where ε as in (2.11). Then, in view of (2.6)-(2.7), we obtain that $|u_L(x) - a_+| < d$, $x \geq x_1$. Similarly, we can show that $\lim_{x\to-\infty} u_L(x) = a_-$.

The proof of the theorem is complete. □

Remark 2.1. The proof of Theorem 1.1 carries over without difficulty to the quasi-linear setting:

$$\left(|u_x|^{p-2}u_x\right)_x = \nabla W(u), \quad \lim_{x\to\pm\infty} u(x) = a_\pm, \quad (p > 2),$$

that was considered very recently in [31], and the references therein, under assumption (1.6). The only essential difference is that one has to modify slightly the proof of the clearing-out lemma of [15] by using the Hölder inequality instead of the Cauchy-Schwarz .

3. Inhomogeneous problems

3.1. The periodic inhomogeneity.

Theorem 3.1. Assume that $h \in C(\mathbb{R})$ is T-periodic and positive. Under assumptions (1.2) and (1.3), there exists a solution to the problem (1.4).

Proof. The proof is completely analogous to that of Theorem 1.1. The only difference is that we take M in (2.14) to be a large multiple of the period T. □

Remark 3.1. In the scalar case ($n = 1$), further assuming that $a_\pm$ are non-degenerate minima of W, this problem was considered in [4], and for W as above in [16].

3.2. The asymptotically constant inhomogeneity.

Theorem 3.2. Assume that $h \in C(\mathbb{R})$ is positive, bounded,

$$\lim_{x\to\pm\infty} h(x) = h_\infty > 0 \quad \text{and} \quad h(x) \leq h_\infty, \ x \in \mathbb{R}. \tag{3.1}$$

Under assumptions (1.2) and (1.3), there exists a solution to the problem (1.4).

Proof. The main difference of the problem at hand with the previous ones is that there is no translation invariance (continuous or discrete).

As before, for $L > 2$, let

$$m_L = \inf_{u \in X_L^- \cap X_L^+} J(u), \tag{3.2}$$

where $X_L^\pm$ are as in (2.3)–(2.4), and

$$J(u) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|u_x|^2 + h(x)W(u) \right\} dx. \tag{3.3}$$

It is easy to show that the infimum is attained at some $u_L \in X_L^- \cap X_L^+$.

Motivated from [14], where ground states to the nonlinear Schrödinger equation with potential h were considered, we will compare m_L with

$$m_{\infty,L} = \inf_{u \in X_L^- \cap X_L^+} \int_{\mathbb{R}} \left\{ \frac{1}{2}|u_x|^2 + h_\infty W(u) \right\} dx,$$

which, as we have already shown in Theorem 1.1, is attained by a classical solution $u_{\infty,L} \in X_L^- \cap X_L^+$ of the problem

$$u_{xx} = h_\infty \nabla W(u), \quad \lim_{x\to\pm\infty} u(x) = a_\pm,$$

provided that L is sufficiently large. We may assume that $h(x) < h_\infty$ somewhere, say that

$$h(x) < h_\infty, \quad x \in (x_-, x_+), \tag{3.4}$$

for some $x_-, x_+ \in \mathbb{R}$. By translating $u_{\infty,L}$, if necessary, we may assume that

$$|u_{\infty,L}(x_-) - a_-| \geq \delta \ \text{ and } \ |u_{\infty,L}(x_-) - a_+| \geq \delta,$$

for large L. Observe that, from the proof of Theorem 1.1, this can be achieved while keeping that $u_{\infty,L} \in X_L^- \cap X_L^+$ (intuitively, $u_{\infty,L}$ has at most $3M$ time to transition from a_- to a_+). Therefore, by the analog of (2.5), it is easy to see that

$$|u_{\infty,L}(x) - a_-| \geq \frac{\delta}{2} \ \text{ and } \ |u_{\infty,L}(x) - a_+| \geq \frac{\delta}{2} \ \text{ for } \ x \in \left[x_-, x_- + \frac{\delta^2}{8C_1}\right], \tag{3.5}$$

(the point being that this interval is independent of large L). Indeed, if $x \in \left[x_-, x_- + \frac{\delta^2}{8C_1}\right]$, letting $\rho_\pm(x) = |u_{\infty,L}(x) - a_\pm|$, we have

$$|\rho_\pm(x) - \rho_\pm(x_-)| \leq \int_{x_-}^{x} |u_{\infty,L} - a_\pm|_t dt \leq \int_{x_-}^{x} |(u_{\infty,L})_t|\, dt \leq |x - x_-|^{\frac{1}{2}} (2C_1)^{\frac{1}{2}} \leq \frac{\delta}{2}.$$

Then, using $u_{\infty,L}$ as a test function, we find that

$$\begin{aligned} m_L &\leq \int_{\mathbb{R}} \left\{\tfrac{1}{2}|(u_{\infty,L})_x|^2 + h(x)W(u_{\infty,L})\right\} dx \\ &= \int_{\mathbb{R}} \left\{\tfrac{1}{2}|(u_{\infty,L})_x|^2 + h_\infty W(u_{\infty,L})\right\} dx + \int_{\mathbb{R}} \left(h(x) - h_\infty\right) W(u_{\infty,L}) dx \\ \text{via (3.4), (3.5)} \quad &\leq m_{\infty,L} - c \end{aligned} \tag{3.6}$$

where $c > 0$ is independent of large L.

Observe that all the properties in the proof of Theorem 1.1 up to (2.15) remain true for this u_L with the obvious changes (with h in from of W); in fact, let us keep the same notation. This time we let

$$L = L_j = jM,$$

with j a sufficiently large integer that is to be determined so that (2.2) holds, which in particular will imply that u_L is a classical solution to

$$u_{xx} = h(x)\nabla W(u). \tag{3.7}$$

Suppose, to the contrary, that there exists a sequence of $L_j \to \infty$ such that (2.2) with $L = L_j$ is violated at some $x \leq -L_j$ (the other case is completely analogous). Then, by the analogous property to (2.6)-(2.7), denoting u_{L_j} by u_j, we would have that

$$|u_j(x) - a_-| < \delta \ \text{ if } \ x \leq -(j+10)M; \quad |u_j(x) - a_+| < \delta \ \text{ if } \ x \geq -(j-10)M. \tag{3.8}$$

From the second part of the above relation (which implies that u_j solves (3.7) for $x > -(j-10)M$), making use of Arczela-Ascoli's theorem and the standard diagonal argument, passing to a subsequence if needed, we find that

$$u_j \to U \ \text{ in } C^1_{loc}(\mathbb{R}, \mathbb{R}^N),$$

where U satisfies

$$U_{xx} - h(x)\nabla W(U) = 0, \quad |U(x) - a^+| \leq \delta, \quad x \in \mathbb{R}. \tag{3.9}$$

Furthermore, from the analog of (2.5), we obtain that

$$J(U) \leq C_1, \tag{3.10}$$

where J is the energy in (3.3). Moreover, from the minimality of u_j, and the second part of (3.8), it follows readily that U is a minimizer of the energy subject to its boundary conditions, that is

$$J(U) \leq J(U + \varphi) \quad \forall\, \varphi \in W_0^{1,2}(I) \text{ and any interval } I,$$

(this can be proven as in [23]). As in the proof of Theorem 1.1, given any $d \in (0, \delta)$, there exists $\varepsilon \in \left(0, \frac{d}{2}\right)$ such that property $(2.6)_+ - (2.7)_+$ holds for U. Combining (2.15), (3.10), and the fact that h is bounded from below by some positive constant, we find that $U \equiv a_+$ (in the case where (1.6) holds, not necessarily with a strict inequality, this can also be deduced by the weak sub-harmonicity of the function $\rho = |U - a_+|$, which follows directly from (3.9)). In particular, we get that

$$W(u_j) \to 0 \text{ in } C_{loc}(\mathbb{R}). \tag{3.11}$$

On the other hand, we have

$$\begin{aligned} m_{L_j} &= \int_{\mathbb{R}} \left\{ \tfrac{1}{2} |(u_j)_x|^2 + h(x) W(u_j) \right\} dx \\ &= \int_{\mathbb{R}} \left\{ \tfrac{1}{2} |(u_j)_x|^2 + h_\infty W(u_j) \right\} dx + \int_{\mathbb{R}} \left(h(x) - h_\infty \right) W(u_j) dx \\ \text{via (3.1) and (3.11)}: &\geq m_{\infty, L_j} + o(1), \end{aligned}$$

where $o(1) \to 0$ as $j \to \infty$, which contradicts (3.6).

Having established that (2.2) holds for sufficiently large L, the rest of the proof proceeds verbatim as that of Theorem 1.1. □

Remark 3.2. Using a different variational argument, the above theorem was proven in the scalar case in [16, Thm. 2.2].

In the vector case, under additional assumptions which include the non-degeneracy of the global minima, related results have been obtained in [21].

3.3. **The diverging inhomogeneity.**

Theorem 3.1. *Assume that $h \in C(\mathbb{R})$ is nonnegative, and*

$$\lim_{x \to \pm\infty} h(x) = \infty. \tag{3.12}$$

Under assumptions (1.2) and (1.3), there exists a solution to the problem (1.4).

Proof. Our strategy remains the same. We consider the constraint minimization problem (3.2)-(3.3) and show that any minimizer u_L (which exists by standard arguments) satisfies (2.2), provided that L is sufficiently large. Clearly, estimate (2.5) holds (abusing notation).

We claim that, for large L, we have that

$$|u_L(x) - a_+| < \delta, \quad x \geq L.$$

Indeed, suppose to the contrary that there exists $x_+ \geq L$ such that $|u_L(x_+) - a_+| = \delta$. Then, arguing as in (3.5), we find that

$$|u_L(x) - a_+| \geq \frac{\delta}{2} \quad \text{for} \quad x \in \left[x_+, x_+ + \frac{\delta^2}{8C_1}\right].$$

In turn, this implies that

$$W(u_L(x)) \geq c > 0, \quad x \in \left[x_+, x_+ + \frac{\delta^2}{8C_1}\right],$$

where *the constant* $c > 0$ *is independent of large* L. On the other hand, if L is sufficiently large, the above relation contradicts the fact that

$$\int_L^\infty W(u_L(t))\, dt \to 0 \quad \text{as} \quad L \to \infty,$$

which follows directly from (2.5) and (3.12). Analogously, we can show that

$$|u_L(x) - a_-| < \delta, \quad x \leq -L.$$

Having established that u_L satisfies (2.2) (and as a consequence (3.7)), for sufficiently large L, we can proceed in a similar manner to show that it also satisfies the desired asymptotic behavior at respective infinities. □

Remark 3.3. If $h(x) > 0,\ x \in \mathbb{R}$, the above theorem is contained in [30].

Remark 3.4. In [38], relying on the oddness of the nonlinearity, we used a shooting argument to show that there exists a unique odd solution to the problem

$$u_{xx} = |x|^\alpha (u^3 - u), \quad \lim_{x\to\pm\infty} u(x) = \pm 1,$$

where $\alpha > 0$. Moreover, this solution is increasing and asymptotically stable. This heteroclinic connection describes the profile of the transition layer, near $x = 0$, of the singular perturbation problem (1.5) with $h \sim |x|^\alpha$ as $x \to 0$ and $h > 0$ elsewhere (here $W(u) = \frac{(u^2-1)^2}{4}$).

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