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ON THE XFEL SCHRÖDINGER EQUATION: HIGHLY OSCILLATORY MAGNETIC POTENTIALS AND TIME AVERAGING

PAOLO ANTONELLI, AGISILLAOS ATHANASSOULIS, HICHEM HAJAIEJ, AND PETER MARKOWICH

ABSTRACT. We analyse a nonlinear Schrödinger equation for the time-evolution of the wave function of an electron beam, interacting selfconsistently through a Hartree-Fock nonlinearity and through the repulsive Coulomb interaction of an atomic nucleus. The electrons are supposed to move under the action of a time dependent, rapidly periodically oscillating electromagnetic potential. This can be considered a simplified effective single particle model for an X-ray Free Electron Laser (XFEL). We prove the existence and uniqueness for the Cauchy problem and the convergence of wave-functions to corresponding solutions of a Schrödinger equation with a time-averaged Coulomb potential in the high frequency limit for the oscillations of the electromagnetic potential.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we investigate the following nonlinear Schrödinger equation

$$(1.1) \qquad i\hbar\partial_t\psi = (i\hbar\nabla - A)^2\psi + c\frac{1}{|x|}\psi + C_1(|\cdot|^{-1} * |\psi|^2)\psi - a|\psi|^\sigma\psi.$$

The coefficients c, C_1, a and the exponent σ are assumed to be nonnegative and $\hbar$ is supposed to be a scaled version of the Planck constant, which - w.l.o.g. - shall be set equal to 1 in the sequel. A solution ψ of this Schrödinger equation can be considered as the wavefunction of an electron beam, interacting self-consistently through the repulsive Coulomb (Hartree) force with strength C_1, the attractive local Fock approximation with strength a (later on we shall comment on the exponent σ) and interacting repulsively with an atomic nucleus, located at the origin, of interaction strength c. The vectorfield A represents an external electromagnetic field, which we shall assume to depend on time t only (not on position x). Clearly, this Schrödinger equation is time-reversible, but for the sake of notational simplicity we consider $t > 0$. For physical reasons we shall only consider the three-dimensional case here, i.e. the spatial variable x is assumed to be in $\mathbb{R}^3$.

Nevertheless, because of the sole dependence of A on time, by a simple change of coordinates and a phase shift, we see that equation (1.1) can be transformed into a similar nonlinear Schrödinger equation, where the electromagnetic Laplacian is replaced by the standard one, but on the other hand a time-dependent Coulomb potential appears. Indeed, by defining

$$(1.2) \qquad u(t, x) = \psi(t, x + b(t))e^{i\int_0^t |A(s)|^2 ds},$$

where $b(t) = 2\int_0^t A(s)ds$, then we can see that u satisfies

$$(1.3) \qquad i\partial_t u = -\Delta u + Vu + C_1(|\cdot|^{-1} * |u|^2)u - a|u|^\sigma u,$$

where now the potential is given by

$$V(t,x) = \frac{c}{|x - b(t)|}.$$

In this paper we are interested in studying the case when $A(t)$ is rapidly oscillating and we investigate the asymptotic behaviour of solutions of (1.3) in the highly oscillating regime. Then the equation (1.1) can be considered as a model for XFEL (X-Ray Free Electron Laser), cf. [3].

As an example we can think of $b(t) = \vec{e}\sin(\omega t)$, where $\omega \gg 1$ is the oscillation frequency, and $\vec{e}$ is a constant vector in $\mathbb{R}^3$, but as we will show this can be extended to the case where the field b can be written as

$$(1.4) \qquad\qquad b(t) = \vec{e}(t)f(\omega t),$$

where $\vec{e} : \mathbb{R} \to \mathbb{R}^3$ is a smooth vector field and f is an arbitrary continuous, $2\pi-$periodic function.

To this end, we shall point out the ω dependence of functions with a superscript, $b^\omega(t) = \vec{e}(t)f(\omega t)$, and

$$(1.5) \qquad\qquad V^\omega(t,x) = \frac{c}{|x - b^\omega(t)|}.$$

We will then study solutions of the Cauchy problem

$$(1.6) \qquad \begin{cases} i\partial_t u^\omega = -\Delta u^\omega + V^\omega u^\omega + C_1(|\cdot|^{-1} * |u^\omega|^2)u^\omega - a|u^\omega|^\sigma u^\omega \\ u^\omega(0) = u_0, \end{cases}$$

and their convergence to solutions of the *averaged* equation

$$(1.7) \qquad \begin{cases} i\partial_t u = -\Delta u + \langle V \rangle u + C_1(|\cdot|^{-1} * |u|^2)u - a|u|^\sigma u \\ u(0) = u_0, \end{cases}$$

where $\langle V \rangle$ is the limiting potential and is given (see Section 2 for details) by

$$(1.8) \qquad \langle V \rangle(t,x) := \int_0^1 \frac{c}{|x - \vec{e}(t)\sin(2\pi\omega\tau)|}d\tau.$$

The main theorem we will prove in this paper is the following one

Theorem 1.1. *Let $0 < \sigma < 4/3$, $u_0 \in L^2(\mathbb{R}^3)$, $u^\omega, u \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3))$ be the unique global solutions of (1.6), (1.7), respectively (see Theorem 3.5 below). Then for each finite time $0 < T < \infty$ and for each admissible Strichartz index pair (q,r), we have*

$$\|u^\omega - u\|_{L^q([0,T];L^r(\mathbb{R}^3))} \to 0 \qquad as \ |\omega| \to \infty.$$

Remark 1.2. *For the statement of this Theorem we restrict ourselves to the case when the power-type nonlinearity is* mass-subcritical *(see [10], [1]). Anyway the physically interesting exponent for this model, i.e. $\sigma = \frac{2}{3}$, is included in the Theorem. However, for its mathematical interest, the case of a* energy-subcritical *nonlinearity will be the object of a future investigation.*

By means of formula (1.2), the main result gives us the asymptotic behavior for ψ, solution of (1.1).

Corollary 1.3. *Let $0 < \sigma < \frac{4}{3}$, $\psi_0 \in L^2(\mathbb{R}^3)$, $A = A^\omega(t)$ be such that*

$$2\int_0^t A^\omega(s)ds = b^\omega(t) = \vec{e}(t)f(\omega t),$$

as in (1.4) and let $\psi^\omega \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3))$ be the unique global solution of the Cauchy problem (1.1). Then for each finite time $0 < T < \infty$ and for each admissible

Strichartz index pair (q, r) (see Section 2), we have

$$\left(\int_0^T \left(\int_{\mathbb{R}^3} \left| \psi^\omega(t, x) - e^{-i \int_0^t |A^\omega(s)|^2 ds} u(t, x - b^\omega(t)) \right|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{r}} = o(1),$$

as $|\omega| \to \infty$, where $u \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3))$ is the solution to (1.7).

In Section 2 we review some results about periodic functions and weak convergence. We also recall the Strichartz inequalities associated to the Schrödinger group, in the spirit of [7] (see also [4]). Such estimates will be then used in Section 3 to perform a fixed point argument and to show the local well-posedness for the Cauchy problems (1.6) and (1.7). By using the conservation of mass in the case of a L^2−subcritical power-type nonlinearity we also prove the global well-posedness (see the seminal paper by Tsutsumi [11], and also the monographs [1], [10], [8]), by obtaining some uniform bounds for $\{u^\omega\}$ in ω.

In Section 4 we prove the main result of this paper, Theorem 1.1. The idea for the proof is as in [2] and can be easily explained in the following way: if we consider the Duhamel's formula for equation (1.6), then the oscillating potential (1.5) appears inside the time integral, thus the weak convergence for (1.5) can be improved to the strong one for $\{u^\omega\}$. This is indeed possible thanks to the uniform bounds in ω we have for $\{u^\omega\}$.

2. PRELIMINARY RESULTS AND NOTATIONS

In this Section we first recall some basic facts about weak convergence and periodic functions, which will then be extended to adapt them to our analysis. Finally, we will also give a very quick overview on dispersive estimates for the Schrödinger equation and on local and global analysis of its solutions.

First of all, let us recall the following theorem about weak limits of rapidly oscillating functions.

Theorem 2.1. *Let $1 \le p \le \infty$ and f be a 2π−periodic function in $L^p(0, 2\pi)$. Let us define*

$$f_n(t) := f(nt), \qquad n \in \mathbb{N}.$$

Then for $1 \le p < \infty$,

$$f_n \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \qquad \text{in } L^p(I), \text{ for any bounded } \Omega \in \mathbb{R},$$

and for $p = \infty$ we have

$$f_n \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \qquad \text{in } L^\infty(\mathbb{R}),$$

where the convergence is weak−∗ in $L^\infty(\mathbb{R})$.

Another very basic fact is that weak convergence is basically the convergence in average for the sequence: indeed the following theorem holds.

Theorem 2.2. *Let $\{f_n\} \subset L^p(\mathbb{R}^N)$ be a uniformly bounded sequence in $L^p(\mathbb{R}^N)$. Then the following are equivalent:*

(1)
$$f_n \rightharpoonup f \qquad \text{in } L^p(\mathbb{R}^N);$$

(2)
$$f_n \rightharpoonup f \qquad \text{in } \mathcal{D}'(\mathbb{R}^N);$$

(3) *for each Borel set $E \subset \mathbb{R}^N$, $0 < |E| < \infty$ we have*

$$\lim_{n \to \infty} \frac{1}{|E|} \int_E f_n(x) dx = \frac{1}{|E|} \int_E f(x) dx;$$

(4) *for each cube $E \subset \mathbb{R}^N$, $0 < |E| < \infty$ we have*

$$\lim_{n \to \infty} \frac{1}{|E|} \int_E f_n(x) dx = \frac{1}{|E|} \int_E f(x) dx;$$

Remark 2.3. *The above Theorem holds for each exponent $1 \leq p \leq \infty$: obviously in the $p = \infty$ case one has to change the weak convergence with the weak$-*$ convergence in L^∞.*

Furthermore, the same result is also valid in the more general case of a function $g \in L^p(\mathbb{R}; X)$, where X is an arbitrary Banach space. Clearly we are interested in the case when X is a Lebesgue space $L^s(\mathbb{R}^d)$. Let $g \in L^p(\mathbb{R}; L^s(\mathbb{R}^d))$, such that $g(t + 2\pi, \cdot) = g(t, \cdot)$ in $L^s(\mathbb{R}^d)$, for each $t \in \mathbb{R}$, then let us define the sequence $\{g_n\} \subset L_t^p L_x^s$ in the following way:

$$g_n(t, x) := g(nt, x).$$

Then we can prove

$$g_n \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} g(t, \cdot) dt \qquad \text{in } L_t^p L_x^s.$$

Indeed, to prove the validity of this weak limit it suffices to prove the convergence in average on sets $(a, b) \times E \subset \mathbb{R} \times \mathbb{R}^d$, where $E \subset \mathbb{R}^d$ is a bounded Borel set in $\mathbb{R}^d$. Since $g \in L_t^p L_x^s$, then the function

$$t \mapsto \frac{1}{|E|} \int_E g(t, x) dx$$

is in $L^p(\mathbb{R})$, is $2\pi-$periodic, thus it weakly converges to its average,

$$\frac{1}{|E|} \int_E g_n(\cdot, x) dx \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|E|} \int_E g(t, x) dx dt \qquad \text{in } L^p(\mathbb{R}).$$

Hence, by the convergence in average, we have that for all $(a, b) \subset \mathbb{R}$

$$\frac{1}{(b-a)|E|} \int_a^b \int_E g_n(t, x) dx dt \to \frac{1}{(b-a)|E|} \int_a^b \int_E \frac{1}{2\pi} \int_0^{2\pi} g(t', x) dt' dx dt,$$

and this clearly means

$$g_n \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} g(t, \cdot) dt \qquad \text{in } L_t^p L_x^s.$$

Now, let us consider a double scale function, i.e. a function depending on a *slow* and a *fast* variable. To best adapt the discussion below to our analysis we consider only a special class amongst those functions, namely

$$\tilde{g}(t, \tau) = g(e(t) f(\tau)).$$

In our specific case t will be the slow variable and τ the fast one. We assume f to be $2\pi-$periodic as before, $e \in \mathcal{C}^\infty(\mathbb{R})$ (or just regular enough) but not periodic in general, and g continuous and such that, as it is defined, it lies in $L^p(\mathbb{R})$. Let us define the sequence

$$g_n(t) := \tilde{g}(t, nt) = g(e(t) f(nt)), \qquad n \in \mathbb{N},$$

then we can show that

$$g_n \rightharpoonup \langle g \rangle(t) := \frac{1}{2\pi} \int_0^{2\pi} g(e(t) f(\tau)) d\tau \qquad \text{in } L^p,$$

where the convergence is weak$-*$ if $p = \infty$. Indeed, let us consider an interval $(a, b) \subset \mathbb{R}$, we have

$$\int_a^b g(e(t)f(nt))dt = \frac{1}{n} \int_{na}^{nb} g\left(e\left(\frac{t}{n}\right)f(t)\right)dt$$

$$= \frac{1}{n} \sum_{k=0}^{\left[\frac{n(b-a)}{2\pi}\right]-1} \int_{na+2k\pi}^{na+2(k+1)\pi} g\left(e\left(\frac{t}{n}\right)f(t)\right)dt + \frac{1}{n} \int_{na+\left[\frac{n(b-a)}{2\pi}\right]2\pi}^{nb} g\left(e\left(\frac{t}{n}\right)f(t)\right)dt.$$

Now, because of the continuity hypothesis on g and e, for n big enough we can approximate the integrals in the sum by

$$\frac{1}{n} \sum_{k=0}^{\left[\frac{n(b-a)}{2\pi}\right]-1} \int_{na+2k\pi}^{2(k+1)2\pi} g\left(e\left(a+\frac{2k\pi}{n}\right)f(t)\right)dt + \frac{1}{n} \int_{na+\left[\frac{n(b-a)}{2\pi}\right]2\pi}^{nb} g\left(e\left(\frac{t}{n}\right)f(t)\right)dt$$

$$= \frac{1}{n} \sum_{k=0}^{\left[\frac{n(b-a)}{2\pi}\right]-1} \int_0^{2\pi} g\left(e\left(a+\frac{2k\pi}{n}\right)f(t)\right)dt + \frac{1}{n} \int_{na+\left[\frac{n(b-a)}{2\pi}\right]2\pi}^{nb} g\left(e\left(\frac{t}{n}\right)f(t)\right)dt.$$

Again, by the continuity hypothesis on g we have, for n going to infinity we can see the expression above converges to

$$\int_a^b \frac{1}{2\pi} \int_0^{2\pi} g(e(t)f(\tau))d\tau dt.$$

Consequently, for each open set (a, b) we show

$$\lim_{n \to \infty} \frac{1}{b-a} \int_a^b g_n(t)dt = \frac{1}{b-a} \int_a^b \frac{1}{2\pi} \int_0^{2\pi} g(e(t)f(\tau))d\tau dt.$$

This clearly implies the same convergence for each Borel set $E \subset \mathbb{R}$:

$$\lim_{n \to \infty} \frac{1}{|E|} \int_E g_n(t)dt = \frac{1}{|E|} \int_E \frac{1}{2\pi} \int_0^{2\pi} g(e(t)f(\tau))d\tau dt,$$

and this, plus the uniform bound on the sequence $\{g_n\} \subset L^p(\mathbb{R})$ proves that

$$g_n \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} g(e(t)f(\tau))d\tau \qquad \text{in } L^p(\mathbb{R}).$$

Now, in a similar way as before, we can extend the same result to the case of $g \in L^p(\mathbb{R}; X)$, where X is a Banach space.

In our specific case, we consider

$$(2.1) \qquad\qquad V^\omega(t, x) = \frac{c}{|x - \vec{e}(t)\sin(\omega t)|}.$$

First of all, we notice $V^\omega \in L^\infty(\mathbb{R} : L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3))$, where p_1, p_2 are two Lebesgue exponents such that $p_1 < 3 < p_2$ and are sufficiently close to 3. Indeed, let B_t be the unit ball in $\mathbb{R}^3$ centered at the point $\vec{e}(t)\sin(\omega t)$, and let χ_{B_t} the its characteristic function. We then write $V^\omega = V_1^\omega + V_2^\omega := V^\omega \chi_{B_t} + V^\omega(1 - \chi_{B_t})$, and $V_1^\omega \in L^\infty(\mathbb{R}; L^{p_1}(\mathbb{R}^3))$, $V_2^\omega \in L^\infty(\mathbb{R}; L^{p_2}(\mathbb{R}^3))$, where $p_1 = \frac{3}{1+3\varepsilon}$, $p_2 = \frac{3}{1-3\varepsilon}$, for some small $\varepsilon > 0$. Let us furthermore notice that the norm $\|V^\omega\|_{L^\infty(\mathbb{R}:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))}$ does *not* depend on ω.

Consequently, from what we said above, we see the sequence $\{V^\omega\}$ converges weakly to the function

$$(2.2) \qquad\qquad \langle V \rangle(t, x) := \frac{1}{2\pi} \int_0^{2\pi} \frac{c}{|x - \vec{e}(t)\sin(\omega\tau)|}d\tau,$$

in $L^\infty(\mathbb{R} : L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3))$.

Here and throughout the paper we shall set $p_1 := \frac{3}{1+3\varepsilon}, p_2 := \frac{3}{1-3\varepsilon}$.

Finally, let us consider also when g is a regular function, in which case we have further convergence properties. More in particular, we consider a smooth function $\zeta \in \mathcal{C}^\infty(\mathbb{R}^3)$, such that it is in $L^\infty(\mathbb{R}^3)$, together with all its derivatives. Let

$$(2.3) \qquad g(t, \tau, x) = \zeta(x - \vec{e}(t)\sin(\tau)),$$

where $t \in [0, T]$, $\tau \in \mathbb{R}$, $x \in \mathbb{R}^3$, $0 < T < \infty$ is fixed. We state a Lemma which will be useful later on this article.

Lemma 2.4. *Let g be defined as in (2.3). Then*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \left| \int_0^t \left(\zeta(x - \vec{e}(t')\sin(\omega t')) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \zeta(x - \vec{e}(t')\sin(\tau)d\tau \right) dt' \right| \to 0,$$

as $|\omega| \to \infty$.

Proof. Let us define

$$g_l(t, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t, \tau, x)e^{-il\tau}d\tau, \qquad l \in \mathbb{Z},$$

so that we can write

$$g(t, \tau, x) = \sum_{l\in\mathbb{Z}} g_l(t, x)e^{il\tau},$$

and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t, \tau, x)|^2 d\tau = \sum_{l\in\mathbb{Z}} |g_l(t, x)|^2.$$

Furthermore, we use the Fourier transform for the slow time variable. For this purpose we extend the function g from $[0, T] \times \mathbb{R} \times \mathbb{R}^3$ to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$, such that it is smooth in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$ and it vanishes outside the slab $(-1, T+1) \times \mathbb{R} \times \mathbb{R}^3$. Thus we have

$$g(t, \tau, x) = \sum_{l\in\mathbb{Z}} e^{il\tau} \int_{\mathbb{R}} e^{i\sigma t}\hat{g}_l(\sigma, x)d\sigma = \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \hat{g}_l(\sigma, x)e^{i(\sigma t + l\tau)}d\sigma.$$

where

$$\hat{g}_l(\sigma, x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\sigma t}g_l(t, x)dt.$$

It is straightforward to see that we can write

$$\int_0^t \left(\zeta(x - \vec{e}(t')\sin(\omega t')) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \zeta(x - \vec{e}(t')\sin(\tau))d\tau \right) dt' = \int_0^t g(t', \tau, x) - g_0(t', x)dt'.$$

Conseuqently, to prove the statement of the Lemma, we must prove

$$\int_0^t \sum_{l\neq 0} \int_{\mathbb{R}} \hat{g}_l(\sigma, x)e^{i(\sigma + l\omega)t'}d\sigma dt' \to 0,$$

as $|\omega| \to \infty$, uniformly in $(t, x) \in [0, T] \times \mathbb{R}^3$.
Let us swap the integration order in the above expression, we then obtain

$$(2.4) \qquad \sum_{l\neq 0} \int_{\mathbb{R}} \hat{g}_l(\sigma, x)\frac{e^{i(\sigma + l\omega)t} - 1}{\sigma + l\omega}d\sigma.$$

Let us consider for the moment each integral in the sum,

$$\int_{\mathbb{R}} \hat{g}_l(\sigma, x)\frac{e^{i(\sigma + l\omega)t} - 1}{\sigma + l\omega}d\sigma,$$

without loss of generality we can consider now the case when $\omega > 0$ and $l > 0$. We split the above integral in two regions, inside and outside the ball centered at the origin of radius $\frac{3}{4}l\omega$.

$$\int_{\mathbb{R}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma$$

$$= \int_{\{|\sigma| \leq \frac{3}{4}l\omega\}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma + \int_{\{|\sigma| \geq \frac{3}{4}l\omega\}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma.$$

For the first one we have that, in this region, $|\sigma + l\omega| \geq \frac{1}{4}l\omega$, hence

$$\left| \int_{\{|\sigma| \leq \frac{3}{4}l\omega\}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma \right| \leq \frac{8}{l\omega} \int_{\mathbb{R}} |\hat{g}_l(\sigma, x)| d\sigma.$$

Now we have

$$(2.5) \quad \int_{\mathbb{R}} |\hat{g}_l(\sigma, x)| d\sigma \leq \left(\int_{\mathbb{R}} \frac{1}{(1 + |\sigma|^2)^{\frac{1}{2}+\varepsilon}} d\sigma \right)^{1/2} \left(\int_{\mathbb{R}} (1 + |\sigma|^2)^{\frac{1}{2}+\varepsilon} |\hat{g}_l(\sigma, x)|^2 d\sigma \right)$$

$$\lesssim \left(\int_{\mathbb{R}} |\langle D_t \rangle^{\frac{1+\varepsilon}{2}} g_l(t, x)|^2 dt \right)^{1/2}.$$

Consequently,

$$\left| \int_{\{|\sigma| \leq \frac{3}{4}l\omega\}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma \right| \lesssim \frac{1}{l\omega} \left(\int_{\mathbb{R}} |\langle D_t \rangle^{\frac{1+\varepsilon}{2}} g_l(t, x)|^2 dt \right)^{1/2},$$

which tells us that the sum of those terms is $O(\frac{1}{\omega})$, as we will see more precisely later on. On the other hand, for the second integral we use the fact that the Fourier transform of a $\mathcal{C}^\infty$ function decays faster than any polynomial, consequently we have that for each $N \in \mathbb{N}$, hence those integrals give us a contribution which is smaller than any power of ω.

$$\left| \int_{\{|\sigma| \geq \frac{3}{4}l\omega\}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma \right| \lesssim \int_{\{|\sigma| \geq \frac{3}{4}l\omega\}} |\hat{g}_l(\sigma, x)| d\sigma$$

$$\lesssim \frac{1}{(1 + (l\omega)^2)^{N/2}} \int_{\mathbb{R}} (1 + |\sigma|^2)^{N/2} |\hat{g}_l(\sigma, x)| d\sigma \lesssim \frac{1}{(1 + (l\omega)^2)^{N/2}} \left(\int_{\mathbb{R}} |\langle D_t \rangle^{\frac{N+1+\varepsilon}{2}} g_l(t, x)|^2 dt \right)^{1/2},$$

where the last inequality follows from (2.5). Thus, by taking the modulus of the sum in (2.4), we obtain

$$\left| \sum_{l \neq 0} \int_{\mathbb{R}} \hat{g}_l(\sigma, x) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma \right|$$

$$\lesssim \frac{1}{\omega} \sum_{l \neq 0} \left(\frac{1}{l} \left(\int_{\mathbb{R}} |\langle D_t \rangle^{\frac{1+\varepsilon}{2}} g_l(t, x)|^2 dt \right)^{1/2} + \frac{1}{(1 + (l\omega)^2)^{1/2}} \left(\int_{\mathbb{R}} |\langle D_t \rangle^{\frac{3+\varepsilon}{2}} g_l(t, x)|^2 dt \right)^{1/2} \right).$$

By Cauchy-Schwartz inequality in the sum above we get

$$\left| \sum_{l \neq 0} \int_{\mathbb{R}} \hat{g}_l(x, \sigma) \frac{e^{i(\sigma+l\omega)t} - 1}{\sigma + l\omega} d\sigma \right| \lesssim \frac{1}{\omega} \left(\sum_{l \neq 0} \frac{1}{l^2} \right)^{1/2} \left(\sum_{l \neq 0} \int_{\mathbb{R}} |\langle D_t \rangle^{\frac{3+\varepsilon}{2}} g(x, t)|^2 dt \right)^{1/2}$$

$$\lesssim \frac{1}{\omega} \left(\int_{-\pi}^{\pi} \int_{\mathbb{R}} |\langle D_t \rangle^{\frac{3+\varepsilon}{2}} g(x, t, \tau)|^2 dt d\tau \right)^{1/2}.$$

Hence we can conclude that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\left|\int_0^t\left(\zeta(x-\vec{e}(t')\sin(\omega t'))-\frac{1}{2\pi}\int_{-\pi}^{\pi}\zeta(x-\vec{e}(t')\sin(\tau))d\tau\right)dt'\right|$$

$$\lesssim\frac{1}{\omega}\sup_{x\in\mathbb{R}^3}\left(\int_{-\pi}^{\pi}\int_{\mathbb{R}}|\langle D_t\rangle^{\frac{3+\varepsilon}{2}}g(t,\tau,x)|^2dtd\tau\right)^{1/2}$$

$$=\frac{1}{\omega}\left(\sup_{x\in\mathbb{R}^3}\int_{-\pi}^{\pi}\int_{-1}^{T+1}|\langle D_t\rangle^{\frac{3+\varepsilon}{2}}g(t,\tau,x)|^2dtd\tau\right)^{1/2}$$

$$\lesssim\frac{2\pi(T+2)}{\omega}\|\langle D\rangle^{\frac{3+\varepsilon}{2}}\zeta\|_{L^\infty(\mathbb{R}^3)},$$

which proves the Lemma. □

2.1. **Review of Strichartz estimates and Local and Global Theory for Schrödinger Equations.** In this subsection we quickly review some basic facts about dispersive estimates for the Schrödinger equation and their application to local and global existence theory of solutions to nonlinear Schrödinger equations. Let $U(t):=e^{it\Delta}$ denote the free Schrödinger group, i.e. if u is solution to

$$\begin{cases} i\partial_t u=-\Delta u \\ u(0)=u_0, \end{cases}$$

then $u(t)=U(t)u_0$.

Definition 2.5. *We say (q,r) is an* admissible pair *of exponents if $2\le q\le\infty, 2\le r\le 6$, and*

$$\frac{1}{q}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{r}\right).$$

Here and throughout the paper we will say $\|\cdot\|_{L_t^q L_x^r}$ is a *Strichartz norm* to mean that it is a norm taken in a space such that (q,r) is an admissible pair in the sense of Definition 2.5. We will also use the notation

$$(2.6)\qquad\qquad \|f\|_{S(I)}:=\sup_{(q,r)}\|f\|_{L_t^q L^r(I\times\mathbb{R}^3)},$$

where the sup is taken over all admissible pairs (q,r).
Furthermore, we also need

$$\|f\|_{S^1(I)}:=\|f\|_{S(I)}+\|\nabla f\|_{S(I)}.$$

Now we write the Strichartz estimates we will need in our paper. Such estimates go back to Strichartz [9] which proved them in a particular case for the wave equation, then Ginibre, Velo [4], through a TT^* argument, extended the result and finally Keel, Tao [7] proved the *endpoint estimate* and general dispersive estimates in an abstract setup. Such estimates hold for a general dispersive equation in arbitrary space dimensions, but for our study we will use (and state) them only for the Schrödinger equation in $\mathbb{R}^3$.

Theorem 2.6 (Keel, Tao [7])**.** *Let $(q,r), (\tilde{q}, \tilde{r})$ be two arbitrary admissible pairs and let $U(\cdot)$ be the free Schrödinger group. Then*

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}$$

$$\|\int_0^t U(t-s)F(s)ds\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\|\int U(-s)F(s)ds\|_{L^2} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Strichartz estimates are very useful to prove existence of local solutions to nonlinear Schrödinger equations through a fixed point argument.

Indeed, let us consider the following Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u + F_1 + \ldots + F_M \\ u(0) = u_0, \end{cases}$$

for some functions $F_1, \ldots, F_M$, then by Duhamel's formula we can write

$$u(t) = U(t)u_0 - i \int_0^t U(t-s)(F_1 + \ldots F_M)(s)ds.$$

Then, by applying Strichartz estimates to the above formula we obtain

$$\|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2} + \|F_1\|_{L_t^{q_1'} L_x^{r_1'}} + \ldots + \|F_M\|_{L_t^{q_M'} L_x^{r_M'}},$$

where $(q,r), (q_1, r_1), \ldots, (q_M, r_M)$ are admissible pairs. Hence it is clear that we can bound each term F_j in an arbitrary dual Strichartz space $L_t^{q_j'} L_x^{r_j'}$. If now the F_j's are different nonlinearities, we further estimate each term $\|F_j\|_{L_t^{q_j'} L_x^{r_j'}}$ to close the fixed point argument. The reader should see the monographs [1], [10] and references therein for details.

3. Local and Global Well-Posedness

In this Section we state the local and global well-posedness results we have for equations (1.6) and (1.7). We first prove a local well-posedness result for (1.6) in the space of energy (i.e. $H^1(\mathbb{R}^3)$), then global well-posedness in the space of mass (i.e. $L^2(\mathbb{R}^3)$). Furthermore we show some uniform bounds on the $S(0,T)$ (see (2.6)) norm of solutions to equation (1.6). As we already mentioned in Section 2, both V^ω and $\langle V \rangle$ belong to the space $L^\infty(\mathbb{R} : L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3))$, where $p_1 = \frac{3}{1+3\varepsilon}, p_2 = \frac{3}{1-3\varepsilon}$, with $\varepsilon > 0$ sufficiently small, and the norm $\|V^\omega\|_{L^\infty(\mathbb{R}:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))}$ does *not* depend on ω. However, here we want to show a local well-posedness theory in H^1, hence we also need to estimate their gradient. We see that

$$\nabla V^\omega, \nabla \langle V \rangle \in L^\infty(\mathbb{R}; L^{\frac{3}{2+3\varepsilon}}(\mathbb{R}^3) + L^{\frac{3}{2-3\varepsilon}}(\mathbb{R}^3)).$$

Indeed, let us consider again the characteristic function χ_{B_t} of the unit ball B_t centered at $\vec{e}(t)\sin(\omega t) \in \mathbb{R}^3$, then $\chi_{B_t} \nabla V^\omega \in L^\infty(\mathbb{R}; L^{\frac{3}{2+3\varepsilon}}(\mathbb{R}^3))$ and $(1 - \chi_{B_t})\nabla V^\omega \in L^\infty(\mathbb{R}; L^{\frac{3}{2-3\varepsilon}}(\mathbb{R}^3))$. Again, the norm $\|\nabla V^\omega\|_{L^\infty(\mathbb{R};L^{\frac{3}{2+3\varepsilon}}(\mathbb{R}^3)+L^{\frac{3}{2-3\varepsilon}}(\mathbb{R}^3))}$ does *not* depend on ω. Consequently, by the weak convergence, we can say that $\langle V \rangle$ is in $L^\infty(\mathbb{R}; L^{\frac{3}{2+3\varepsilon}}(\mathbb{R}^3) + L^{\frac{3}{2-3\varepsilon}}(\mathbb{R}^3))$, too.

Consequently here we investigate the Cauchy problem

$$(3.1) \qquad \begin{cases} i\partial_t u = -\Delta u + Vu + C_1(|\cdot|^{-1} * |u|^2)u - a|u|^\sigma u \\ u(0) = u_0, \end{cases}$$

where V is a general potential such that $V = V_1 + V_2$, where $V_1 \in L^\infty(\mathbb{R}; L^{p_1}(\mathbb{R}^3))$ and $V_2 \in L^\infty(\mathbb{R}; L^{p_2}(\mathbb{R}^3))$, and $\nabla V = \nabla V_1 + \nabla V_2$, where $\nabla V_1 \in L^\infty(\mathbb{R}; L^{\frac{3}{2+3\varepsilon}}(\mathbb{R}^3))$

and $\nabla V_2 \in L^\infty(\mathbb{R}; L^{\frac{3}{2-3\varepsilon}}(\mathbb{R}^3))$. In this way we have the well-posedness results below apply both to equation (1.6) and (1.7).

Theorem 3.1. *Assume $0 < \sigma < 4$, $u_0 \in H^1(\mathbb{R}^3)$. Then, there exists a unique local solution to (3.1). Furthermore, we have $u \in S^1(0,T)$.*

The proof of the Theorem above is standard, based on a fixed point argument, see for example [1], [10], [5], [6]. Nevertheless, for the sake of clarity and completeness, we sketch the main steps.
For this purpose. let us first write two techincal lemmas which will be used in what follows.

Lemma 3.2. *Let $k = 0, 1$. Then*

$$(3.2) \qquad \|\nabla^k(|\cdot|^{-1} * |f|^2)f\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^3)} \lesssim T^{\frac{1}{2}}\|f\|_{L_t^6 L_x^{18/7}}\|\nabla^k f\|_{L_t^6 L_x^{18/7}};$$

$$(3.3) \qquad \|\nabla^k(|f|^\sigma f)\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}} \lesssim T^{\frac{2(\sigma+2)}{4-\sigma}}\|f\|_{L_t^\infty H_x^1}^\sigma\|\nabla^k f\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\sigma+2}};$$

$$(3.4) \quad \|(|\cdot|^{-1} * |f|^2)f - (|\cdot|^{-1} * |g|^2)g\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^3)}$$
$$\lesssim T^{\frac{1}{2}}\left(\|f\|_{L_t^6 L_x^{18/7}}^2 + \|g\|_{L_t^6 L_x^{18/7}}^2\right)\|f - g\|_{L_t^6 L_x^{18/7}};$$

$$(3.5) \quad \||f|^\sigma f - |g|^\sigma g\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}}$$
$$\lesssim T^{\frac{2(\sigma+2)}{4-\sigma}}\left(\|f\|_{L_t^\infty H_x^1}^\sigma + \|g\|_{L_t^\infty H_x^1}^\sigma\right)\|f - g\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\sigma+2}};$$

Proof. (3.2) is a consequence of the following inequality

$$(3.6) \qquad \|(|\cdot|^{-1} * (f_1 f_2))f_3\|_{L_t^1 L_x^2} \lesssim T^{\frac{1}{2}}\prod_{i=1}^3 \|f_i\|_{L_t^6 L_x^{18/7}},$$

which can be proved by using Hölder's inequality and Hardy-Littlewood-Sobolev inequality.
In a similar way by Hölder's inequality we get

$$(3.7) \qquad \||f_1|^\sigma f_2\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}} \lesssim T^{\frac{2(\sigma+2)}{4-\sigma}}\|f_1\|_{L_t^\infty L_x^{\sigma+2}}^\sigma\|f_2\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\sigma+2}},$$

and then, since $0 < \sigma < 4$, we can use Sobolev embedding to show that

$$\||f_1|^\sigma f_2\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}} \lesssim T^{\frac{2(\sigma+2)}{4-\sigma}}\|f_1\|_{L_t^\infty H_x^1}^\sigma\|f_2\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\sigma+2}}.$$

Now, (3.3) follows from the fact that $|\nabla(|f|^\sigma f)| \lesssim |f|^\sigma|\nabla f|$.
The proof of inequalitites (3.4) and (3.5) are similar to (3.2) and (3.3), respectively. In particular, by using some algebra and (3.6) yields (3.4), whereas (3.5) follows from (3.7) and

$$(3.8) \qquad \||f|^\sigma f - |g|^\sigma g| \lesssim (|f|^\sigma + |g|^\sigma)|f - g|.$$

$\square$

The second technical lemma estimates the terms with the Coulomb potentials, both in (1.6) and in (1.7).

Lemma 3.3.

$$(3.9) \qquad \|\nabla V f\|_{L_t^2 L_x^{6/5}([0,T]\times\mathbb{R}^3)} \leq T^{\frac{3\varepsilon}{2}}\|\nabla V_1\|_{L_t^\infty L_x^{\frac{3}{2+3\varepsilon}}}\|f\|_{L_t^{\frac{2}{1-3\varepsilon}} L^{\frac{6}{1+6\varepsilon}}};$$

$$(3.10) \qquad \|V_1(1 + |\nabla|)f\|_{L_t^2 L_x^{6/5}} \leq T^{\frac{1-3\varepsilon}{2}}\|V_1\|_{L_t^\infty L_x^{\frac{3}{1+3\varepsilon}}}\|(1 + |\nabla|)f\|_{L_t^{\frac{2}{3\varepsilon}} L_x^{\frac{2}{1-2\varepsilon}}};$$

(3.11) $\|V_2(1+|\nabla|)f\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^3)} \le T^{\frac{1+3\varepsilon}{2}}\|V_2\|_{L_t^\infty L_x^{\frac{3}{1-3\varepsilon}}}\|(1+|\nabla|)f\|_{L_t^{\frac{2}{1-3\varepsilon}} L_x^{\frac{6}{1+6\varepsilon}}}.$

Remark 3.4. *Let us notice that the pair of exponents* $(6,\frac{18}{7})$, $\left(\frac{4(\sigma+2)}{3\sigma},\sigma+2\right)$, $\left(\frac{2}{1-3\varepsilon},\frac{6}{1+6\varepsilon}\right)$, $\left(\frac{2}{3\varepsilon},\frac{2}{1-2\varepsilon}\right)$, *are all Schrödinger admissible, thus the norms in those spaces are all bounded by the* $S(I)$ *norm.*

We can now sketch the proof of Theorem 3.1

Proof. Let $u_0 \in H^1(\mathbb{R}^3)$ be given. By the Duhamel's formula we have

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}\left[Vu + C_1(|\cdot|^{-1}*|u|^2)u - a|u|^\sigma u\right](s)ds.$$

We want to show that, for $0 < T$ sufficiently small,

$$\Phi(w) := e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}\left[Vu + C_1(|\cdot|^{-1}*|u|^2)u - a|u|^\sigma u\right](s)ds$$

maps a ball $B_R \subset S^1(0,T)$ (the radius R will be chosen later) into itself, and that in this ball Φ is a contraction in the $S(0,T)$ metric.
By Strichartz estimates we have

$$\|\Phi(w)\|_{S^1(0,T)} \lesssim \|u_0\|_{H^1} + \|\nabla V w\|_{L_t^2 L_x^{6/5}} + \|V_1(1+|\nabla|)w\|_{L_t^2 L_x^{6/5}} + \|V_2(1+|\nabla|)w\|_{L_t^1 L_x^2}$$
$$+ \|(1+|\nabla|)\left((|\cdot|^{-1}*|w|^2)w\right)\|_{L_t^1 L_x^2} + \||w|^\sigma(1+|\nabla|)w\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}}.$$

Now we use inequalities (3.9), (3.10), (3.11), (3.2), (3.3) to obtain

$$\|\Phi(w)\|_{S^1(0,T)} \lesssim \|u_0\|_{H^1} + T^{\frac{3\varepsilon}{2}}\|\nabla V_1\|_{L_t^\infty L_x^{\frac{3}{2+3\varepsilon}}}\|w\|_{S^1}$$
$$+ T^{\frac{1-3\varepsilon}{2}}\|\nabla V_2\|_{L_t^\infty L_x^{\frac{3}{2-3\varepsilon}}}\|w\|_{S^1}$$
$$+ T^{\frac{1-3\varepsilon}{2}}\|V_1\|_{L_t^\infty L_x^{\frac{3}{1+3\varepsilon}}}\|w\|_{S^1}$$
$$+ T^{\frac{1+3\varepsilon}{2}}\|V_2\|_{L_t^\infty L_x^{\frac{3}{1-3\varepsilon}}}\|w\|_{S^1}$$
$$+ T^{\frac{1}{2}}\|w\|_{S^1}^3$$
$$+ T^{\frac{2(\sigma+2)}{4-\sigma}}\|w\|_{S^1}^{\sigma+1}.$$

Thus, if we take $0 < T \le 1$ sufficiently small, we have $\Phi : B_r \to B_R \subset S^1(0,T)$, for some radius depending on $\|u_o\|_{H^1}$. Furthermore, by using (3.4) and (3.5), we also have

$$\|\Phi(w_1) - \Phi(w_2)\|_{S^1(0,T)} \lesssim + T^{\frac{3\varepsilon}{2}}\|\nabla V_1\|_{L_t^\infty L_x^{\frac{3}{2+3\varepsilon}}}\|w_1 - w_2\|_{S(0,T)}$$
$$+ T^{\frac{1-3\varepsilon}{2}}\|\nabla V_2\|_{L_t^\infty L_x^{\frac{3}{2-3\varepsilon}}}\|w_1 - w_2\|_{S(0,T)}$$
$$+ T^{\frac{1-3\varepsilon}{2}}\|V_1\|_{L_t^\infty L_x^{\frac{3}{1+3\varepsilon}}}\|w_1 - w_2\|_{S(0,T)}$$
$$+ T^{\frac{1+3\varepsilon}{2}}\|V_2\|_{L_t^\infty L_x^{\frac{3}{1-3\varepsilon}}}\|w_1 - w_2\|_{S(0,T)}$$
$$+ T^{\frac{1}{2}}\left(\|w_1\|_{S^1}^2 + |w_2\|_{S^1}^2\right)\|w_1 - w_2\|_{S(0,T)}$$
$$+ T^{\frac{2(\sigma+2)}{4-\sigma}}\left(\|w_1\|_{S^1}^\sigma + \|w_2\|_{S^1}^\sigma\right)\|w_1 - w_2\|_{S(0,T)}.$$

Once again, if we take $0 < T \le 1$ small enough, then we have $\Phi : B_R \to B_R$ is a contraction in the $S(0,T)$ metric. Thus there exists a fixed point for Φ which is hence a local solution for (3.1).

$\square$

Next Theorem deals with the well-posedness issue in $L^2(\mathbb{R}^3)$ for (1.6) and (1.7). We show that when the power σ is *mass-subcritical*, i.e. $0 < \sigma < \frac{4}{3}$, then for any initial datum in L^2 we have a *global* solution. We also stress here that the uniform bound we obtain for the $S(0,T)$ norm of the solution does *not* depend on ω.

Theorem 3.5. *Assume $0 < \sigma < 4/3$ and consider $u_0 \in L^2(\mathbb{R}^3)$. The solution for (3.1) is global, $u \in C(\mathbb{R} : L^2(\mathbb{R}^3))$. Furthermore for each finite time $0 < T < \infty$ we have*

$$(3.12) \qquad \|u\|_{S(0,T)} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}, T),$$

where the constant in the right hand side depends only on the L^2-norm of the initial datum and the time T, in particular it does not depend on ω.

Proof. The proof works as for Theorem 3.1 at a local level and then we use the conservation of mass to extend the solution globally.

Let us consider the Duhamel's formula

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} \left(Vu + C_1(|\cdot|^{-1} * |u|^2)u - a|u|^\sigma u \right)(s) ds,$$

then by applying the Strichartz estimates to the expression above in the time interval $[0, T_1]$, we get

$$\|u\|_{S(0,T_1)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)} + T^{\frac{1-3\varepsilon}{2}} \|V_1\|_{L_t^\infty L_x^{\frac{3}{1+3\varepsilon}}} \|u\|_{S(0,T_1)} + T^{\frac{1+3\varepsilon}{2}} \|V_2\|_{L_t^\infty L_x^{\frac{3}{1-3\varepsilon}}} \|u\|_{S(0,T_1)}$$
$$+ T_1^{1/2} \|u\|_{S(0,T_1)}^3 + T_1^{\frac{4-3\sigma}{4}} \|u\|_{S(0,T_1)}^{\sigma+1}.$$

Now we can see that if we choose $T_1 = T_1(\|u_0\|_{L^2})$ small enough, then by a standard fixed point argument we have

$$(3.13) \qquad \|u\|_{S(0,T_1)} \leq C\|u_0\|_{L^2}.$$

Furthermore, the total mass is conserved at all times, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Thus we can repeat the argument to continue the solution also in the time interval $[T_1, 2T_1]$, and so on. Consequently the solution is global, and for any finite time $0 < T < \infty$, we consider $[0, T] \subset [0, T_1] \cup \ldots \cup [(N-1)T_1, NT_1]$, $N = \left[\frac{T}{T_1}\right] + 1$, where here $[\cdot]$ denotes the integer part of the number. Thus

$$\|u\|_{S(0,T)} \leq CN\|u_0\|_{L^2(\mathbb{R}^3)},$$

where C is the constant appearing in (3.13). Consequently

$$\|u\|_{S(0,T)} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}, T),$$

for each finite time $0 < T < \infty$. $\square$

Remark 3.6. *Here is a couple of remarks about the Theorem above.*

- *Regarding the case when the power nonlinearity is mass-supercritical, energy-subcritical (i.e. $\frac{4}{3} < \sigma < 4$), we cannot establish a global well-posedness result in $H^1(\mathbb{R}^3)$, not even in the defocusing case (i.e. $a < 0$), because the energy is not conserved in our model, and the bound on the H^1-norm of the solution $u(t)$ at time t in general would depend on ω.*

- *On the other hand, the time dependence of the constant in (3.12) is unavoidable, because of the mass-subcriticality of the power-type nonlinearity. This is indeed what also happens for the usual NLS equation (see [10] for instance).*

4. CONVERGENCE RESULT

In this Section we will prove the main result of this paper, namely Theorem 1.1. As already introduced in Section 1, we want to show the convergence of solutions for

$$(4.1) \quad \begin{cases} i\partial_t u^\omega = -\Delta u^\omega + V^\omega u^\omega + C_1(|\cdot|^{-1} * |u^\omega|^2)u^\omega - a|u^\omega|^\sigma u^\omega \\ u^\omega(0) = u_0, \end{cases}$$

where V^ω is defined in (2.1), to solutions of

$$(4.2) \quad \begin{cases} i\partial_t u = -\Delta u + \langle V \rangle u + C_1(|\cdot|^{-1} * |u|^2)u - a|u|^\sigma u \\ u(0) = u_0, \end{cases}$$

where the averaged potential $\langle V \rangle$ has been defined in (2.2).

Let us recall the definition of the Strichartz space, already given in Section 2, which is

$$\|f\|_{S(0,T)} := \sup_{(q,r)} \|f\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^3)},$$

where the sup is taken over all admissible pairs (q, r).

The key point of the convergence result stated in Theorem 1.1 is the Lemma below: indeed the weak convergence of V^ω towards $\langle V \rangle$ improves to strong convergence for u^ω towards u because, by considering the difference of the Duhamel's formulas for (4.1) and (4.2), the term $V^\omega - \langle V \rangle$ appears inside the time integral, and thus the convergence in average for the oscillating potential yields the strong convergence for the solutions. This fact, together with the uniform bounds (3.12), provides us the right convergence result. A similar result is considered also in [2], where the authors study the solutions of a nonlinear Schrödinger equation with an oscillating nonlinearity and their asymptotic behavior when the oscillations are increasing more and more. The Lemma above is heavily inspired by Proposition 2.5 in [2].

Lemma 4.1. *Let* $0 < T < \infty$, $f \in S(0,T)$, *and let* $V^\omega, \langle V \rangle$ *be defined in* (2.1), (2.2). *Then*

$$(4.3) \quad \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle)f \right)(s)ds \|_{S(0,T)} \to 0,$$

as $|\omega| \to \infty$.

Proof. First of all let us point out that the norm appearing in (4.3) is uniformly bounded. Indeed by using Strichartz estimates we have

$$(4.4) \quad \| \int_0^t e^{i(t-s)\Delta}(Vf)(s)ds \|_{S(0,T)} \lesssim \|V_1 f\|_{L_t^2 L_x^{6/5}} + \|V_2 f\|_{L_t^1 L_x^2}$$

$$\lesssim T^{\frac{1-3\varepsilon}{2}} \|V_1\|_{L_t^\infty L^{p_1}} \|f\|_{L_t^{\frac{2}{3\varepsilon}} L_x^{\frac{2}{1-2\varepsilon}}} + T^{\frac{1+3\varepsilon}{2}} \|V_2\|_{L_t^\infty L_x^{p_2}} \|f\|_{L_t^{\frac{2}{1-3\varepsilon}} L_x^{\frac{6}{1+6\varepsilon}}}$$

$$\lesssim T^\alpha \|V\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))} \|f\|_{S(0,T)}.$$

where V can be either V^ω or $\langle V \rangle$, and $\alpha > 0$. For the integral in (4.3) we have the following uniform bound

$$\| \int_0^t e^{i(t-s)\Delta} \left[(V^\omega - \langle V \rangle)f \right](s)ds \|_{S(0,T)} \lesssim$$

$$T^\alpha (\|V^\omega\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))} + \|\langle V \rangle\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))}) \|f\|_{S(0,T)}.$$

Thanks to this a priori bound, by using a standard density argument it suffices to prove (4.3) only for $V^\omega, \langle V \rangle, f \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^3)$. For the sake fo clarity we explain the

last statement more in detalis. Let $\{f_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^{1+3})$ be a sequence of compactly supported smooth functions such that $f_n \to f$ in $S(0,T)$. Furthermore, set

$$V_n^\omega := \phi_{\frac{1}{n}} * V^\omega,$$

where $\phi_{\frac{1}{n}}(x)$ is a Gaussian with variance equal to $\frac{1}{n}$, and the convolution is only in space. The following properties for $\{V_n^\omega\}$ hold true:

- $V_n^\omega \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^3)$ and $D^\alpha V_n^\omega \in L^\infty(\mathbb{R}^3)$, for each multlindex α;
- $V_n^\omega \to V^\omega$ in $L^\infty(\mathbb{R}; L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3))$, as $n \to \infty$;
- $V_n^\omega \rightharpoonup \langle V \rangle_n$, where $\langle V \rangle_n = \phi_{\frac{1}{n}} * \langle V \rangle$ is the regularisation of the averaged potential.

Clearly we also have $\langle V \rangle_n \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^3)$, $\langle V \rangle_n \to \langle V \rangle$ in $L^\infty(\mathbb{R}; L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3))$, as $n \to \infty$. We now consider the integral in (4.3), we split it into four parts

$$\| \int_0^t e^{i(t-s)\Delta} \left[(V^\omega - \langle V \rangle) f \right] (s) ds \|_{S(0,T)}$$

$$\leq \| \int_0^t e^{i(t-s)\Delta} \left[(V^\omega - \langle V \rangle)(f - f_n) \right] (s) ds \|_{S(0,T)}$$

$$+ \| \int_0^t e^{i(t-s)\Delta} \left[(V^\omega - V_n^\omega) f_n \right] (s) ds \|_{S(0,T)}$$

$$+ \| \int_0^t e^{i(t-s)\Delta} \left[(\langle V \rangle_n - \langle V \rangle) f_n \right] (s) ds \|_{S(0,T)}$$

$$+ \| \int_0^t e^{i(t-s)\Delta} \left[(V_n^\omega - \langle V \rangle_n) f_n \right] (s) ds \|_{S(0,T)}$$

$$=: I_1 + I_2 + I_3 + I_4.$$

Let us apply the estimate (4.4) to the terms I_1, I_2, I_3, we then obtain

$$I_1 \leq T^\alpha (\|V^\omega\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))} + \|\langle V \rangle\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))}) \|f - f_n\|_{S(0,T)},$$
$$I_2 \leq T^\alpha \|V^\omega - V_n^\omega\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))} \|f_n\|_{S(0,T)},$$
$$I_3 \leq T^\alpha \|\langle V \rangle_n - \langle V \rangle\|_{L^\infty([0,T]:L^{p_1}(\mathbb{R}^3)+L^{p_2}(\mathbb{R}^3))} \|f_n\|_{S(0,T)}.$$

Thus it only remains to estimate the integral

$$(4.5) \qquad \int_0^t e^{i(t-s)\Delta} \left[(V_n^\omega - \langle V \rangle_n) f_n \right] (s) ds.$$

By integrating by parts we have

$$\int_0^t e^{i(t-s)\Delta} \left\{ \frac{d}{ds} \left[\int_0^s V_n^\omega - \langle V_n \rangle ds' f_n(s) \right] - \int_0^s V_n^\omega - \langle V_n \rangle ds' \partial_s f_n(s) \right\} ds,$$

and then again we have

$$\int_0^t \frac{d}{ds} \left[e^{i(t-s)\Delta} \left(\int_0^s V_n^\omega - \langle V_n \rangle ds' f_n(s) \right) \right] ds$$

$$+ i \int_0^t e^{i(t-s)\Delta} \Delta \left(\int_0^s V_n^\omega - \langle V_n \rangle ds' f_n(s) \right) ds - \int_0^t e^{i(t-s)\Delta} \left(\int_0^s V_n^\omega - \langle V_n \rangle ds' \partial_s f_n(s) \right) ds.$$

Hence, we get the integral in (4.5) equals

$$\int_0^t V_n^\omega - \langle V_n \rangle ds \, f_n(t)$$

$$+ i \int_0^t e^{i(t-s)\Delta} \Big[\int_0^s \Delta V_n^\omega - \Delta \langle V_n \rangle ds' \, f_n(s) + 2 \int_0^s \nabla V_n^\omega - \nabla \langle V_n \rangle ds' \cdot \nabla f_n$$

$$+ \int_0^s V_n^\omega - \langle V_n \rangle ds' \Delta f_n(s) \Big] ds$$

$$- \int_0^t e^{i(t-s)\Delta} \left(\int_0^s V_n^\omega - \langle V_n \rangle ds' \partial_s f_n(s) \right) ds.$$

Now we estimate each term in the expression above in the space $S(0,T)$. By Strichartz estimates the first term is bounded by

$$\| \int_0^t V_n^\omega - \langle V \rangle ds f_n(t) \|_{S(0,T)} \leq \| \int_0^t (V_n^\omega - \langle V_n \rangle) \, ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| f_n \|_{S(0,T)}.$$

The other terms are estimated similarly, let us consider the next one for example

$$\| \int_0^t e^{i(t-s)\Delta} \Big[\int_0^s \Delta V^\omega - \langle \Delta V_n \rangle ds' f_n(s) \Big] ds \|_{S(0,T)}$$

$$\lesssim \| \int_0^t (\Delta V_n^\omega - \langle \Delta V_n \rangle) \, ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| f_n \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

for some admissible pair $(\tilde{q}, \tilde{r})$.

Consequently, by putting everything together we obtain

$$\| \int_0^t e^{i(t-s)\Delta} \left[(V_n^\omega - \langle V \rangle_n) f_n \right](s) ds \|_{S(0,T)}$$

$$\lesssim \| \int_0^t V_n^\omega - \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| f_n \|_{S(0,T)}$$

$$+ \| \int_0^t \Delta V_n^\omega - \Delta \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| f_n \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$+ \| \int_0^t \nabla V_n^\omega - \nabla \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| \nabla f_n \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$+ \| \int_0^t V_n^\omega - \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \| (i\partial_t + \Delta) f_n \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

We notice that thanks to the properties of $V_n^\omega, \langle V \rangle_n$, we can apply Lemma 2.4 to say that

$$\| \int_0^t V_n^\omega - \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \to 0, \qquad \text{as } |\omega| \to \infty.$$

Furthermore, the same Lemma applies to any derivatives of $V_n^\omega, \langle V \rangle_n$,

$$\| \int_0^t D^\alpha V_n^\omega - D^\alpha \langle V \rangle_n ds \|_{L_{t,x}^\infty([0,T] \times \mathbb{R}^3)} \to 0, \qquad \text{as } |\omega| \to \infty,$$

for any multi-index $\alpha \in \mathbb{N}^3$.

Consequently, for each fixed $n \in \mathbb{N}$, the term I_4 converges to zero in $S(0,T)$ as $|\omega| \to \infty$.

Now, let $\varepsilon > 0$ be arbitrarily small, thus we can choose $n^* \in \mathbb{N}$ big enough so that

$$I_1 + I_2 + I_3 \leq \varepsilon, \qquad \forall \, n \geq n^*.$$

Thus, for the integral in (4.3), we have

$$\| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) f \right) (s) ds \|_{S(0,T)} \leq \varepsilon + I_4,$$

and in the limit as $|\omega| \to \infty$, we obtain

$$\lim_{|\omega| \to \infty} \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) f \right) (s) ds \|_{S(0,T)} \leq \varepsilon, \quad \forall \, \varepsilon > 0.$$

Hence the limit is zero and the Lemma is proved.

$\square$

We are now ready to prove Theorem 1.1.
Let u^ω, u be solutions to (4.1), (4.2), respectively, with the same initial datum, and let us consider the equation for the difference $v := u^\omega - u$, which reads
(4.6)
$$\begin{cases} i\partial_t v = -\Delta v + (V^\omega - \langle V \rangle) u + V^\omega v + C_1[(|\cdot| * |u^\omega|^2) u^\omega - (|\cdot| * |u|^2) u] - a[|u^\omega|^\sigma u^\omega - |u|^\sigma u] \\ v(0) = 0. \end{cases}$$

By the Duhamel's formula and by applying Strichartz estimates to each term we obtain By applying Strichartz estimates we obtain

$$\begin{aligned} \|v\|_{S(0,T)} \lesssim & \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) u \right) (s) ds \|_{S(0,T)} \\ & + \|V_1^\omega v\|_{L_t^2 L_x^{6/5}([0,T] \times \mathbb{R}^3)} + \|V_2^\omega v\|_{L_t^1 L_x^2([0,T] \times \mathbb{R}^3)} \\ & + \|(|\cdot|^{-1} * |u^\omega|^2) u^\omega - (|\cdot|^{-1} * |u|^2) u\|_{L_t^1 L_x^2([0,T] \times \mathbb{R}^3} \\ & + \||u^\omega|^\sigma u^\omega - |u|^\sigma u\|_{L_t^{\frac{4(\sigma+2)}{\sigma+8}} L_x^{\frac{\sigma+2}{\sigma+1}}}. \end{aligned}$$

Now we use inequalities (3.10), (3.11), (3.4), (3.5), to get

$$\begin{aligned} \|v\|_{S(0,T)} \lesssim & \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) u \right) (s) ds \|_{S(0,T)} \\ & + T^{\frac{1-3\varepsilon}{2}} \|V_1^\omega\|_{L_t^\infty L_x^{\frac{3}{1+3\varepsilon}}} \|v\|_{S(0,T)} \\ & + T^{\frac{1+3\varepsilon}{2}} \|V_2^\omega\|_{L_t^\infty L_x^{\frac{3}{1-3\varepsilon}}} \|v\|_{S(0,T)} \\ & T^{\frac{1}{2}} \left(\|u^\omega\|_{S(0,T)}^2 + \|u\|_{S(0,T)}^2 \right) \|v\|_{S(0,T)} \\ & T^{\frac{2(\sigma+2)}{4-\sigma}} \left(\|u^\omega\|_{S(0,T)}^\sigma + \|u\|_{S(0,T)}^\sigma \right) \|v\|_{S(0,T)} \\ \lesssim & \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) u \right) (s) ds \|_{S(0,T)} + T^\alpha C(\|u^\omega\|_{S(0,T)}, \|u\|_{S(0,T)}) \|v\|_{S(0,T)}. \end{aligned}$$

Let us now recall that, by (3.12), the norms $\|u^\omega\|_{S(0,T)}$ are uniformly bounded, indipendently on ω. Consequently the constant C in the inequality above depends only on T and $\|u_0\|_{L^2(\mathbb{R}^3)}$. Thus we can use Gronwall's inequality to obtain

$$\|v\|_{S(0,T)} \lesssim e^{cT} \| \int_0^t e^{i(t-s)\Delta} \left((V^\omega - \langle V \rangle) u \right) (s) ds \|_{S(0,T)}.$$

By applying Lemma 4.1 we finally prove that

$$\|u^\omega - u\|_{S(0,T)} \to 0, \qquad \text{as } |\omega| \to \infty,$$

for all finite times $0 < T < \infty$. Hence the Theorem is proved.

$\square$

References

[1] T. Cazenave, *Semilinear Schrödinger Equations*. Courant Lecture Notes in Mathematics vol. 10, New York University, Courant Institute of Mathematical Sciences, AMS, 2003.

[2] T. Cazenave, M. Scialom, *A Schrödinger equation with time-oscillating nonlinearity*, Rev. Mat. Univ. Complut. Madrid **23**, no. 2 (2010), 321–339.

[3] A. Fratalocchi, G. Ruocco, *Molecular imaging with X-ray Free Electron Lasers: dream or reality?*, Phys. Rev. Lett **106** (2011), 105504.

[4] J. Ginibre, G. Velo, *Smoothing Properties and Retarded Estimates for Some Dispersive Evolution Equations*, Comm. Math. Phys. **123** (1989), 535–573.

[5] J. Ginibre, G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poinc. Anal. Non Linéaire **2** (1985), 309–327.

[6] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poinc. Phys. Théor. **46** (1987), 113–129.

[7] M. Keel and T. Tao, *Endpoint Strichartz Estimates*. Amer. J. Math. **120** (1998), 955–980.

[8] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*. Springer-Verlag, New York, 2009.

[9] R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44**, no. 3 (1977), 705–714.

[10] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*. CBMS Regional Conference Series in Mathematics, AMS 2006.

[11] Y. Tsutsumi, L^2-*Solutions for Nonlinear Schödinger Equations and Nonlinear Groups*, Funkcial. Ekvac., **30** (1987), 115–125.

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